

**A BOUNDARY ELEMENT FORMULATION
FOR FRICTIONAL CONTACT AND CENTRIFUGAL
FORCES IN ORTHOTROPIC MATERIALS**

LEONARDO BERNARDO E SILVA

**TESE DE DOUTORADO EM CIÊNCIAS MECÂNICAS
DEPARTAMENTO DE ENGENHARIA MECÂNICA**

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Leonardo Bernardo e Silva

ABSTRACT

Title: A Boundary Element Formulation for Frictional Contact and Centrifugal Forces in Orthotropic Materials

This thesis presents a Boundary Element Method (BEM) formulation for the analysis of two-dimensional frictional contact problems in orthotropic materials under fretting fatigue conditions and centrifugal loading. The main novelty is the exact transformation of the centrifugal body-force domain integral into an equivalent boundary integral, preserving the boundary-only character of the BEM. The boundary is discretized using discontinuous quadratic elements, and the nodal displacements and tractions are expressed in a local normal–tangent reference system. A node-to-node contact strategy is adopted, in which the normal contact conditions, the tangential friction law, and force equilibrium are enforced incrementally to accurately describe the evolution of the contact regions.

A key contribution of this work is the extension of a boundary-only BEM formulation to frictional contact problems in orthotropic materials subjected to centrifugal body forces. This is achieved by incorporating radial integration into the contact formulation, thereby avoiding domain discretization, internal points, and radial basis function interpolation while preserving the computational advantages of the BEM.

The proposed methodology is validated through representative benchmark problems and engineering applications. First, the analysis of a rotating orthotropic disk segment verifies the correct representation of centrifugal body forces through comparison with analytical solutions. Second, a flat-ended punch contact problem involving orthotropic materials demonstrates excellent agreement with published benchmark numerical results. Third, a benchmark isotropic fretting problem is used to validate the incremental contact algorithm and frictional contact strategy through comparison with analytical and finite element solutions. Finally, a dovetail joint in a turbine blade subjected to centrifugal loading is analyzed to demonstrate the applicability of the proposed formulation to an engineering problem and to investigate the influence of friction and material orientation on the contact response.

The results demonstrate that the proposed formulation is an accurate and robust tool for the analysis of frictional contact problems, providing a consistent framework for engineering applications involving anisotropic materials and centrifugal loading conditions.

Keywords: Boundary element method, Frictional contact, Orthotropic materials, Contact mechanics, Centrifugal body forces, Radial integration.

RESUMO

Título: Uma Formulação de Elementos de Contorno para Contato com Atrito e Forças Centrífugas em Materiais Ortotrópicos

Esta tese apresenta uma formulação do Método dos Elementos de Contorno (MEC) para a análise de problemas bidimensionais de contato com atrito em materiais ortotrópicos submetidos a condições de fretting e carregamentos centrífugos. A principal novidade consiste na transformação exata da integral de domínio associada às forças de corpo centrífugas em uma integral equivalente de contorno, preservando a característica de discretização apenas do contorno do MEC. O contorno é discretizado por elementos quadráticos descontínuos, e os deslocamentos e as trações nodais são representados em um sistema de referência local normal-tangente. Adota-se uma estratégia de contato nó a nó, na qual as condições de contato na direção normal, a lei de atrito na direção tangencial e o equilíbrio de forças são impostos de forma incremental para descrever adequadamente a evolução das regiões de contato.

Uma das principais contribuições deste trabalho é a extensão de uma formulação do MEC baseada exclusivamente na discretização do contorno para problemas de contato com atrito em materiais ortotrópicos sujeitos a forças de corpo centrífugas. Essa extensão é obtida pela incorporação da técnica de integração radial à formulação de contato, eliminando a necessidade de discretização do domínio, de pontos internos e da interpolação por funções de base radial, preservando, ao mesmo tempo, as vantagens computacionais inerentes ao MEC.

A metodologia proposta é validada por meio de problemas de referência e aplicações de engenharia representativas. Inicialmente, a análise de um segmento de disco rotativo ortotrópico comprova a correta representação das forças de corpo centrífugas por meio da comparação com soluções analíticas. Em seguida, um problema de contato do tipo flat-ended punch envolvendo materiais ortotrópicos apresenta excelente concordância com resultados numéricos de referência disponíveis na literatura. Posteriormente, um problema clássico de fretting em material isotrópico é empregado para validar o algoritmo incremental de contato e a estratégia de contato com atrito por meio da comparação com soluções analíticas e de elementos finitos. Por fim, uma junta do tipo dovetail em pás de turbinas, submetida a carregamento centrífugo, é analisada para demonstrar a aplicabilidade da formulação proposta a um problema de engenharia, bem como investigar a influência do atrito e da orientação do material na resposta de contato.

Os resultados demonstram que a formulação proposta constitui uma ferramenta precisa e robusta para a análise de problemas de contato com atrito, fornecendo uma estrutura consistente para aplicações de engenharia envolvendo materiais anisotrópicos e condições de carregamento centrífugo.

Palavras-chave: Método dos Elementos de Contorno, contato com atrito, materiais ortotrópicos; mecânica do contato, forças de corpo centrífugas, integração radial.

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List of Symbols

Latin Symbols

A_{jk}, B_{jk} Complex coefficients associated with Lekhnitskii's stress functions ($j, k = 1, 2$)

b_i Body force vector component

$C_{ij}(\xi)$ Tensor of geometric free-coefficients of the boundary integral equation

$C_{\alpha\beta}$ Elastic stiffness constants (Voigt notation)

E Young's modulus (longitudinal elasticity)

F_{ij}^* Auxiliary tensor for the Radial Integration Method (RIM)

g_N, g_T Normal contact gap and relative tangential slip displacement

G Shear modulus (transverse elasticity)

G_{ij}, H_{ij} Global influence matrices of the Boundary Element Method

h Thickness

J Jacobian matrix / determinant of the coordinate transformation

n_i Outward normal vector component

N Number of boundary nodes

P, Q Total normal and tangential contact loads

P_{max}, Q_{max} Peak or maximum normal and tangential forces in cyclic loading

$p(x), q(x)$ Local normal pressure and tangential friction stress distributions

p_k Complex roots of the Lekhnitskii characteristic equation ($k = 1, 2$)

$Q_{\alpha\beta}$ Transformed elastic stiffness matrix for anisotropic materials

r Radial distance between the source point and the field point ($r = |x - \xi|$)

$S_{\alpha\beta}$ Elastic compliance constants

t_i Boundary traction vector component

U_{ij}^*, T_{ij}^* Anisotropic displacement and traction fundamental solutions (Green's functions)

u_i Displacement vector component

$\mathcal{U}$ Potential function for conservative body forces

x_i, ξ_i Cartesian coordinates of the field point and source point, respectively

Greek Symbols

β_{ij} Reduced elastic compliance coefficients for plane problems

δ_{ij} Kronecker delta tensor

$\delta(\cdot)$ Dirac delta function

Δ Displacement jump / increment variation

ε_{ij} Strain tensor component

ζ Chentsov's coefficient

θ Crystallographic orientation angle

λ_i Interpolation or shape functions utilized in boundary elements

f Coulomb friction coefficient

ν Poisson's ratio

ξ Source point coordinate / Local element intrinsic coordinate

ρ Specific mass (density)

σ_{ij} Stress tensor component

σ_n Interface normal contact stress (pressure)

τ_{nt} Interface tangential (shear) contact stress

ω Angular rotation velocity (centrifugal loading)

Γ Global boundary of the structural domain

Γ_C Contact interface boundary ($\Gamma_C = \Gamma_{stick} \cup \Gamma_{slip}$)

$\Gamma_{stick}, \Gamma_{slip}$ Subregions of the contact interface in stick and slip states

Ω Physical domain of the body

Chapter 1

Introduction

1.1 General Considerations

Throughout the twentieth century, the consolidation of advances in Mathematics and Physics (initiated in the seventeenth century) provided the theoretical foundations that, combined with the advent of digital computing, enabled the establishment of numerical modeling as a robust and indispensable discipline. Since then, numerical modeling has expanded significantly and is now widely used in scientific and technological fields due to its ability to efficiently address complex engineering problems.

The development of reliable and computationally efficient models for the analysis of engineering systems remains a central concern in several industrial sectors. In particular, the mechanical, aerospace, and automotive industries require increasingly lightweight components without compromising structural integrity and performance. In this context, accurate prediction of failure mechanisms is essential for the safe and efficient design of engineering structures.

As engineering applications evolve, reducing the gap between numerical models and real-world behavior requires the adoption of less restrictive assumptions and more realistic representations of material behavior. Although linear elastic anisotropy can be efficiently handled by domain-based methods such as the Finite Element Method (FEM), its incorporation into Boundary Element Method (BEM) formulations becomes considerably more challenging, particularly when frictional contact and body forces are simultaneously involved. These challenges arise mainly from the derivation of anisotropic fundamental solutions and from the treatment of body-force domain integrals while preserving the boundary-only character of the BEM.

In aerospace applications, components operating under extreme conditions—such as turbine blades in aircraft engines—are commonly manufactured from nickel-based single-crystal superalloys. These materials exhibit excellent mechanical strength and chemical stability at elevated temperatures while presenting strongly direction-dependent elastic properties. Consequently, accurate representation of their anisotropic behavior becomes essential for

reliable stress prediction under complex loading conditions.

A typical configuration of an aircraft engine is illustrated in Figure 1.1, where regions subjected to severe thermal and mechanical loads can be identified. In these regions, components are exposed not only to high temperatures but also to centrifugal body forces and frictional contact interactions, making them representative engineering applications for the formulation proposed in this work.

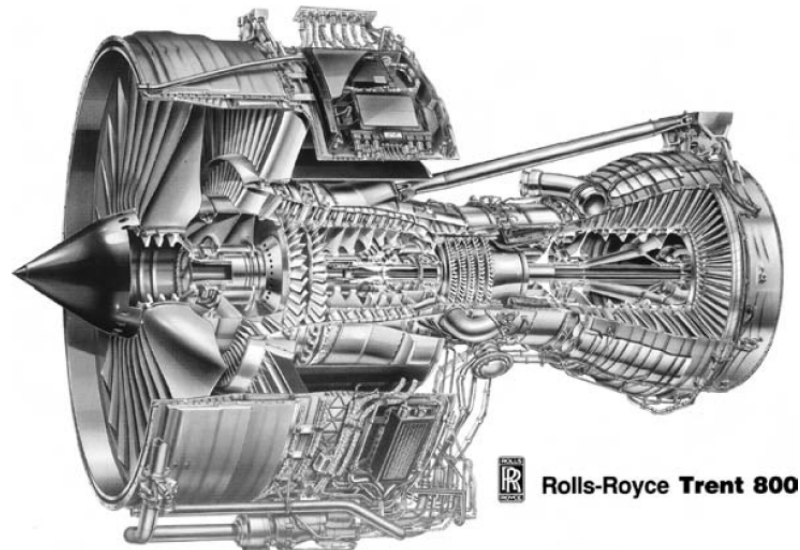


Figure 1.1: Cross-sectional view of the Rolls-Royce Trent 800 turbomachinery, highlighting regions subjected to extreme operating conditions where nickel-based superalloys are typically employed (adapted from (REED, 2006)).

To improve high-temperature performance, turbine blades are often manufactured from single crystals, eliminating grain boundaries and enhancing resistance to creep. As a result, these materials exhibit pronounced anisotropic behavior, with mechanical properties strongly influenced by crystallographic orientation. Accurately capturing this directional dependence is essential for realistic numerical modeling.

In addition to material anisotropy, contact interactions play a critical role in the structural integrity of engineering systems. In many engineering applications, components are subjected to quasi-static cyclic loading involving repeated contact and small-amplitude relative motion, leading to complex stress distributions at the interface. Under these conditions, fretting fatigue may occur, arising from small-amplitude oscillatory motion between contacting surfaces, which promotes crack initiation and propagation and significantly reduces fatigue life (HILLS; NOWELL, 1994).

In high-performance systems such as aircraft engines, contact phenomena often occur simultaneously with elevated temperatures and complex loading conditions. Although high-temperature components may exhibit inelastic effects such as plasticity and creep under severe operating conditions, the present work is restricted to linear elastic behavior in order to isolate the influence of material anisotropy, frictional contact, and centrifugal body forces within the

proposed Boundary Element formulation. Within this linear elastic framework, the interaction between contact stresses, cyclic loading, and material anisotropy becomes a key factor in determining the durability of components. Therefore, accurate evaluation of stress and strain fields in contact regions is essential for a reliable structural assessment.

The analysis of such problems typically requires the use of numerical methods, since analytical solutions are restricted to simplified geometries and boundary conditions. Among the available techniques, the Boundary Element Method (BEM) is particularly attractive for contact problems, as it requires discretization only of the boundary, reducing the dimensionality of the problem, and improving computational efficiency.

Due to its formulation in terms of boundary integrals, BEM provides a natural framework for the analysis of contact problems, which are inherently boundary-dominated. In addition, it offers high accuracy in evaluating stresses and displacements at interfaces, which is especially important in contact mechanics (LEONEL; VENTURINI, 2011; CAVALCANTE, 2016; RODRÍGUEZ-TEMBLEQUE, 2012; RODRÍGUEZ-TEMBLEQUE, 2014).

However, while frictional contact algorithms developed for isotropic materials can generally be extended to anisotropic elasticity without substantial modifications, the treatment of centrifugal body forces within the Boundary Element Method remains a significant challenge. Conventional BEM formulations typically require domain discretization, internal cells, or radial basis function interpolation to account for body forces, partially compromising one of the main advantages of the method, namely its boundary-only discretization. To overcome this limitation, the present work proposes a Boundary Element formulation that incorporates centrifugal body forces through the Radial Integration Method (RIM), allowing the exact transformation of body-force domain integrals into boundary integrals without requiring domain discretization, internal points, or radial basis function interpolation. The resulting formulation provides a unified framework for the analysis of frictional contact problems in anisotropic materials subjected to centrifugal loading while preserving the computational advantages of the Boundary Element Method.

1.2 Motivation and Objectives

The accurate analysis of contact and fretting phenomena in anisotropic materials is of fundamental importance in the design of critical engineering components, particularly in the aerospace industry, where structures operate under severe mechanical and thermal conditions. Failures associated with fretting fatigue or high stress concentrations at contact interfaces can lead to significant performance degradation or even catastrophic failure. Therefore, reliable prediction of these effects is essential to ensure structural integrity and operational safety.

Despite the widespread use and continuous development of numerical methods for contact mechanics, most analyses are still performed using domain-based approaches, particularly the Finite Element Method (FEM). Although these methods are robust and widely adopted,

they require full domain discretization, resulting in increased computational cost and mesh-generation complexity, especially for problems involving localized boundary phenomena. In contrast, the Boundary Element Method (BEM) provides an attractive alternative through its boundary-only formulation. However, the incorporation of body forces into BEM formulations remains a significant challenge, since conventional approaches generally rely on domain discretization or auxiliary interpolation techniques, partially compromising one of the main computational advantages of the method.

The Boundary Element Method (BEM) represents an attractive alternative due to its boundary-only formulation, which reduces the dimensionality of the problem and provides high accuracy in the evaluation of stresses and displacements at interfaces. However, when body forces such as centrifugal or gravitational loads are present, traditional BEM formulations typically rely on internal discretization or auxiliary interpolation techniques, which partially compromise the computational advantages of the method.

In the case of single-crystal superalloys, the anisotropic elastic behavior has a strong influence on the stress distribution near the contact regions. Accurately capturing these effects becomes particularly challenging when contact interactions and body forces are present simultaneously. To the best of our knowledge, the combined treatment of frictional contact, anisotropic elasticity, and centrifugal body forces has not yet been fully addressed within a unified and computationally efficient BEM framework.

To address this limitation, the present work proposes a Boundary Element Method formulation for the analysis of frictional contact problems in anisotropic materials subjected to centrifugal body forces. The proposed approach incorporates body forces through the Radial Integration Method (RIM), avoiding the need for internal discretization or radial basis function interpolation, thereby preserving the boundary-only character of the method.

The main objectives of this work are:

- Develop a Boundary Element formulation for frictional contact problems in anisotropic elastic materials subjected to centrifugal body forces;
- Incorporate centrifugal body forces using the RIM;
- Model frictional contact conditions within the proposed framework;
- Develop a computational implementation of the proposed formulation and evaluate its accuracy and computational performance;
- Validate the proposed approach through comparisons with available solutions in the literature.

1.3 Publications Related to This Thesis

The research developed in this thesis has resulted in the following scientific publications in journals and conference proceedings:

- Leonardo B. e Silva; Eder L. de Albuquerque; Thiago de C. R. Doca; Lucas S. Campos. *Analysis of Fretting Problems in Anisotropic Materials Using the Boundary Element Method*. 7th Brazilian Conference on Composite Materials (BCCM 7), 2024.
- E. L. Albuquerque; L. S. Campos; W. R. Niz; L. B. Silva; T. Doca; F. M. Loyola. *Fretting Fatigue Analysis Using Boundary Element Method. Numerical Modeling of Crack Growth Under Fretting Conditions*. XLVI Ibero-Latin American Congress on Computational Methods in Engineering (CILAMCE), 2025.
- L. Bernardo; W. R. Niz; F. M. Loyola; T. Doca; L. S. Campos; E. L. Albuquerque. *Analysis of Frictional Contact Problems in Anisotropic Materials under Centrifugal Loads Using the Boundary Element Method*. Engineering Computations, 2026.

1.4 Thesis Outline

This work is organized into seven chapters, as outlined below:

- **Chapter 1: Introduction.** This chapter presents the motivation for the study, highlighting the challenges associated with anisotropic bodies subjected to centrifugal body forces and contact conditions. A brief review of the literature is provided that identifies existing limitations and the research gap addressed in this work. The objectives and scope of the thesis are also defined.
- **Chapter 2: Anisotropic Elasticity Theory.** This chapter introduces the fundamental concepts of elasticity, including stress and strain tensors, equilibrium equations, and constitutive relations, with emphasis on anisotropic materials.
- **Chapter 3: Contact Mechanics.** This chapter presents the theoretical framework of elastic contact problems, including classical formulations and analytical solutions, providing the basis for their numerical treatment.
- **Chapter 4: BEM Formulation for Anisotropic Elasticity with Body Forces.** This chapter develops the Boundary Element Method formulation for anisotropic materials, including the derivation of integral equations and the treatment of centrifugal body forces.

- **Chapter 5: Contact Formulation and Numerical Implementation.** This chapter describes the numerical implementation of the proposed formulation, including discretization procedures, contact conditions, and solution strategies.
- **Chapter 6: Results.** This chapter presents verification studies and numerical applications that demonstrate the accuracy, robustness, and applicability of the proposed formulation.
- **Chapter 7: Conclusions.** This chapter summarizes the main contributions of the work, discusses its limitations, and outlines directions for future research.

1.5 Bibliographic revision

1.5.1 Boundary Element Method

The Boundary Element Method (BEM) is a well-established numerical technique widely used to solve boundary value problems in engineering. Since its development, it has been successfully applied to a broad range of fields, including elasticity, fracture mechanics, acoustics, electromagnetism, heat transfer, and contact mechanics. Among these applications, the analysis of anisotropic elastic materials has received increasing attention due to the growing use of advanced materials in engineering.

In 1978, the BEM was established as an alternative approach to traditional domain discretization techniques (BREBBIA; DOMINGUEZ, 1996). Its great advantage is the reduction of the dimension of the problem, based on the boundary integral equations and on the fundamental solutions of the differential equations that govern the phenomenon to be studied (KANE, 1994). Mesh reduction is beneficial for problems that have infinite or semi-infinite domains present in acoustics, soil mechanics, and fluid mechanics (WROBEL, 2002).

BEM is widely used for many engineering applications, such as: crack growth (GOMES; MIRANDA, 2018), electromagnetism (OUESLATI, 2021), fatigue (OLIVEIRA, 2019), acoustics (GILVEY, 2020), multi-body contact (LOYOLA, 2020) just to name a few recent works. These are evidences of the versatility of the method.

This technique is developed based on the transformation of the governing partial differential equations of the problem, which involve unknown fields on the boundary and inside a domain, into an integral equation involving only the fields on the boundary. The BEM has very desirable characteristics for modeling many problems, making it possible to model problems with high stress and strain gradients with only the boundary discretization.

The use of BEM for solving problems with anisotropic materials has become more common in recent years. As the anisotropy increases, the number of elastic constants of the material increases, resulting in difficulties in the development of formulations. In particular,

in the boundary element method, a greater number of variables means much more difficulty is finding the fundamental solutions. This aspect is evident in the literature; the number of references in which the BEM is applied to anisotropic structures is significantly less than the number of isotropic structures. However, in recent years, important advances have been published in the application of boundary element techniques to anisotropic materials, for example, plane elasticity problems were analyzed by Tyagnii (2015), Cordeiro e Leonel (2020), problems of contact in elastic solids by Nguyen e Hwu (2019), crack propagation problems by Cordeiro e Leonel (2016), three-dimensional problems by Gu, He, Chen e Zhang (2018), Shiah e Hematiyan (2018) and by Shiah e Hematiyan (2021). In cases of isotropic materials, where the references are larger, we can cite plane elasticity problems in Vodička (2013), Wang e Qin (2017), frictional contact problems Gun (2014), three-dimensional problems Chen, Yu, Liu e Zhou (2023), thermoelastic problems Zhao, Dang, Li, Fan e Xu (2016), Yoon, Lee e Koo (2020), elastic damage Li, Yang, Jiang, Zhao, Zhao e Xu (2023), among others.

In the general formulation of the BEM with body forces, domain integrals arise in the formulation due to the body forces. In order to evaluate these integrals, a cell integration scheme can be used to give accurate results, as performed by Shi e Bezzine (1988) for anisotropic plate bending problems. However, the discretization of the domain in cells nullifies one of the main advantages of the BEM, which is the discretization of the boundary only. An alternative to this procedure was presented by Rajamohan (1999), who proposes the use of particular solutions to avoid domain discretization. However, using particular solutions requires finding a suitable function that satisfies the governing equation. Depending on how complicated the governing equation is, this function can be quite difficult to find.

Domain integrals arising from distributed loads can be transformed into boundary integrals by exact transformation using the radial integration method. This method was first presented by Venturini (1988) for isotropic plate bending problems. Later, in Gao (2002) it was extended to three-dimensional isotropic elastic problems. The most attractive feature of the method is its simplicity, since only the radial variable is integrated. For domain integrals that include unknown variables, the proposed procedure can be performed using a radial basis function as in the dual reciprocity method suggested by Gao (2002). In the case of centrifugal loads, since body forces do not depend on unknown variables, no radial basis function is used. Consequently, the use of internal points is unnecessary, preserving the boundary-only character of the formulation and simplifying the numerical implementation.

1.5.2 Numerical analysis of fretting problems

Fretting is a phenomenon that occurs at the contact interface of structural components subjected to vibratory excitation or tangential oscillatory forces while constrained (SZOLWINSKI; FARRIS, 1996; WATERHOUSE, 1981; REED, 2008). In such situations, contact analysis is essential to accurately evaluate stress fields, which provide the

basis for understanding crack initiation, fretting fatigue, and other damage mechanisms that ultimately affect the structural integrity and service life of engineering components.

Nickel-based single-crystal superalloys (NBSX) are widely used in turbine blades due to their exceptional mechanical performance at elevated temperatures (REED, 2008). Over the years, directional solidification technology and the effect of crystal orientation in NBSX superalloys have been exploited to improve blade design and manufacturing (FLEURY, 2014). Metallic single crystals, including nickel-based superalloys, exhibit elastic anisotropy resulting from their crystallographic structure and orientation. In the present work, only the elastic anisotropy associated with crystallographic orientation is considered, whereas anisotropy induced by plastic deformation or material damage lies beyond the scope of the proposed formulation. For the plane elasticity formulation adopted in this work, the anisotropic elastic behavior is represented through an equivalent orthotropic constitutive model. Accounting for crystallographic orientation in fretting simulations significantly improves the prediction of deformation and stress fields in the near-surface regions where damage initiates (HUANG, 2009). Unlike polycrystalline alloys, single-crystal materials exhibit pronounced direction-dependent elastic behavior, making the alignment of the crystal lattice with the geometry of the component a critical factor in structural analysis (MOROSO, 1999).

The anisotropic nature of materials introduces additional complexity into structural analysis, restricting analytical solutions to relatively simple cases. Consequently, numerical methods are essential for analyzing more complex structures. In recent years, the Boundary Element Method (BEM) has been increasingly applied to problems involving anisotropic materials. Nevertheless, the higher number of elastic constants in anisotropic media poses challenges for the modeling and numerical formulation. In particular, deriving fundamental solutions becomes more difficult, which is reflected in the relatively smaller number of BEM studies on anisotropic structures compared to the number of isotropic ones. However, significant advances have been made over the past decade. For example, plane elasticity problems have been analyzed by Tyagnii (2015) and Cordeiro e Leonel (2020), contact problems in elastic solids by Nguyen e Hwu (2019), and three-dimensional problems by Gu, He, Chen e Zhang (2017), Shiah e Hematiyan (2018) and Shiah e Hematiyan (2021).

Since contact is essentially a boundary phenomenon, BEM offers a natural and efficient framework for its analysis. Its features have been further exploited in various studies on contact mechanics, both with and without fretting, such as Leonel e Venturini (2011), Rodríguez-Tembleque (2014), Cavalcante, Shaterzadeh-Yazdi, Sollero, Albuquerque e Doca (2017), Doca, Loyola e Albuquerque (2022), Loyola, Doca, Campos, Trevelyan e Albuquerque (2022). Recent advances have further refined the numerical frameworks and solution strategies for BEM-based contact problems, as demonstrated by Nguyen, Nguyen e Duc (2025), Nguyen, Nguyen, Nguyen e Nguyen (2026), Nguyen, Chen e Hwu (2023), Nguyen e Hwu (2020), showing the method's versatility across different engineering applications. However, while these studies contribute significantly to the numerical efficiency of contact solvers, they typically do not account for coordinate-dependent body forces.

Most BEM formulations for body force problems rely on discretization of the domain in cells with direct integration into each cell's area (SHI; BEZINE, 1988) or transformation of domain integrals into boundary integrals using techniques such as the dual reciprocity method (BREBBIA; NARDINI, 1983), the radial integration method (GAO, 2002; ALBUQUERQUE, 2006), or the direct integration method (CAMPOS, 2023). Each approach has advantages and drawbacks. Domain discretization compromises the main strength of BEM: its boundary-only discretization. The dual reciprocity method requires particular solutions, which may not always be readily available and can complicate code implementation when new approximation functions are needed (ALBUQUERQUE, 2002). The radial integration method is accurate, but computationally expensive, when interpolation with radial basis functions is required. The direct integration method is less costly but still depends on radial basis functions to interpolate body forces over the domain. However, when body forces depend only on the spatial coordinates, such as centrifugal or gravitational loads, the corresponding domain integrals do not involve unknown field variables. Consequently, the Radial Integration Method can transform these domain integrals exactly into boundary integrals, as demonstrated by Gao (2002) for three-dimensional elasticity and by Albuquerque, Sollero, Venturini e Aliabadi (2006) for plate bending. This approach eliminates the need for radial basis function interpolation, internal cells, or internal points, thereby preserving the boundary-only character of the Boundary Element Method.

Previous studies have investigated contact and fretting in anisotropic materials using different numerical strategies. For example, Nguyen e Hwu (2019) employed incremental loading to address nonlinearities, Blázquez, Mantić e París (2006) used an augmented Lagrangian formulation with a Uzawa predictor-corrector scheme to enforce contact constraints, and Rodríguez-Tembleque (2012) analyzed similar problems using a generalized plane formulation to reduce dimensional complexity. In contrast, the present work extends the boundary element methodology to contact problems with body forces, focusing on centrifugal loads, which are critical in turbine blade–rotor interactions. Although the main application is single-crystal materials, the methodology is applicable to a wide range of two-dimensional anisotropic contact problems. The formulation employs fundamental solutions that explicitly account for material anisotropy and models frictional contact using a node-to-node contact formulation. By incorporating body forces directly through the radial integration method, the approach avoids discretization of cells, internal points, and interpolation of radial basis functions, which are commonly required in conventional formulations.

This novel framework enables the analysis of problems that involve mixed contact modes, diverse boundary conditions, body forces, and contact between dissimilar anisotropic materials. The numerical results confirm that the method is accurate and efficient, while offering a simpler and original alternative for modeling fretting contact problems with body forces. To the best of the authors' knowledge, no previous work has applied BEM to the analysis of anisotropic bodies under centrifugal loads, underscoring the novelty of the present formulation.

Chapter 2

Anisotropic Elasticity Theory

2.1 The governing differential equations

The general formulation of anisotropic elasticity requires the use of tensor notation to describe stress, strain, and their relationships.

Taking into account an infinitesimal element in a domain Ω , the balance of forces can be expressed:

$$\sigma_{ij,j} + \rho b_i = 0. \quad (2.1)$$

The equilibrium of moments is expressed as:

$$\sigma_{ij} = \sigma_{ji} \quad (2.2)$$

where σ_{ij} is the stress tensor and b_i is the body force vector, ρ is the density of the material.

The vector of surface forces t_i at a point on the boundary Γ of a domain Ω is expressed in the following form:

$$t_i = \sigma_{ij} n_j \quad (2.3)$$

where n_j is the normal boundary vector Γ at the point.

To establish the relationship between stress and strain, we begin by considering the Lagrangian and Eulerian strain tensors. Assuming that the displacement vector and its spatial derivatives are infinitesimal, both tensors reduce to the infinitesimal (Cauchy) strain tensor, which can be expressed as:

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (2.4)$$

To ensure the uniqueness of the displacements, the components of the strain tensor cannot

be arbitrarily designated and must satisfy certain compatibility and integrability conditions. The compatibility equation is given by:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \quad (2.5)$$

which in the two-dimensional case is reduced to the form:

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \quad (2.6)$$

The constitutive relations for a linear elastic material, describing the relationship between stress and strain in an infinitesimal element, are given by the generalized Hooke's law:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \quad (2.7)$$

$$\varepsilon_{ij} = S_{ijkl} \sigma_{kl}. \quad (2.8)$$

The terms σ_{ij} and ε_{ij} are the second-order stress and strain tensors, respectively. Each tensor has nine components, whose indices are $i, j = 1, 2, 3$. In Eq. (2.7), the linearity coefficient C_{ijkl} is a fourth-order tensor with $3^4 = 81$ elements, known as the elastic stiffness tensor, which, according to LIU (1990), satisfies Green's symmetry conditions.

$$C_{ijkl} = C_{klij} = C_{ijlk} = C_{jilk}. \quad (2.9)$$

In order for Eq. (2.9) to exist, Hwu (2010) explains that, since the stress components are symmetric, an exchange of the indices i and j in Eq. (2.7) does not alter the formula, so that $C_{ijkl} = C_{jikl}$. Therefore, it can be stated that Eq. (2.7) reduces the number of independent elastic constants to $3 \times 3 \times 6 = 54$. Furthermore, by means of strain tensor symmetry, another reduction in the number of elastic constants can be made. That is, $\varepsilon_{ij} = \varepsilon_{ji}$, and therefore $\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = C_{ijlk}\varepsilon_{lk} = C_{ijlk}\varepsilon_{kl}$, which leads to $(C_{ijkl} - C_{ijlk})\varepsilon_{kl} = 0$. Since this equality must hold for arbitrary values of ε_{kl} , it follows that $C_{ijkl} = C_{ijlk}$.

Similarly, S_{ijkl} in Eq. (2.8) is a fourth-order tensor, called the compliance tensor, which satisfies the same minor and major symmetry conditions as the stiffness tensor. These symmetry properties reduce the number of independent elastic constants from 81 to 21 in classical linear elasticity (LEKHNITSKII, 1963). Thus, Eq. (2.7) and Eq. (2.8) can be expressed in matrix form using the contracted Voigt notation, as presented in (2.10) and (2.11).

Using the Voigt notation, the fourth-order tensors can be represented in a matrix form, reducing the complexity of the constitutive relations.

$$\begin{pmatrix} \sigma_{11} = \sigma_1 \\ \sigma_{22} = \sigma_2 \\ \sigma_{33} = \sigma_3 \\ \sigma_{23} = \sigma_4 \\ \sigma_{13} = \sigma_5 \\ \sigma_{12} = \sigma_6 \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_{11} = \varepsilon_1 \\ \varepsilon_{22} = \varepsilon_2 \\ \varepsilon_{33} = \varepsilon_3 \\ 2\varepsilon_{23} = \varepsilon_4 \\ 2\varepsilon_{13} = \varepsilon_5 \\ 2\varepsilon_{12} = \varepsilon_6 \end{pmatrix} \quad (2.10)$$

$$\begin{pmatrix} \varepsilon_{11} = \varepsilon_1 \\ \varepsilon_{22} = \varepsilon_2 \\ \varepsilon_{33} = \varepsilon_3 \\ 2\varepsilon_{23} = \varepsilon_4 \\ 2\varepsilon_{13} = \varepsilon_5 \\ 2\varepsilon_{12} = \varepsilon_6 \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{pmatrix} \sigma_{11} = \sigma_1 \\ \sigma_{22} = \sigma_2 \\ \sigma_{33} = \sigma_3 \\ \sigma_{23} = \sigma_4 \\ \sigma_{13} = \sigma_5 \\ \sigma_{12} = \sigma_6 \end{pmatrix} \quad (2.11)$$

Elastic coefficients can be expressed in terms of engineering constants like (LEKHNITSKII, 1963):

$$\begin{aligned} S_{11} &= 1/E_1 & S_{12} &= \nu_{12}/E_1 = -\nu_{21}/E_2 \\ S_{13} &= -\nu_{31}/E_1 = -\nu_{13}/E_3 & S_{14} &= \eta_{23,1}/E_1 = \eta_{1,23}/G_{23} \\ S_{15} &= \eta_{32,1}/E_1 = \eta_{1,32}/G_{23} & S_{16} &= \eta_{12,1}/E_1 \\ S_{22} &= 1/E_2 & S_{23} &= \nu_{32}/E_2 = -\nu_{23}/E_3 \\ S_{24} &= \eta_{23,1}/E_2 = \nu_{23,3}/G_{23} & S_{25} &= \eta_{31,2}/E_2 = \eta_{2,31}/G_{13} \\ S_{26} &= \eta_{12,2}/E_2 = \eta_{2,12}/G_{12} & S_{33} &= 1/E_3 \\ S_{34} &= \eta_{23,3}/E_3 = \eta_{3,23}/G_{23} & S_{35} &= \eta_{31,1}/E_3 = \eta_{3,31}/G_{13} \\ S_{36} &= \eta_{12,3}/E_3 = \eta_{3,12}/G_{12} & S_{44} &= 1/G_{23} \\ S_{45} &= \zeta_{32,23}/G_{23} = \zeta_{23,31}/G_{13} & S_{46} &= \zeta_{12,23}/G_{23} = \zeta_{23,12}/G_{12} \\ S_{55} &= 1/G_{13} & S_{56} &= \zeta_{12,31}/G_{13} = \zeta_{31,12}/G_{12} \\ S_{66} &= 1/G_{12} \end{aligned} \quad (2.12)$$

According to Hwu (2010), where E_k is the longitudinal modulus of elasticity, or Young's modulus, referring to the x_k axes, G_{ij} is the transverse modulus of elasticity, or Coulomb modulus, for the planes defined by the axes $x_i x_j$. The coefficients ν_{ij} are called Poisson coefficients. The constants $\eta_{jk,l}$ are called coefficients of mutual influence of the first kind that characterize extensions in the directions of the principal axes, produced by tangential stresses acting on the principal planes. The constants $\eta_{l,jk}$ are the coefficients of mutual influence of the second kind, which express tangential deformations in the principal planes, caused by stresses normals acting on the principal planes. Finally, $\zeta_{ij,kl}$ are the Chentsov coefficients, which characterize tangential deformations in planes parallel to the main planes of elasticity, caused by tangential stresses acting in other planes, parallel to the main planes of elasticity.

Due to the symmetry of the stiffness and compliance tensors, it is possible to define the number of independent constants for different types of materials. Table 2.1 shows five of the most common materials, according to (CHRISTENSEN, 2012).

Table 2.1: Material type and independent elastic constants.

Material model	Independent constants
Anisotropic	21
Monoclinic	13
Orthotropic	9
Transversely isotropic	5
Isotropic	2

For the particular case of plane stress in Eq. (2.10), the condition occurs when the normal stress on a specific direction is zero. Similarly, for the plane strain condition in Eq. (2.11), the normal strain on a specific direction is zero. This restriction limits these models to two-dimensional problems, Eq. (2.13) and Eq. (2.14). However, for generally anisotropic materials, the plane strain assumption is not always strictly satisfied, except for particular material symmetries and geometrical configurations, because the equilibrium equations cannot, in general, be satisfied simultaneously (LEKHNITSKII, 1963).

However, for two-dimensional analyses, all stress and displacement components are assumed to depend only on the x_1 - and x_2 -coordinates, implying no variation in the x_3 -direction. Thus, under the plane stress condition ($\sigma_3 = \sigma_4 = \sigma_5 = 0$), the material can be represented using only six independent elastic constants. Therefore, Eq. (2.11) can be written as:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \quad (2.13)$$

In the plane strain state ($\sigma_3 = -1/S_{33}(S_{13}\sigma_1 + S_{23}\sigma_2 + S_{36}\sigma_6)$, $\sigma_4 = \sigma_5 = \varepsilon_3 = 0$) Eq. (2.11) can be written as:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{26} \\ \beta_{16} & \beta_{26} & \beta_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \quad (2.14)$$

where the coefficients β_{ij} are expressed in terms of the compliance coefficients (S_{ij}), according to Lekhnitskii (1968), as follows:

$$\beta_{ij} = S_{ij}, \text{ for plane stress state,} \quad (2.15)$$

$$\beta_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}}, \text{ for plane deformation state, being } i, j = 1, 2, 6. \quad (2.16)$$

2.2 Plane deformation by the Lekhnitskii formalism

The Lekhnitskii formalism provides an efficient framework for solving two-dimensional anisotropic elasticity problems using complex variable methods.

As presented in Section 2.1, the governing equations of anisotropic elasticity consist of the balance of linear momentum (Eq. (2.1)), the balance of angular momentum (Eq. (2.2)), the strain-displacement relations for small deformations (Eq. (2.4)), and the constitutive equations of linear elasticity (Eq. (2.7)). Based on these fundamental equations, the Lekhnitskii formalism provides an efficient framework for solving two-dimensional anisotropic elasticity problems using complex variable methods.

In Lekhnitskii's formulation, the constitutive equations are expressed in terms of the compliance tensor. Thus, by rewriting Eq. (2.13), the strain components, assumed to depend only on the coordinates x_1 and x_2 , is expressed in terms of the complex stress functions $\phi(x_1 + \mu x_2)$, known as Lekhnitskii stress functions.

$$\begin{aligned}\varepsilon_1 &= S_{11} \frac{\partial^2 \phi}{\partial x_2^2} + S_{12} \frac{\partial^2 \phi}{\partial x_1^2} - S_{16} \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \\ \varepsilon_2 &= S_{12} \frac{\partial^2 \phi}{\partial x_2^2} + S_{22} \frac{\partial^2 \phi}{\partial x_1^2} - S_{26} \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \\ \varepsilon_6 &= S_{16} \frac{\partial^2 \phi}{\partial x_2^2} + S_{26} \frac{\partial^2 \phi}{\partial x_1^2} - S_{66} \frac{\partial^2 \phi}{\partial x_1 \partial x_2}.\end{aligned}\tag{2.17}$$

Using the set of Eqs. (2.17), and substituting them into the compatibility conditions given by Eq. (2.6). Substituting Eqs. (2.17) into the compatibility equation (Eq. (2.6)) and collecting like terms yields Eq. (2.18):

$$\begin{aligned}& S_{11} \frac{\partial^4 \phi}{\partial x_2^4} + S_{12} \left(\frac{\partial^4 \phi}{\partial x_2^2 \partial x_1^2} \right) - S_{16} \frac{\partial^4 \phi}{\partial x_1 \partial x_2^3} + S_{12} \frac{\partial^4 \phi}{\partial x_2^2 \partial x_2} + S_{22} \frac{\partial^4 \phi}{\partial x_1^4} - S_{26} \frac{\partial^4 \phi}{\partial x_1^3 \partial x_2} = \\ & S_{16} \frac{\partial^4 \phi}{\partial x_1 \partial x_2^3} + S_{26} \frac{\partial^4 \phi}{\partial x_1^3 \partial x_2} - S_{66} \frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} + S_{11} \frac{\partial^4 \phi}{\partial x_2^4} + 2S_{12} \frac{\partial^4 \phi}{\partial x_2^2 \partial x_1^2} - 2S_{16} \frac{\partial^4 \phi}{\partial x_1^3 \partial x_2} + \\ & S_{22} \frac{\partial^4 \phi}{\partial x_1^4} - 2S_{26} \frac{\partial^4 \phi}{\partial x_1^3 \partial x_2} + S_{66} \frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} = 0 \quad \Leftrightarrow \quad L_4 \phi = 0.\end{aligned}\tag{2.18}$$

Here, L_4 is the fourth-order differential operator, expressed as:

$$L_4 = S_{22} \frac{\partial^4}{\partial x_1^4} - 2S_{26} \frac{\partial^4}{\partial x_1^3 \partial x_2} + (2S_{12} + S_{66}) \frac{\partial^4}{\partial x_1^2 \partial x_2^2} - 2S_{16} \frac{\partial^4}{\partial x_1 \partial x_2^3} + S_{11} \frac{\partial^4}{\partial x_2^4}.\tag{2.19}$$

and its solution can be obtained by the method of characteristics (Hwu (2010)), written as:

$$S_{11}\mu^4 - 2S_{16}\mu^3 + (2S_{12} + S_{66})\mu^2 - 2S_{26}\mu + S_{22} = 0.\tag{2.20}$$

The terms of Eq. (2.20) can be decomposed into four first-order linear operators, written as:

$$D_1 D_2 D_3 D_4 \phi = 0, \quad \text{where:} \quad D_k = \frac{\partial}{\partial x_2} - \mu_k \frac{\partial}{\partial x_1}. \quad (2.21)$$

According to Lekhnitskii (1968), the roots of Eq. (2.20) are always complex or purely imaginary, occurring in conjugate pairs. Therefore, the roots can be expressed in two particular cases:

$$\begin{cases} \text{Case 1:} & \mu_1 = \alpha_1 + i\beta_1, \quad \mu_2 = \alpha_2 + i\beta_2, \quad \mu_3 = \bar{\mu}_1, \quad \mu_4 = \bar{\mu}_2, \\ \text{Case 2:} & \mu_1 = \mu_2 = \alpha + i\beta, \quad \mu_3 = \mu_4 = \bar{\mu}_1. \end{cases} \quad (2.22)$$

However, the second case rarely occurs, and it can be shown that it corresponds to an isotropic formulation. Therefore, Case 2 is ignored, as it is not the focus of this investigation. Hence, using the roots of Case 1 and substituting them into Eq. (2.21) after decomposition, the following system is obtained:

$$\begin{aligned} D_1 \phi_1 &= 0, \\ D_2 D_3 D_4 \phi_2 &= \phi_1, \\ D_3 D_4 \phi_3 &= \phi_2, \\ D_4 \phi_4 &= \phi_3. \end{aligned} \quad (2.23)$$

it is possible to obtain the general solution written in complex variable form, as shown in Eq. (2.24):

$$\phi(z_1, z_2) = 2\text{Re}[\phi_1(z_1) + \phi_2(z_2)], \quad z_1 = x_1 + \mu_1 x_2, \quad z_2 = x_1 + \mu_2 x_2. \quad (2.24)$$

To express the elasticity equations in terms of complex potential functions, we adopt the following notation:

$$\frac{d\phi_k(z_i)}{dz_i} = \Phi_i(z_i). \quad (2.25)$$

Using the complex potential functions defined in Eqs. (2.24) and (2.25), the stress components can be expressed as follows:

$$\text{Stress} = \begin{cases} \sigma_1 = 2\text{Re}\left[\mu_1^2 \Phi_1^{(1)}(z_1) + \mu_2^2 \Phi_2^{(1)}(z_2)\right], \\ \sigma_2 = 2\text{Re}\left[\Phi_1^{(1)}(z_1) + \Phi_2^{(1)}(z_2)\right], \\ \sigma_6 = -2\text{Re}\left[\mu_1 \Phi_1^{(1)}(z_1) + \mu_2 \Phi_2^{(1)}(z_2)\right], \end{cases} \quad (2.26)$$

where $\Phi_k^{(1)}$ represents the first derivative of Φ_k . The displacement components are given by:

$$\text{Displacement} = \begin{cases} u_1 = 2\text{Re}[q_{11}\Phi_1(z_1) + q_{12}\Phi_2(z_2)] , \\ u_2 = 2\text{Re}[q_{21}\Phi_1(z_1) + q_{22}\Phi_2(z_2)] , \end{cases} \quad (2.27)$$

where q_{ik} is the complex parameter matrix, expressed as:

$$q_{ik} = \begin{bmatrix} \mu_k^2 S_{11} + S_{12} - \mu_k S_{16} \\ \mu_k S_{12} - S_{26} + (S_{22}/\mu_k) \end{bmatrix}. \quad (2.28)$$

2.3 Anisotropy of Single-crystal Ni Alloys

In order to connect the theoretical formulation with engineering applications, this section focuses on the anisotropic behavior of single-crystal nickel-based superalloys used in turbine blades.

Polycrystalline materials (such as metals, ceramics, and alloys, including nickel-based superalloys) are widely used in engineering applications. Their microstructure is characterized by an aggregate of multiple crystals or grains with distinct orientations and crystal planes. In recent decades, turbine blade technology has advanced not only in terms of higher temperature capability but also in several other critical aspects. Significant progress has been achieved in creep and fatigue resistance, oxidation and corrosion protection, and cooling efficiency through complex internal channels and coatings, as well as in the development of single-crystal and directionally solidified alloys. Together, these advances have substantially increased the reliability and service life of turbine components (CARON; KHAN, 1999).

The mechanical performance of turbine blades is directly related to the control of their microstructure. In conventional casting, the resulting polycrystalline structure contains numerous equiaxed grains, and the presence of grain boundaries oriented in different directions often limits the resistance of the alloy under creep and fatigue conditions at elevated temperatures (KHOEI, 2021). Such limitations stimulated the development of advanced solidification techniques capable of tailoring the crystallographic orientation of the blades.

Directional solidification represented a major step in this technological evolution. By promoting grain growth along the primary stress axis, this process reduces the number of misoriented interfaces and effectively eliminates transverse grain boundaries, which are critical sites for crack initiation and propagation. The natural progression of this approach was the complete suppression of grain boundaries, giving rise to single-crystal blades. In these materials, the absence of interfaces between adjacent grains significantly restricts dislocation activity, leading to remarkable strengthening and enhanced creep resistance, which are essential for reliable operation in the extreme conditions of modern gas turbines (REED, 2006).

Figure 2.1 shows the evolution of the microstructures of turbine blades: (a) equiaxed grains

produced by conventional casting, which contain numerous grain boundaries that reduce creep resistance; (b) columnar grains obtained by directional solidification, which eliminate transverse grain boundaries and improve mechanical performance; (c) monocrystalline blades, completely free of grain boundaries, providing superior creep and fatigue resistance at high temperatures.

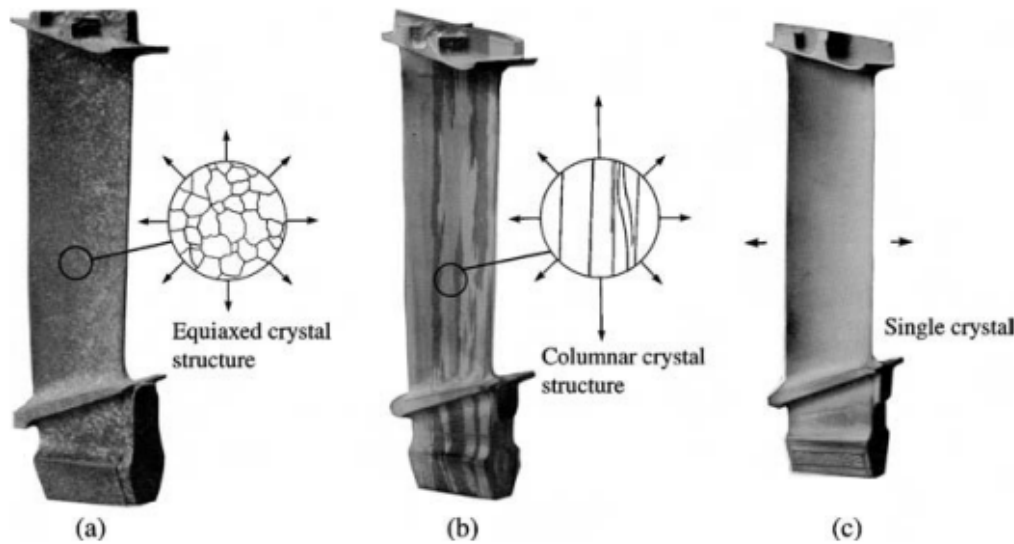


Figure 2.1: Turbine blades manufactured with distinct solidification routes, illustrating: (a) equiaxed crystal structure; (b) columnar crystal structure; (c) single-crystal forms (Rolls-Royce plc, 1992).

According to Reed (2006), monocrystalline superalloys are increasingly being used in gas engines, especially when looking for superior creep performance. However, it is important to note that columnar and equiaxed cast components are still used in various applications.

The microstructural evolution depicted in Figure 2.1 is directly reflected in the mechanical performance of the turbine blades under high-temperature conditions. This correlation is illustrated in Figure 2.2, which presents creep curves for blades produced through different solidification routes. Equiaxed alloys exhibit the lowest creep resistance, fracturing in shorter times because of the high density of grain boundaries that facilitate crack initiation and propagation. Solidified directions demonstrate an improved creep life, as the elimination of the transverse grain boundaries significantly reduces weak points in microstructure (KHOEI, 2021). Superior performance is observed in single-crystal blades, which, because they are entirely free of grain boundaries, maintain prolonged exposure before fracture and provide the highest resistance to creep deformation.

Following the general discussion on the influence of solidification routes and microstructural evolution on creep behavior, as illustrated in Figure 2.1 and Figure 2.2, attention now is focused on the specific alloy investigated in this work, the MD2 single-crystal superalloy. The crystal is a face-centred cubic (FCC) as shown in Figure 2.3. Developed by GE Power, MD2 represents a more advanced generation of nickel-based superalloys, incorporating higher concentrations of refractory elements to enhance creep resistance and thermal stability. In order

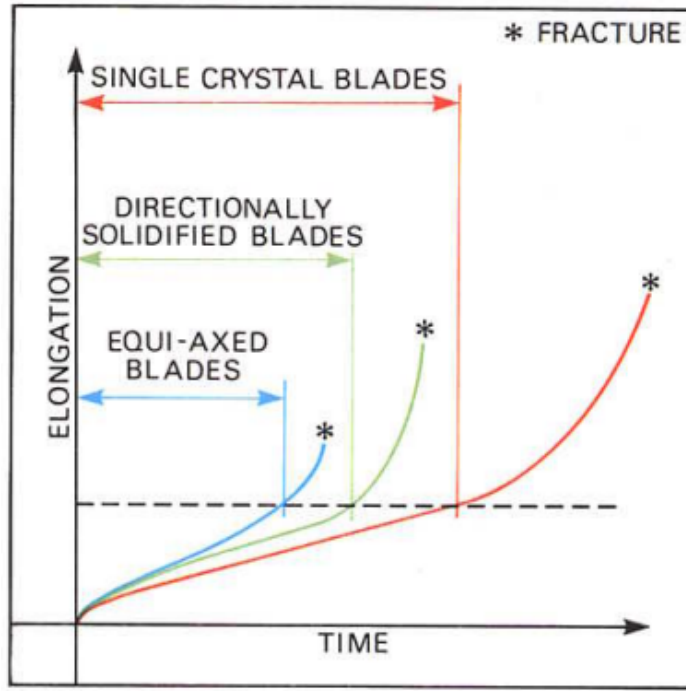


Figure 2.2: Creep behavior of turbine blades with different microstructures: equiaxed, directionally solidified, and single-crystal forms (Rolls-Royce plc, 1992).

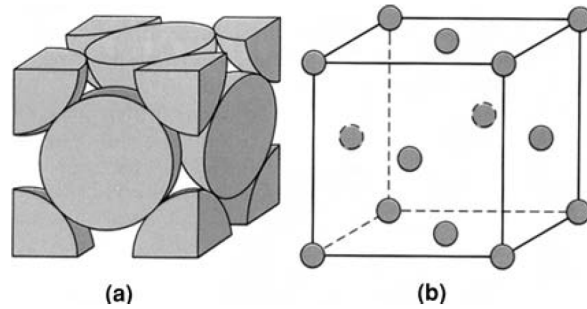


Figure 2.3: The unit cell of the face-centred cubic (FCC) crystal structure, which is displayed by nickel.

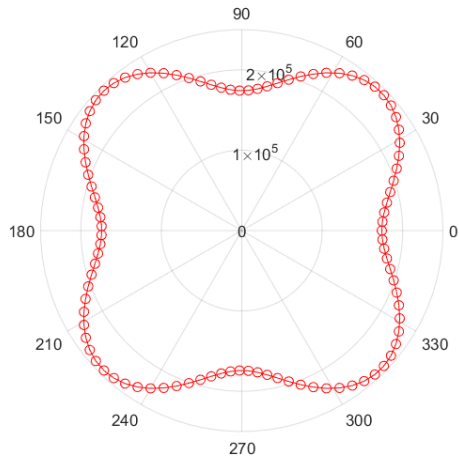
to characterize its mechanical response, particular emphasis is placed on its elastic stiffness constants, namely C_{11} , C_{12} , and C_{44} , which govern the anisotropic behavior of the crystal lattice. These constants provide a fundamental basis for understanding the elastic anisotropy of MD2 and are essential for modeling its performance under service conditions (ZHANG, 2020).

In a single-crystal cubic system such as MD2, the elastic behavior is fully described by three independent stiffness constants: C_{11} , C_{12} , and C_{44} . This reduction arises from the high degree of symmetry in the cubic lattice, which constrains many of the off-diagonal terms of the stiffness tensor to vanish. As a result, the general 6×6 stiffness matrix is simplified to:

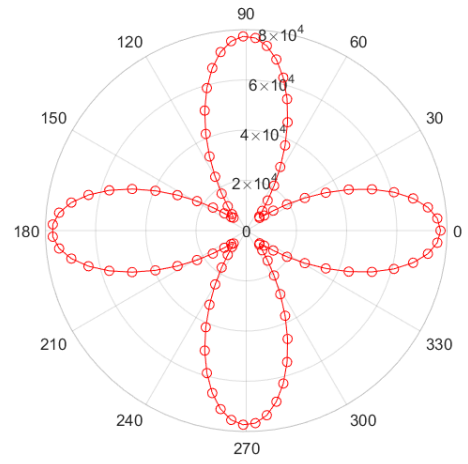
$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}$$

Therefore, other constants such as C_{14} do not exist as independent elastic stiffness terms in cubic systems, since they are strictly zero due to crystal symmetry. The three constants C_{11} , C_{12} , and C_{44} are sufficient for cubic materials to fully characterize the elastic anisotropy of cubic crystals (HUNTINGTON, 1958). However, when the coordinate system rotates with respect to the crystallographic axes, additional components such as C_{14} may appear as a result of the tensor transformation, without representing new independent properties (NORRIS, 2008). Figure 2.4 shows the polar representation of the elastic stiffness constants of the MD2 superalloy. The angle of the polar coordinate θ is the rotation of the crystal with respect to one of the axes of the cubic system. For $\theta = 0^\circ$, the properties of this alloy can be expressed by the anisotropic stiffness constants C_{ij} , defined as: $C_{11} = 174.03$ GPa, $C_{12} = 77.38$ GPa, and $C_{44} = 118.52$ GPa, and $C_{14} = 0$.

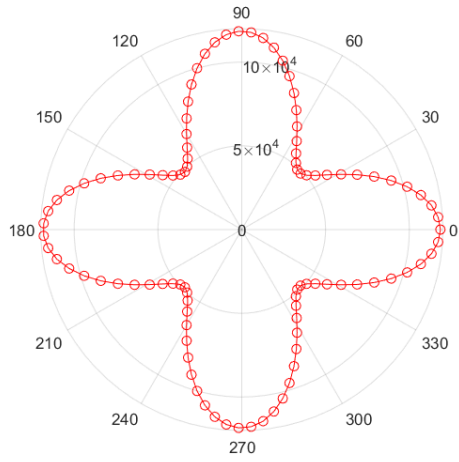
The combined analysis of the constants C_{11} , C_{12} , and C_{44} provides a comprehensive description of the elastic anisotropy of cubic single-crystal alloys such as MD2. These constants capture the essential mechanical responses under longitudinal loading, coupled deformations, and shear stresses, respectively. The representation of their directional dependence reinforces the strong influence of crystallographic orientation on the mechanical behavior of the material (ZHANG, 2020). Therefore, the use of these three independent elastic stiffness terms is sufficient to model the elastic behavior of MD2 and to establish a direct link between its crystallographic symmetry and macroscopic under elastic regime.



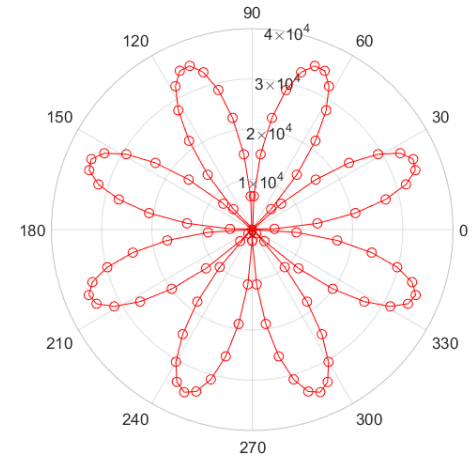
(a) C_{11} (GPa)



(b) C_{12} (GPa)



(c) C_{44} (GPa)



(d) C_{14} (GPa)

Figure 2.4: Polar representation of the elastic stiffness constants of the MD2 superalloy: (a) C_{11} , (b) C_{12} , (c) C_{44} , and (d) C_{14} .

Chapter 3

Contact Mechanics

Contact mechanics is a fundamental branch of solid mechanics that investigates the local interaction between bodies in contact under external loading. These interactions govern critical phenomena such as friction, wear, fatigue, and localized stress concentrations, which are essential in the analysis of mechanical components such as gears, bearings, joints, and metallic interfaces.

In this chapter, the fundamental principles of contact mechanics are presented, including the different contact regimes and the formulation of the plane problem. The discussion then extends to contact behavior under partial slip, as well as the effects associated with cyclic loading and the influence of the linear anisotropic elastic response of the bulk material near the contact region.

3.1 Fundamentals of Contact Mechanics

The analytical solutions available in the literature for contact problems provide a useful reference for verification numerical formulations. These solutions describe the main variables of contact mechanics, such as the contact area, the adhesion zone, the tangential load variation, and the stress distribution within the contact region.

With that in mind, this chapter aims to describe the development of the main contact mechanics equations and to present the fundamental principles of the area. The analysis of a contact problem starts with its classification. Figure 3.1 presents the contact classification according to its nature.

The first configuration, shown in Fig. 3.1(a), consists of a rigid cylinder of radius R pressed against a semi-plane. As the bodies meet, contact is made along a line and, as the load increases, the contact half-width, a , increases. This type of contact is referred to as incomplete because the contacting bodies do not share a conforming geometry, leading to a contact region that is limited and evolves with the applied load. Furthermore, the contact pressure is zero at the ends of the contact. In Fig. 3.1(b), the size of the contact is not affected by the load applied

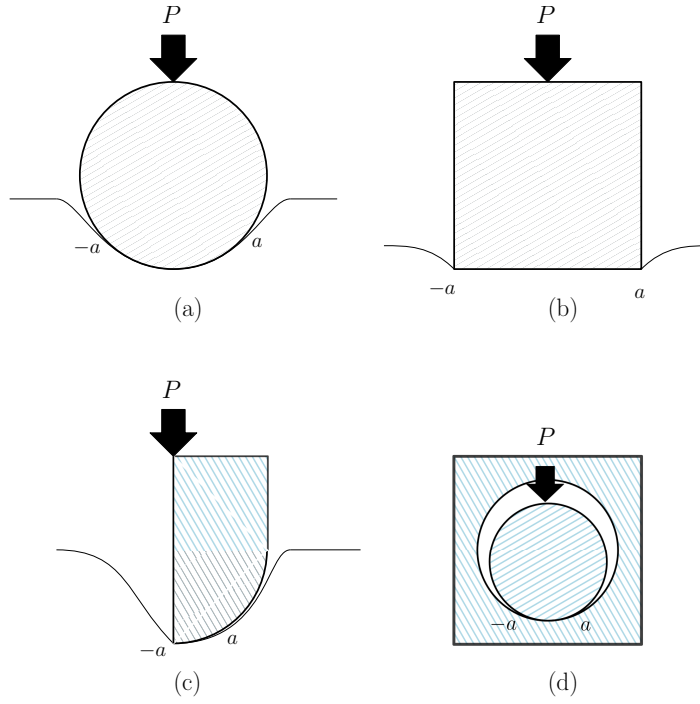


Figure 3.1: Characterization of contacts: (a) Incomplete and non-compliant; (b) Complete; (c) Incomplete with singularity; (d) Incomplete and compliant.

to the rigid flat indenter and the contact pressure presents edge singularities since the two bodies do not have a common tangent at the edge of the contact. In fact, the surface derivative of the elastic half-plane is not continuous at the contact boundaries. The case of Fig. 3.1(c) is a mixture of the configurations presented previously, where one edge has a singular pressure that is distributed through the contact line and drops continuously to zero on the other.

In the case of conformity, in Fig. 3.1(a), the contact area is smaller than the dimensions of the bodies, $a \ll R$, and an elastic half-plane simplification can be done to the problem, whereas in Fig. 3.1(d), as the load is applied to the cylinder, it deforms along the hole, and a becomes a good part of the geometry of the situation. Therefore, the complete elastic field must be considered, and the half-plane simplification is no longer applicable.

The formulation treats contact as a one-sided constraint problem and is generally used when it is of paramount importance to apply geometric constraints correctly (WRIGGERS; LAURSEN, 2006).

The behavior present in the contact zone follows certain conditions that apply throughout the calculation of the contact variables, and it is important that they are presented as in classical works on computational contact mechanics (ALART; CURNIER, 1991; KIKUCHI; ODEN, 1988). The conditions relate the contact pressure p_N and the relative normal distance between the bodies, also called *the gap*, v_N :

$$\begin{aligned}
v_N &\geq 0, \\
p_N &\leq 0, \\
p_N v_N &= 0.
\end{aligned} \tag{3.1}$$

These are known as the Kuhn-Karush-Tucker (KKT) conditions. The first statement of Eq. 3.1 is the non-penetration condition, according to which either the *gap* is greater than zero and the bodies are separated, or it is zero and the bodies are in contact. The second inequality states that the bodies in contact undergo compression when they are in contact, or the normal contact pressure is zero. Finally, the product of the contact pressure and the relative normal distance must be zero, since there are instants when the bodies are in contact ($v_N = 0, p_N < 0$) and instants when they are separated and do not interact mechanically ($v_N > 0, p_N = 0$).

Moving on to the stress distribution on the contact surface, suppose that two elastically similar bodies are placed in normal contact as shown in Fig. 3.2. When the contact load is applied, the contact pressure compresses both bodies and displacement occurs parallel to the contact zone. However, since bodies have the same elastic properties, the corresponding particles of bodies will suffer the same relative displacement in x , that is, even in the presence of a non-zero coefficient of friction, no shear stresses develop due to the absence of relative tangential displacement.

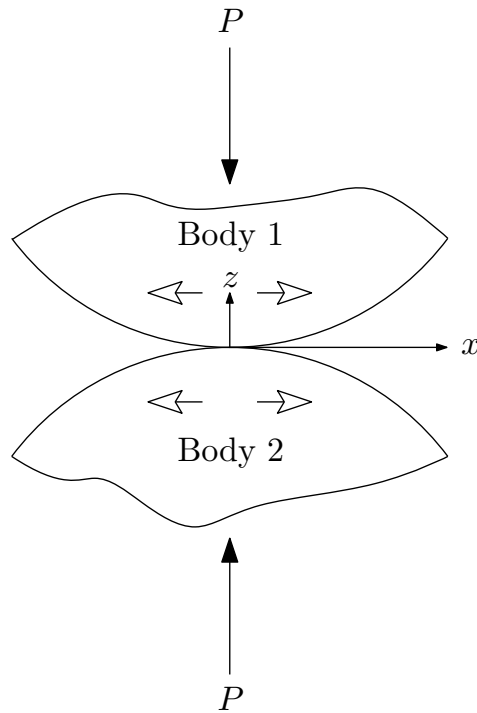


Figure 3.2: Normal contact between elastically similar bodies.

When a tangential load is applied, shear stresses develop along the contact interface. Their magnitude is bounded by the Coulomb friction law, and local sliding begins when the friction limit is reached, i.e.,

$$|q(x, y)| = -fp(x, y) \quad (3.2)$$

where $p(x, y)$ is the normal contact pressure and $q(x, y)$ is the distribution of shear stress. The presence of these shear stresses will cause a normal displacement of the surfaces of the bodies, and because both are elastically similar, the normal displacements will have the same value, causing the contact pressure to remain unchanged. In *fretting* problems, the shear load, Q , is usually too small to cause a full sliding of one body into another:

$$|Q| < -fP \quad (3.3)$$

where P is the normal load. In this case, the contact area consists of a mixture of regions where the particles cannot move relative to each other, called the adhesion zone, and regions where the particles slide past each other and the surface forces are limited by friction, called the slip zone. These zones are illustrated in Fig. 3.3.

In the slip zone, the shear stress reaches the Coulomb friction limit and is given by Eq. (3.2). In contrast, in the stick zone no relative tangential motion occurs, and the shear stress remains below the Coulomb friction limit, i.e.,

$$|q(x, y)| < -fp(x, y) \quad (3.4)$$

where $p(x, y)$ is negative under compression according to the adopted sign convention.

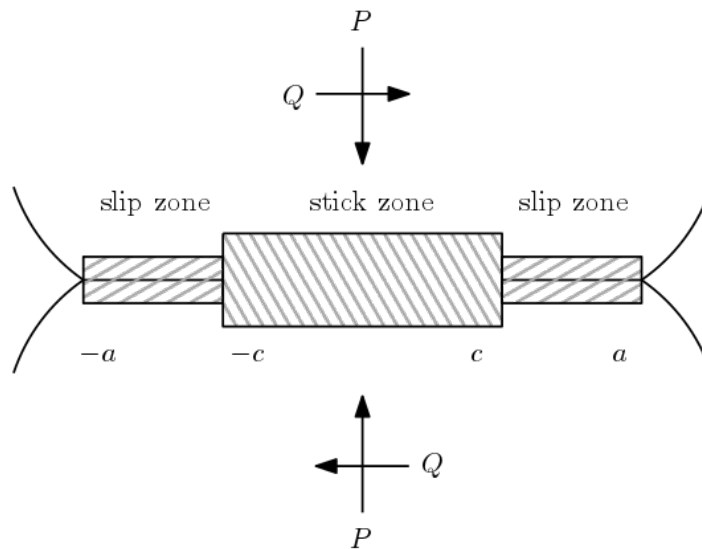


Figure 3.3: Zones in the contact area.

It is worth mentioning again that the shear stresses of the bodies have the same magnitude but in opposite directions. In contact problems, the load application must be iterative, since

the adhesion and slip zones are functions of the applied load. Zones must be recalculated for each iteration. In other words, both the adhesion and slip zones and the effective contact region can change from one iteration to another.

3.1.1 Contact Regimes

In contact simulations, it is usual to choose a region of probable contact because of the constant evolution of the contact area resulting from loading. The nodes inside this region form the so-called contact pairs (a and b) as illustrated in Fig. 3.4, and can present three contact modes. There are different relationships for normal and tangential surface forces, t_t and t_n , and for normal and tangential displacements, u_t and u_n . The values of surface forces and local displacements are shown in Tab. 3.1 and the contact modes can be summarized as follows:

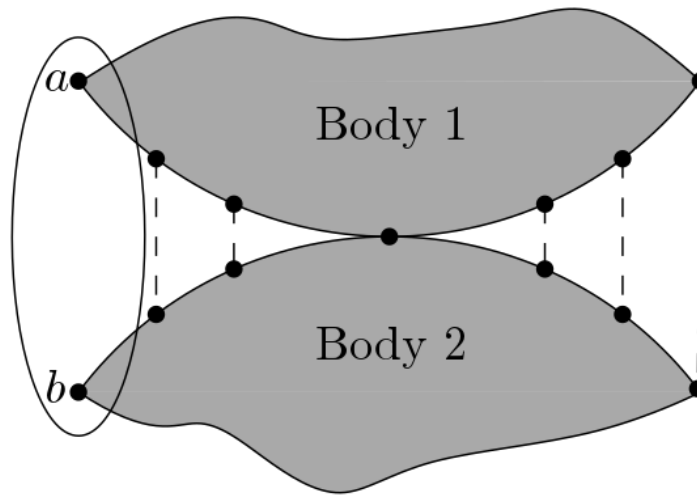


Figure 3.4: Region of probable contact.

For each contact pair, one of three possible contact regimes may occur depending on the normal gap and the tangential contact conditions: separation, sliding, or adhesion (stick). These regimes are governed by the contact constraints and the Coulomb friction law, and they may change during the iterative solution process as the external loading evolves.

- Separation: The nodes have a positive and non-zero distance between them;
- Sliding: The contact pair is not constrained in the tangential direction but free to slide over each other;
- Adhesion: Node pairs are constrained in the tangential and normal directions.

Table 3.1: Contact Regimes

Separation	Sliding	Adhesion
$t_t^a - t_t^b = 0$	$t_t^a - t_t^b = 0$	$t_t^a - t_t^b = 0$
$t_n^a - t_n^b = 0$	$t_n^a - t_n^b = 0$	$t_n^a - t_n^b = 0$
$t_t^a = 0$	$t_t^a \pm f t_n^a = 0$	$u_t^a - u_t^b = 0$
$t_n^a = 0$	$u_n^a - u_n^b = gap^{ab}$	$u_n^a - u_n^b = gap^{ab}$

3.2 Plane Problem Formulation

To determine the subsurface stress field, it is first necessary to obtain the normal contact pressure and the tangential shear stress distributions along the contact interface. These quantities are determined by solving the singular integral equations that relate the normal pressure $p(x)$ to the normal surface displacement $h(x)$, and the shear stress distribution $q(x)$ to the relative tangential displacement $g(x)$, as illustrated in Fig. 3.5. The derivation of these singular integral equations is presented in detail by (HILLS; NOWELL, 1994).

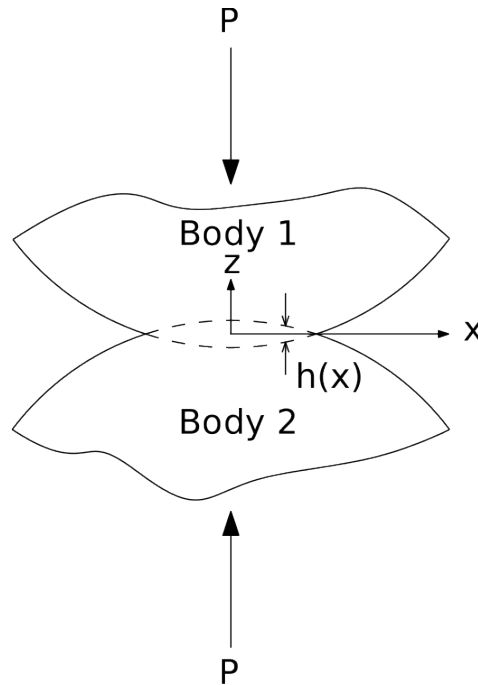


Figure 3.5: Plane normal contact problem between two elastically similar bodies.

For points belonging to the contact region, the normal pressure distribution is governed by

the singular integral equation:

$$\frac{1}{A} \frac{\partial h}{\partial x} = \frac{1}{\pi} \int_{-a}^a \frac{p(\xi)}{x - \xi} d\xi \quad (3.5)$$

Similarly, the tangential contact problem is governed by

$$\frac{1}{A} \frac{\partial g}{\partial x} = \frac{1}{\pi} \int_{-a}^a \frac{q(\xi)}{x - \xi} d\xi \quad (3.6)$$

These singular integral equations are valid only over the effective contact region. The corresponding boundary conditions depend on the contact regime (adhesion or sliding). In the separation region, no contact tractions are transmitted; therefore, $p(x) = q(x) = 0$.

$$A = 2 \left(\frac{k + 1}{4G} \right) \quad (3.7)$$

where k is the Kolosov constant. Since only plane strain conditions are considered throughout this work, $k = 3 - 4\nu$, where ν is Poisson's ratio and G is the shear modulus.

3.3 Contact under Partial Slip

In previous sections, the existence of adhesion and sliding regions was discussed as a result of a shear load that does not promote the total displacement of one body over the other. In this case, the adhesion region extends from the center of the contact to a distance c and the sliding region extends from the end of the adhesion zone to the half-width of the contact a . It is important to note that this behavior is symmetric with respect to the center of the contact.

Since the materials are assumed to behave as linear elastic solids, the governing integral equations are linear. Consequently, the principle of superposition can be applied, allowing the shear stress distribution to be represented as the sum of a full sliding solution and a corrective term associated with the adhesion (stick) zone, following Mindlin's solution.

Combining Eq. (3.2) with the Hertzian normal contact pressure distribution and representing the shear stress distribution as the superposition of the full sliding solution and a corrective term associated with the adhesion zone, the following expression is obtained:

$$q(x) = fp_0 \sqrt{1 - \left(\frac{x}{a}\right)^2} - q'(x) \quad (3.8)$$

In the adhesion zone, the value of the term $q'(x)$ is obtained. However, there is no relative movement in the x direction between the corresponding points in the adhesion region ($g'(x) = 0, |x| \leq c$). Thus, Eq. (3.8) results in:

$$q'(x) = fp_0 \frac{c}{a} \sqrt{1 - \left(\frac{x}{c}\right)^2} \quad |x| \leq c \quad (3.9)$$

In the slip zone ($c < |x| \leq a$) there is no disturbance of the total slip solution, resulting:

$$q'(x) = 0 \quad c < |x| \leq a \quad (3.10)$$

Considering tangential equilibrium, one can find the size of the adhesion zone c :

$$\frac{c}{a} = \sqrt{1 - \left(\frac{Q}{fP} \right)} \quad (3.11)$$

For the Hertzian contact problem between elastically similar bodies under partial slip conditions, the resulting normal pressure and shear stress distributions are illustrated in Fig. 3.6 for different values of the normalized tangential load $Q/(fP)$.

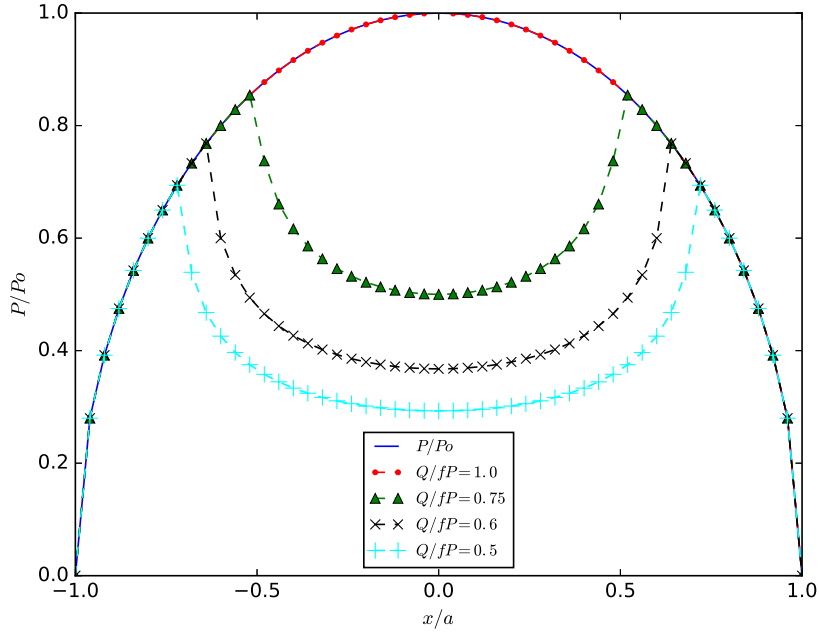


Figure 3.6: Normal pressure and shear stress distributions for a Hertzian contact under partial slip conditions.

Figure 3.6 shows the Hertzian normal pressure distribution and the corresponding shear stress distributions for a Hertzian contact under partial slip conditions. The curves correspond to different values of the normalized tangential load $Q/(fP)$. The shear stress reaches the Coulomb friction limit in the slip regions and remains below this limit within the adhesion region. As the normalized tangential load increases, the stick (adhesion) region decreases while the slip regions expand, approaching the full sliding solution when $Q/(fP) = 1$.

This behavior illustrates the redistribution of shear stresses as the tangential load increases, particularly in the transition between the adhesion and slip regions.

3.3.1 Cyclic load

Under cyclic loading conditions, the contact response exhibits path-dependent behavior as a result of the evolution of adhesion and slip zones throughout the loading cycle.

One of the objectives of this work is to evaluate shear stresses when the applied tangential load is oscillatory, as described in Fig. 3.7. The expressions developed previously are valid for when the tangential force is at its maximum value $Q_{\acute{m}ax}$, illustrated by the point A in Fig. 3.7. At this point, the shear stresses point in the opposite direction to the movement of the body, the behavior described in Eq. (3.2) and Eq. (3.4). This phenomenon is repeated throughout the cycle, therefore:

$$sgn(q(x)) = -sgn\left(\frac{\partial g}{\partial t}\right) \quad (3.12)$$

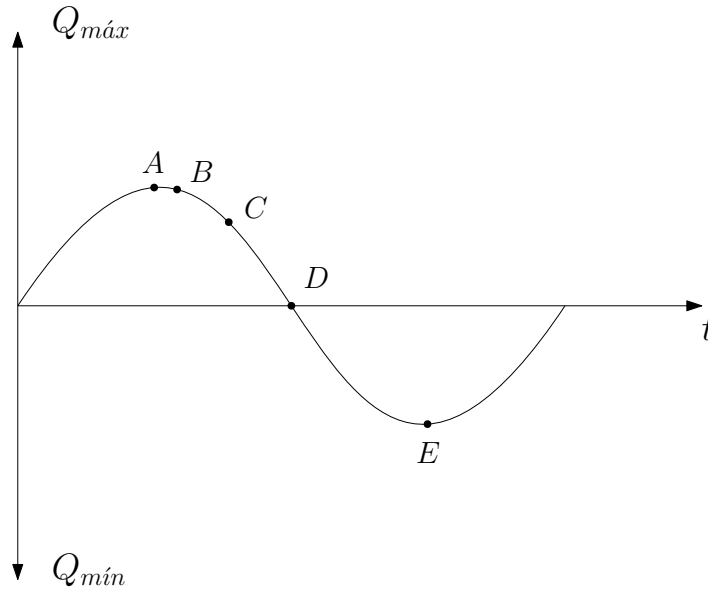


Figure 3.7: Cyclic variation of the tangential load applied during one fretting cycle.

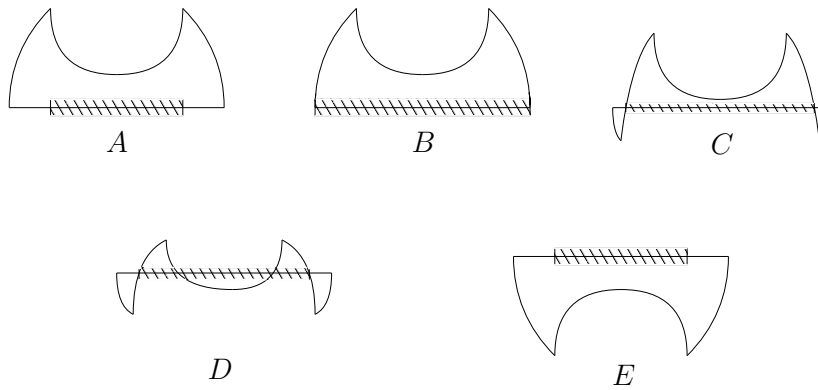


Figure 3.8: Shear stresses during the loading cycle shown in Figure 3.7.

Subsequently, the assumption is made that the tangential load has been infinitely reduced from the maximum value, a situation illustrated by the point B (Fig. 3.7). The shear stresses

also have an infinitesimal reduction and the tangential displacement drops. At this point, the direction of motion and existing shear stresses create a state of full adhesion across the entire contact. As the load decreases (Point C), reverse slip occurs at the boundaries of the contact region. The behavior of shear stresses in these phenomena can be seen in Fig. 3.8.

In the new adhesion zones, the shear stresses will have changed direction, from $fp(x)\sqrt{1 - \left(\frac{x}{a}\right)^2}$ to $-fp(x)\sqrt{1 - \left(\frac{x}{a}\right)^2}$. After the reversal of the tangential load, a new adhesion region forms. The boundary between the adhesion and slip regions is now located at $x = \pm c'$, where c' represents the half-width of the newly formed adhesion region. This notation distinguishes the new adhesion region from the original one, whose half-width is denoted by c . Analogously, it can be deduced that the perturbation necessary to avoid slipping in the new adhesion zone, c' , is given by:

$$q''(x) = 2fp_0 \frac{c'}{a} \sqrt{1 - \left(\frac{x}{c'}\right)^2} \quad (3.13)$$

The factor given in Eq. (3.13) should cancel the relative displacement when the stresses of the slip zone vary $2fp_0\sqrt{1 - \left(\frac{x}{a}\right)^2}$ instead of $fp_0\sqrt{1 - \left(\frac{x}{a}\right)^2}$ as in monotonic loading. The table presents the distribution of shear stresses for each region in the unloading phase.

Table 3.2: Surface shear stresses for each region during the variation of tangential loading with time.

$q(x)/fp_0$	Application Zone
$-\sqrt{1 - \left(\frac{x}{a}\right)^2}$	$c' < x \leq a$
$-\sqrt{1 - \left(\frac{x}{a}\right)^2} + 2\frac{c'}{a}\sqrt{1 - \left(\frac{x}{c'}\right)^2}$	$c < x \leq c'$
$-\sqrt{1 - \left(\frac{x}{a}\right)^2} + 2\frac{c'}{a}\sqrt{1 - \left(\frac{x}{c'}\right)^2} - \frac{c}{a}\sqrt{1 - \left(\frac{x}{c}\right)^2}$	$ x \leq c$

The size of the new adhesion zone is calculated from the equilibrium condition, resulting in:

$$\frac{c'}{a} = \sqrt{1 - \left| \frac{Q_{max} - Q(t)}{2fP} \right|} \quad (3.14)$$

Figure 3.9 gives a clearer view of the variation of shear stresses under cyclic loading. As Q decreases with time from its maximum value to its minimum value, it is noticeable the appearance of reverse slips and the stress distributions for Q_{max} and Q_{min} are equal in magnitude. It is notable that in both Fig. 3.8 and Fig. 3.9, the total complete unloading does not eliminate shear stresses, but they are in equilibrium. This means that the stress distribution

over time is not linear, and consequently both the shear stresses and the stresses and strains of the bodies in contact depend on the loading history.

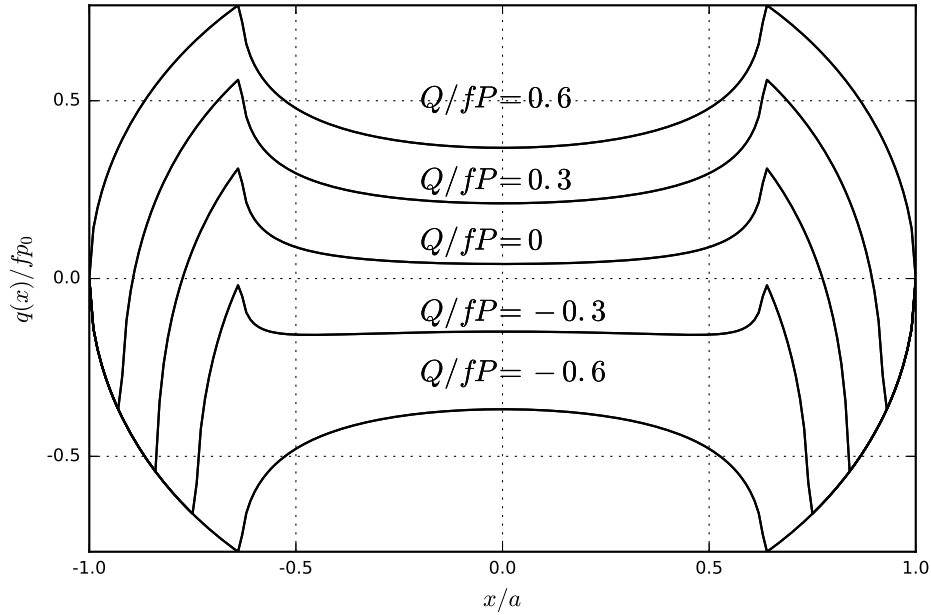


Figure 3.9: Surface shear stresses during cyclic loading, with $Q_{\max}/fP = 0.6$.

3.3.2 Bulk Load Effect

In practical applications, components subjected to fretting conditions are often affected by an additional remote bulk stress, as shown in Fig. 3.10.

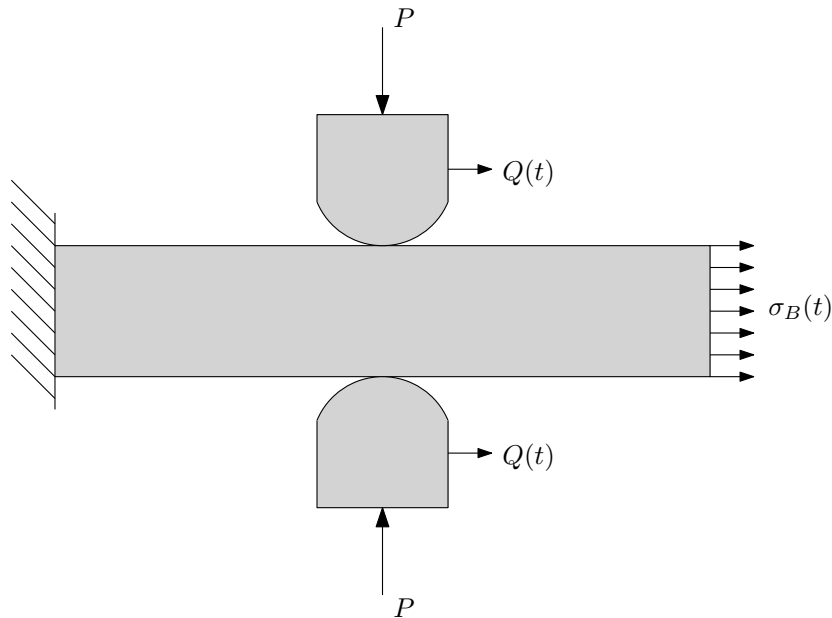


Figure 3.10: Typical configuration of fatigue loading applications by *fretting*.

The application of remote fatigue stress σ_B is generally in phase with the tangential load. A consequence of this is the displacement of the adhesion zone in the loading and unloading

phases of the tangential force. An adhesion zone shift factor, e , is performed similarly to c , where:

$$\frac{e}{a} = \frac{\sigma_B}{4fp_0} \quad (3.15)$$

$$\frac{e'(t)}{a} = \frac{\sigma_B - \sigma_B(t)}{8fp_0} \quad (3.16)$$

Equation (3.15) is valid for the moments when the tangential load has its maximum module, and the adhesion zone becomes $|x - e| \leq c$. Eq. (3.16) is valid for the loading and unloading moments, and the contact adhesion is present in the region $|x - e'| \leq c$. This analytical solution is only valid for limited values of σ_B , as there must be no reverse slippage in the contact zone, that is, $e + c' \leq a$.

$$\sigma_B \leq 4fp_0 \left(1 - \sqrt{1 - \frac{Q}{fP}} \right) \quad (3.17)$$

Figure 3.11 illustrates shear stresses with the presence of fatigue loading and the variation of the bond zone. For higher remote loads, there is a need for numerical techniques to obtain the shear stress distribution.

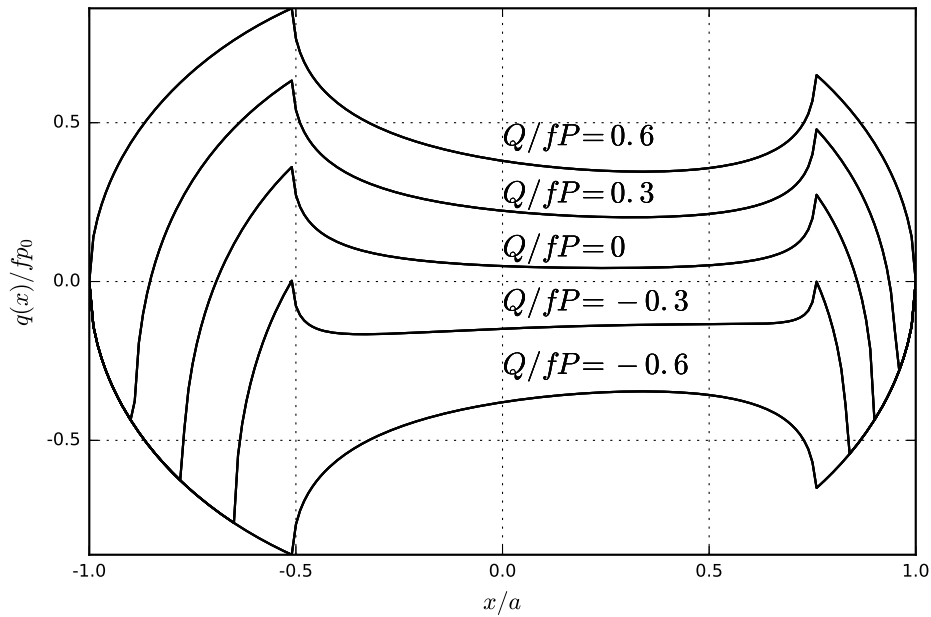


Figure 3.11: Cyclic loading with fatigue loading in phase with tangential loading, with $Q_{\text{máx}}/fP = 0.6$ and $\sigma_B/fp_0 = 0.5$.

Chapter 4

BEM Formulation for Anisotropic Elasticity with Body Forces

The Boundary Element Method (BEM) is a powerful numerical tool widely used to solve elasticity problems, particularly in anisotropic media, where the mechanical properties vary with direction. Unlike domain-based methods, BEM requires only the discretization of the boundary of the body, which significantly reduces the dimensionality of the problem and enables an efficient formulation for computing stresses and displacements.

However, the inclusion of body forces in the BEM framework represents a significant challenge, as it typically requires domain discretization, compromising the main advantage of the method. This limitation becomes particularly relevant in problems involving centrifugal loading, where body forces play a fundamental role in the mechanical response.

In this chapter, the BEM formulation for anisotropic elasticity is presented as the basis for a formulation that overcomes this limitation. The classical framework is first introduced, including anisotropic fundamental solutions, Betti's reciprocity theorem, and the formulation of integral equations. Transformation of domain integrals into boundary integrals is then achieved through the radial integration method, allowing the incorporation of body forces without internal discretization.

This extension, fully developed in Section 4.11, constitutes a key contribution of this work, enabling efficient analysis of anisotropic problems subjected to centrifugal body forces within a purely boundary-based framework.

4.1 Anisotropic fundamental solutions

To obtain static fundamental solutions to two-dimensional problems in anisotropic materials, the domain Ω will be assigned to a complex plane, using the following variable change:

$$\mathbf{z}' = \begin{Bmatrix} z'_1 \\ z'_2 \end{Bmatrix} = \begin{Bmatrix} x'_1 + \mu_1 x'_2 \\ x'_1 + \mu_2 x'_2 \end{Bmatrix} \quad (4.1)$$

and

$$\mathbf{z} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} x_1 + \mu_1 x_2 \\ x_1 + \mu_2 x_2 \end{Bmatrix} \quad (4.2)$$

where μ_k are the complex roots of Eq. (2.20), x'_1 and x'_2 are the coordinates of the source point (the point of application of the unit concentrated load) and x_1 and x_2 are the coordinates of the field point (the point obtained where the response is evaluated due to the application of the unit load).

If we consider a closed boundary Γ around the source point and if we use the surface forces defined by Eq. (2.24) and the stresses defined by Eq. (2.26), we have:

$$\begin{aligned} \int_{\Gamma} t_1 d\Gamma &= 2\text{Re}[[\mu_1 \Psi_1 + \mu_2 \Psi_2]] \\ \int_{\Gamma} t_2 d\Gamma &= 2\text{Re}[[\Psi_1 + \Psi_2]] \end{aligned} \quad (4.3)$$

where the double square brackets represent the jump in the function to a closed loop around the source point. If the boundary Γ includes $\mathbf{z}'$, then the result of Eqs. (4.3) will differ from zero.

The fundamental solutions on an infinite anisotropic plane can be found using the Airy stress function resulting from the fundamental surface forces, given by Eqs. (4.3), and the force balance equation (2.1) considering zero body forces and inertia effects.

The Airy stress function for a loaded point in the x_i direction can be represented by Ψ_{ik} . Since the boundary integral equations (4.3) have opposite signs to the applied load, it can be expressed for a source point as:

$$\begin{aligned} 2\text{Re}[[\mu_1 \Psi_{i1} + \mu_2 \Psi_{i2}]] &= -\delta_{i1} \\ 2\text{Re}[[\Psi_{i1} + \Psi_{i2}]] &= \delta_{i2} \end{aligned} \quad (4.4)$$

Eqs. (4.4) can be satisfied for any closed boundary $\mathbf{z}'$, taking:

$$\Psi_{ik} = A_{ik} \ln(\mathbf{z} - \mathbf{z}') \quad (4.5)$$

where A_{ik} are complex constants. Using properties of complex functions, it can be shown

that, for any closed boundary enclosing the point z' , the jump of the complex logarithm across its branch cut satisfies:

$$[[\ln(z - z')]] = 2\pi i. \quad (4.6)$$

Using Eqs. (4.4), (4.5) and (4.6), two equations can be obtained for unknown constants A_{ik} :

$$\begin{aligned} A_{i1} - \bar{A}_{i1} + A_{i2} - \bar{A}_{i2} &= \delta_{i2}/(2\pi i) \\ \mu_1 A_{i1} - \bar{\mu}_1 \bar{A}_{i1} + \mu_2 A_{i2} - \bar{\mu}_2 \bar{A}_{i2} &= -\delta_{i1}/(2\pi i) \end{aligned} \quad (4.7)$$

The other two equations needed to determine A_{ik} result from the requirement that displacements have unique values, that is,

$$[[u_i]] = 0 \quad (4.8)$$

Using displacement equations (2.27), Eq. (4.5) and Eq. (4.6), Eq. (4.8) can be expanded as:

$$\begin{aligned} q_{11}A_{i1} - \bar{q}_{11}\bar{A}_{i1} + q_{12}A_{i2} - \bar{q}_{12}\bar{A}_{i2} &= 0 \\ q_{21}A_{i1} - \bar{q}_{21}\bar{A}_{i1} + q_{22}A_{i2} - \bar{q}_{22}\bar{A}_{i2} &= 0 \end{aligned} \quad (4.9)$$

Writing Eqs. (4.7) and (4.9) in matrix form, we have the following:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ \mu_1 & -\bar{\mu}_1 & \mu_2 & -\bar{\mu}_2 \\ q_{11} & -\bar{q}_{11} & q_{12} & -\bar{q}_{12} \\ q_{21} & -\bar{q}_{21} & q_{22} & -\bar{q}_{22} \end{bmatrix} \begin{Bmatrix} A_{j1} \\ \bar{A}_{j1} \\ A_{j2} \\ \bar{A}_{j2} \end{Bmatrix} = \begin{Bmatrix} \delta_{j2}/(2\pi i) \\ -\delta_{j1}/(2\pi i) \\ 0 \\ 0 \end{Bmatrix} \quad (4.10)$$

which is sufficient to find the complex constants A_{ik} .

The fundamental solutions for displacements are obtained by adding the stress function given by Eq. (4.5) to the displacement equations (2.27). In this way, one has:

$$U_{ji}(\mathbf{z}', \mathbf{z}) = 2\text{Re}[q_{i1}A_{j1} \ln(z - z'_1) + q_{i2}A_{j2} \ln(z - z'_2)] \quad (4.11)$$

Similarly, the fundamental solutions for surface forces are obtained by substituting Eq. (4.5) into the stress equations (2.26) and using Eq. (2.3), we get the following.

$$T_{ij}(\mathbf{z}', \mathbf{z}) = 2\text{Re} \left[\frac{1}{(z_1 - z_1')} g_{i1}(\mu_1 n_1 - n_2) A_{j1} + \frac{1}{(z_2 - z_2')} g_{i2}(\mu_2 n_1 - n_2) A_{j2} \right] \quad (4.12)$$

where:

$$[g_{ji}] = \begin{bmatrix} \mu_1 & \mu_2 \\ -1 & -1 \end{bmatrix} \quad (4.13)$$

and n_k are the components of the external normal vector.

Note that both the fundamental solution of displacements and that of surface forces are singular when the source point approaches the field point. In the case of the fundamental displacement solution, the singularity is weak ($\ln r$). In the case of the fundamental solution of surface forces, there is a strong singularity ($1/r$). The ways in which these singularities will be treated are shown in Section 4.6.

4.2 Betti's Reciprocity Theorem

The first step in presenting the BEM formulation is related to the transformation of a differential equation into an integral equation (GAUL, 2003). For specific problems, whether in isotropic or anisotropic media, the integral formulation is based on Betti's reciprocal work theorem (BETTI, 1873).

Starting from the equilibrium equation (2.1), let $\dot{u}_i$ be a continuous vector weighting function defined over the domain Ω . Multiplying Eq. (2.1) by $\dot{u}_i$ and integrating over the domain Ω , one obtains:

$$\int_{\Omega} \sigma_{ij,j} \dot{u}_i d\Omega + \int_{\Omega} \rho b_i \dot{u}_i d\Omega - \int_{\Omega} \rho \ddot{u}_i \dot{u}_i d\Omega = 0 \quad (4.14)$$

By the rule for the derivation of the product of two functions we have:

$$(\sigma_{ij} \dot{u}_i)_{,j} = \sigma_{ij,j} \dot{u}_i + \sigma_{ji} \dot{u}_{i,j} \quad (4.15)$$

One can write $\dot{u}_{i,j}$ as the sum of a symmetric and an antisymmetric tensor, of the form:

$$\dot{u}_{i,j} = \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}) + \frac{1}{2}(\dot{u}_{i,j} - \dot{u}_{j,i}) = \varepsilon_{ij}^{\bullet} + \omega_{ij}^{\bullet} \quad (4.16)$$

where $\varepsilon_{ij}^{\bullet}$ and $\omega_{ij}^{\bullet}$ represent the tensors of stress (symmetric) and rotation (anti-symmetric), respectively, of the elastic state $\bullet$.

Substituting Eq. (4.16) into Eq. (4.15) we have:

$$(\sigma_{ji}u_i^\bullet)_{,j} = \sigma_{ij,j}u_i^\bullet + \sigma_{ij}\varepsilon_{ij}^\bullet + \sigma_{ij}\omega_{ij}^\bullet \quad (4.17)$$

where σ_{ij} is a symmetric tensor. The product of a symmetric tensor by an antisymmetric one is zero. In this way, Eq. (4.17) becomes:

$$\sigma_{ji,j}u_i^\bullet = (\sigma_{ij}u_i^\bullet)_{,j} - \sigma_{ij}\varepsilon_{ij}^\bullet \quad (4.18)$$

Substituting Eq. (4.18) into Eq. (4.14) we have:

$$-\int_{\Omega} \sigma_{ij}\varepsilon_{ij}^* d\Omega + \int_{\Omega} (\sigma_{ij}u_i^*)_{,j} d\Omega + \int_{\Omega} \rho b_i u_i^* d\Omega - \int_{\Omega} \rho \ddot{u}_i u_i^* d\Omega = 0. \quad (4.19)$$

By Green's theorem, we have:

$$\int_{\Omega} (\sigma_{ij}u_i^\bullet)_{,j} d\Omega = \int_{\Gamma} (\sigma_{ij}u_i^\bullet)n_j d\Gamma = \int_{\Gamma} t_i u_i^\bullet d\Gamma \quad (4.20)$$

where

$$t_i = \sigma_{ij}n_j \quad (4.21)$$

Substituting Eq. (4.20) into Eq. (4.19), we have:

$$\int_{\Omega} \sigma_{ij}\varepsilon_{ij}^\bullet d\Omega = \int_{\Gamma} t_i u_i^\bullet d\Gamma + \int_{\Omega} \rho b_i u_i^\bullet d\Omega - \int_{\Omega} \rho \ddot{u}_i u_i^\bullet d\Omega \quad (4.22)$$

If we start from Eq. (2.1) as corresponding to the state $u_i^\bullet$ and the weight function of Eq. (4.14) being u_i , we obtain, similarly to the previous one:

$$\int_{\Omega} \sigma_{ij}^\bullet \varepsilon_{ij} d\Omega = \int_{\Gamma} t_i^\bullet u_i d\Gamma + \int_{\Omega} \rho b_i^\bullet u_i d\Omega - \int_{\Omega} \rho \ddot{u}_i^\bullet u_i d\Omega \quad (4.23)$$

By the reciprocal theorem, two states of the same material can be related by $\sigma_{ij}^\bullet \varepsilon_{ij} = \sigma_{ij}\varepsilon_{ij}^\bullet$. In this way, equating Eqs. (4.23) and (4.22), we have:

$$\int_{\Gamma} t_i u_i^\bullet d\Gamma + \int_{\Omega} \rho u_i^\bullet (b_i - \ddot{u}_i) d\Omega = \int_{\Gamma} t_i^\bullet u_i d\Gamma + \int_{\Omega} \rho u_i (b_i^\bullet - \ddot{u}_i^\bullet) d\Omega \quad (4.24)$$

Equation (4.24) relates any two stress states. In order to deal with problems of elasticity in a continuous medium, it will be assumed that one of these states is known and the other is to be determined.

4.3 Integral Equation Formulation

To demonstrate the boundary integral equation, first consider that the body force can be represented by a Dirac delta function:

$$b_i = \delta(\mathbf{z} - \mathbf{z}')e_i, \quad (4.25)$$

where the following properties can be defined:

$$\begin{cases} \delta(\mathbf{z} - \mathbf{z}') = \infty, & \text{if } \mathbf{z} = \mathbf{z}', \\ \delta(\mathbf{z} - \mathbf{z}') = 0, & \text{if } \mathbf{z} \neq \mathbf{z}', \\ \int_{-\infty}^{\infty} \delta(\mathbf{z} - \mathbf{z}')d\Omega = 1. \end{cases} \quad (4.26)$$

According to Albuquerque (2001), the reason for choosing body force as a Dirac delta function is the possibility of reducing the number of domain integrals in our problem. This is possible due to the following property:

$$\int_{\Omega} f(\mathbf{x})\delta(\mathbf{z} - \mathbf{z}')d\Omega(\mathbf{z}) = f(\mathbf{z}'). \quad (4.27)$$

To obtain the boundary integral equation from Eq. (4.24), the state $(u_i^{\bullet}, t_i^{\bullet}, b_i^{\bullet})$ is defined as the fundamental state and the state (u_i, t_i, b_i) is related to the desired solution. Using the fundamental solutions for displacement $U_{ij}(\mathbf{z}', \mathbf{z})$ and for surface tractions $T_{ij}(\mathbf{z}', \mathbf{z})$, given by Eqs. (4.11) and (4.12), respectively, the fundamental state can be rewritten as:

$$\begin{aligned} u_i^{\bullet} &= u_j^{\bullet}\delta_{ij} = U_{ij}(\mathbf{z}', \mathbf{z})\delta_{ji}e_i, \\ t_i^{\bullet} &= t_j^{\bullet}\delta_{ij} = T_{ij}(\mathbf{z}', \mathbf{z})\delta_{ji}e_i, \end{aligned}$$

where δ_{ij} is the Kronecker delta. Since the present formulation is restricted to elastostatic problems, the inertial term vanishes, i.e., $\ddot{u}_i = 0$. Consequently, Eq. (4.24) reduces to:

$$\int_{\Gamma} T_{ij}(\mathbf{z}, \mathbf{z}')\delta_{ij}^{\bullet}e_i u_i d\Gamma + \int_{\Omega} b_i^{\bullet} u_i d\Omega = \int_{\Gamma} t_i U_{ij}(\mathbf{z}, \mathbf{z}')\delta_{ij}^{\bullet}e_i d\Gamma + \int_{\Omega} b_i U_{ij}(\mathbf{z}, \mathbf{z}')\delta_{ij}^{\bullet}e_i d\Omega. \quad (4.28)$$

By using the sifting property of the Kronecker delta, $\delta_{ij}e_i = e_j$, and recognizing that the resulting expressions correspond to the components of the displacement and body-force vectors, Eq. (4.28) can be rewritten in the compact tensor form:

$$\int_{\Gamma} T_{ij}(\mathbf{z}, \mathbf{z}')u_j(\mathbf{z})d\Gamma(\mathbf{z}') + u_i(\mathbf{z}') = \int_{\Gamma} t_j(\mathbf{z})U_{ij}(\mathbf{z}, \mathbf{z}')d\Gamma(\mathbf{z}) + \int_{\Omega} b_j(\mathbf{z})U_{ij}(\mathbf{z}, \mathbf{z}')d\Omega(\mathbf{z}). \quad (4.29)$$

Rearranging Eq. (4.29), we obtain:

$$u_i(\mathbf{z}') + \int_{\Gamma} T_{ij}(\mathbf{z}, \mathbf{z}') u_j(\mathbf{z}) d\Gamma(\mathbf{z}) = \int_{\Gamma} t_j(\mathbf{z}) U_{ij}(\mathbf{z}, \mathbf{z}') d\Gamma(\mathbf{z}) + \int_{\Omega} b_j(\mathbf{z}) U_{ij}(\mathbf{z}, \mathbf{z}') d\Omega(\mathbf{z}). \quad (4.30)$$

Equation (4.30) is known as the Somigliana identity for the interior points of a body. Through Betti's reciprocal theorem, Somigliana (1885) obtained the integral representation of the solution to plane elasticity problems, which is equivalent to the second Green's identity. For convenience, Eq. (4.30) can be rewritten by explicitly denoting the source point as z_0 :

$$u_i(\mathbf{z}_0) + \int_{\Gamma} T_{ij}(\mathbf{z}, \mathbf{z}_0) u_j(\mathbf{z}) d\Gamma(\mathbf{z}) = \int_{\Gamma} t_j(\mathbf{z}) U_{ij}(\mathbf{z}, \mathbf{z}_0) d\Gamma(\mathbf{z}) - \int_{\Omega} b_j(\mathbf{z}) U_{ij}(\mathbf{z}, \mathbf{z}') d\Omega(\mathbf{z}). \quad (4.31)$$

According to Sollero e Aliabadi (1995), this equation is not considered a well-posed boundary value problem. Since surface tractions and boundary displacements are not known for all points on the boundary, it is necessary to establish a relationship between traction and displacement on the boundary by moving the source point to the boundary itself. This process is not trivial and requires some adjustments, as presented by (GAUL, 2003).

First, one must analyze the nonsingular behavior (when $\mathbf{z} - \mathbf{z}' \neq 0$) of the integrals in Eq. (4.31). Since r represents the distance between the source point and the field point, it is necessary to analyze the behavior of the integrand as the source approaches the boundary ($\mathbf{z} - \mathbf{z}' = 0$). To achieve this, a semicircular region Γ_{ε}^* of radius ε , centered at the source point, is removed from the boundary. Equation (4.31) is then rewritten as:

$$u_i(\mathbf{z}') + \lim_{\varepsilon \rightarrow 0} \int_{\Gamma - \Gamma_{\varepsilon} + \Gamma_{\varepsilon}^*} T_{ij}(\mathbf{z}, \mathbf{z}') u_j(\mathbf{z}) d\Gamma(\mathbf{z}) = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma - \Gamma_{\varepsilon} + \Gamma_{\varepsilon}^*} U_{ij}(\mathbf{z}, \mathbf{z}') t_j(\mathbf{z}) d\Gamma(\mathbf{z}). \quad (4.32)$$

Equation (4.32) contains two singular integrals. The integral on the right-hand side is known as a weakly singular integral, because the fundamental solution to displacement varies as $\log(r)$. As discussed by Gaul, Koegl e Wagner (2003), the corresponding traction fundamental solution exhibits a stronger singularity, varying as $1/r$ as the source point is approached.

To isolate the singularity of the traction kernel, the displacement field is decomposed into its value at the source point and the remaining regular part, i.e.,

$$u_j(z) = u_j(z') + [u_j(z) - u_j(z')].$$

Since $u_j(z) - u_j(z') = \mathcal{O}(r)$ as $z \rightarrow z'$, the product $T_{ij}(z, z') [u_j(z) - u_j(z')]$ remains bounded. Consequently, the singular part of the integral is isolated, leading to:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Gamma - \Gamma_\varepsilon^+ + \Gamma_\varepsilon^+} T_{ij}(\mathbf{z}_0, \mathbf{z}) u_j(\mathbf{z}) d\Gamma(\mathbf{z}) &= \lim_{\varepsilon \rightarrow 0} \int_{\Gamma - \Gamma_\varepsilon^+} T_{ij}(\mathbf{z}_0, \mathbf{z}) [u_j(\mathbf{z}') - u_j(\mathbf{z})] d\Gamma(\mathbf{z}) \\ &+ u_j(\mathbf{z}') \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon^+} T_{ij}(\mathbf{z}_0, \mathbf{z}) d\Gamma(\mathbf{z}). \end{aligned} \quad (4.33)$$

According to the standard limiting process for Somigliana's identity (see, e.g., (ALIABADI, 2002)), the first integral in Eq. (4.33) vanishes. The remaining integral yields the displacement jump, which is expressed by the coefficients $A_{ij}(\mathbf{z})u_j$. These coefficients depend on the local boundary geometry and the elastic constants (SOLLERO; ALIABADI, 1995).

The coefficients $A_{ij}(\mathbf{z})$ depend on the local geometry of the boundary and on the elastic constants Sollero e Aliabadi (1995). The remaining integral is improper and must be interpreted in the sense of the Cauchy principal value. Finally, the limiting boundary integral equation can be written as:

$$c_{ij}(\mathbf{z}_0)u_i(\mathbf{z}_0) + \oint_{\Gamma} T_{ij}(\mathbf{z}_0, \mathbf{z})u_j(\mathbf{z}) d\Gamma(\mathbf{z}) = \int_{\Gamma} U_{ij}(\mathbf{z}_0, \mathbf{z})t_j(\mathbf{z}) d\Gamma(\mathbf{z}) + \int_{\Omega} b_j(\mathbf{z})U_{ij}(\mathbf{z}, \mathbf{z}')d\Omega(\mathbf{z}) \quad (4.34)$$

where $c_{ij}(\mathbf{z}_0)$ is known as the *free term coefficient*, given by:

$$c_{ij}(\mathbf{z}_0) = \begin{cases} 1 - \frac{\alpha}{2\pi}, & \text{for } \mathbf{z}_0 \in \Gamma, \\ 1 & \text{for } \mathbf{z}_0 \in \Omega, \\ 0 & \text{for } \mathbf{z}_0 \notin \Gamma, \mathbf{z}_0 \notin \Omega. \end{cases} \quad (4.35)$$

This equation is considered the starting point for initiating the numerical solution of the Boundary Element Method (BEM). Thus, the next step consists of discretizing the boundary into small segments so that Eq. (4.34) can be applied to each segment. Consequently, a linear system is assembled from each portion of the discretized boundary.

4.4 Domain-to-Boundary Transformation via RIM

In this section, the radial integration method will be presented, which is quite general for the treatment of domain forces and was originally proposed by Gao (2002).

Consider the term p_i from Eq. (4.31). One can use the method of radial integration to

transform the domain integral into a boundary integral, as follows:

$$\int_{\Omega} p_i U_{ik} d\Omega = \int_{\theta_1}^{\theta_2} \underbrace{\int_0^r U_{ik} p_i [x(\rho, \theta), y(\rho, \theta)] \rho d\rho}_{F_k} d\theta \quad (4.36)$$

For complex geometries, the computation of the radial distance $r(\theta)$ follows the procedure proposed by Gao (2002), in which the domain is partitioned into visible subregions whenever multiple boundary intersections occur.

Defining:

$$F_k = \int_0^r U_{ik} p_i [x(\rho, \theta), y(\rho, \theta)] \rho d\rho \quad (4.37)$$

which results in:

$$\int_{\Omega} U_{ik} p_i(x, y) d\Omega = \int_{\theta_1}^{\theta_2} F_k d\theta \quad (4.38)$$

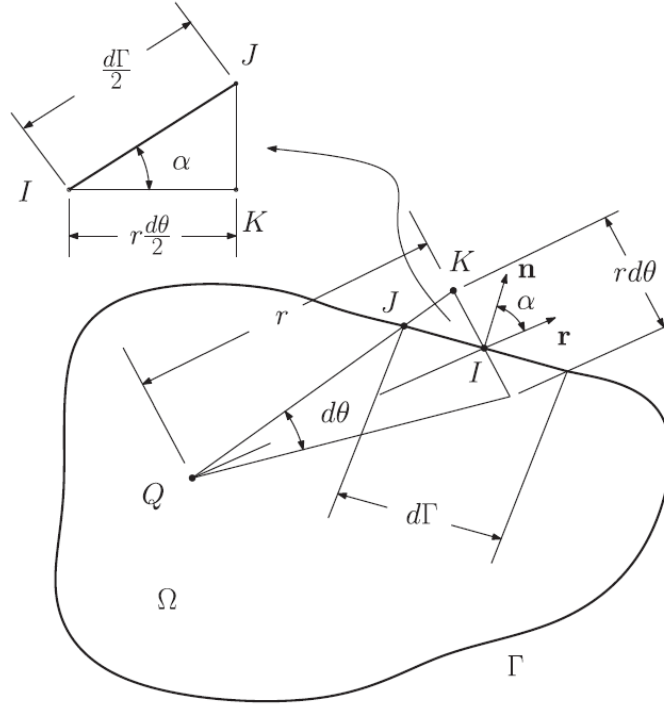


Figure 4.1: Transformation of domain integral into boundary integral.

From the triangle of Fig. (4.1), it follows:

$$\begin{aligned} \cos \alpha &= \frac{r \frac{d\theta}{2}}{\frac{d\Gamma}{2}} \\ d\theta &= \frac{\cos \alpha}{r} d\Gamma \end{aligned} \quad (4.39)$$

As $\mathbf{n}$ and $\mathbf{r}$ are unit vectors, we have:

$$\cos \alpha = \mathbf{n} \mathbf{r} \quad (4.40)$$

Substituting Eq. (4.40) in Eq. (4.39) and then in Eq. (4.38), we have:

$$\int_{\Omega} U_{ik} p_i(x, y) d\Omega = \int_{\Gamma} F_k(x, y) \frac{\mathbf{n} \mathbf{r}}{r} d\Gamma \quad (4.41)$$

Substituting Eq. (4.41) in Eq. (4.31), follows:

$$c_{ij}(\mathbf{z}_o) u_j(\mathbf{z}_o) + \int_{\Gamma} T_{ij}(\mathbf{z}, \mathbf{z}_o) u_j(\mathbf{z}) d\Gamma(\mathbf{z}) = \int_{\Gamma} U_{ij}(\mathbf{z}, \mathbf{z}_o) t_j(\mathbf{z}) d\Gamma(\mathbf{z}) + \int_{\Gamma} F_i(\mathbf{z}, \mathbf{z}_o) \frac{\mathbf{n} \mathbf{r}}{r} d\Gamma(\mathbf{z}) \quad (4.42)$$

Equation (4.42) serves as the foundational equation for the boundary element method. In particular, all integrals of the domain have been successfully eliminated.

4.5 Formulation of discretized boundary elements

To obtain the solution to the problem and calculate the unknown variables in Eq. (4.34), the boundary is divided into boundary elements. In this work, discontinuous quadratic elements (3 nodes per element) will be used (elements whose end nodes are not shared with neighboring elements).

In this section, it will be more convenient to work with matrix notation rather than index notation. According to Albuquerque (2001), we have:

$$\begin{aligned} \mathbf{u} &= \phi \mathbf{u}^{(i)} \\ \mathbf{t} &= \phi \mathbf{t}^{(i)} \\ \hat{\mathbf{u}} &= \phi \hat{\mathbf{u}}^{(i)} \\ \hat{\mathbf{t}} &= \phi \hat{\mathbf{t}}^{(i)} \end{aligned} \quad (4.43)$$

where bold symbols denote vectors or matrices. Here, $\mathbf{u}^{(i)}$ and $\mathbf{t}^{(i)}$ represent the nodal values of surface displacements and tractions, respectively, ϕ is the matrix of shape functions, $\mathbf{u}$ and $\mathbf{t}$ denote the displacement and traction vectors along the element, and $\hat{\mathbf{u}}^{(i)}$ and $\hat{\mathbf{t}}^{(i)}$ are the nodal vectors associated with the particular solutions for displacements and tractions, respectively.

Assume that the domain has been divided into NE boundary elements. Substituting Eqs. (4.43) into Eq. (4.34), we have:

$$\mathbf{c}^l \mathbf{u}^l + \sum_{j=1}^{NE} \left\{ \oint_{\Gamma_j} \mathbf{T} \phi d\Gamma \right\} \mathbf{u}^j = \sum_{j=1}^{NE} \left\{ \int_{\Gamma_j} \mathbf{U} \phi d\Omega \right\} \mathbf{t}^j + \int_{\Gamma} \mathbf{F} \frac{\mathbf{n} \mathbf{r}}{r} d\Gamma \quad (4.44)$$

The element matrices are defined as:

$$\mathbf{G}^{(j)} = \int_{\Gamma_j} \mathbf{U} \phi d\Gamma, \quad (4.45)$$

and

$$\mathbf{H}^{(j)} = \int_{\Gamma_j} \mathbf{T} \phi d\Gamma. \quad (4.46)$$

after assembling the element contributions, the following system is obtained:

$$\sum_{j=1}^N H^{lj} u^j = \sum_{j=1}^N G^{lj} t^j + p_i \quad (4.47)$$

The free-term coefficient matrix is added to the diagonal entries of $\mathbf{H}$. Therefore, the final assembled system can be written as:

$$\mathbf{H} \mathbf{u} = \mathbf{G} \mathbf{t} + \mathbf{p} \quad (4.48)$$

4.6 Boundary Element Interpolation

Displacements and surface forces are represented in a discontinuous quadratic element as:

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} & 0 \\ 0 & \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{Bmatrix} = \phi \mathbf{u}^{(n)} \quad (4.49)$$

and

$$\mathbf{t} = \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = \begin{bmatrix} \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} & 0 \\ 0 & \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} \end{bmatrix} \begin{Bmatrix} t_1^{(1)} \\ t_2^{(1)} \\ t_1^{(2)} \\ t_2^{(2)} \\ t_1^{(3)} \\ t_2^{(3)} \end{Bmatrix} = \phi \mathbf{t}^{(n)} \quad (4.50)$$

where $u_i^{(n)}$ and $t_i^{(n)}$ are the nodal values of the surface displacements and forces, respectively, and $\phi^{(i)}$ are the functions of quadratic forms defined by:

$$\phi^{(1)} = \xi\left(\frac{9}{8}\xi - \frac{3}{4}\right) \quad (4.51)$$

$$\phi^{(2)} = 1 - \frac{9}{4}\xi^2 \quad (4.52)$$

$$\phi^{(3)} = \xi\left(\frac{9}{8}\xi + \frac{3}{4}\right) \quad (4.53)$$

where ξ represents a dimensionless coordinate along the element (Fig. 4.2).

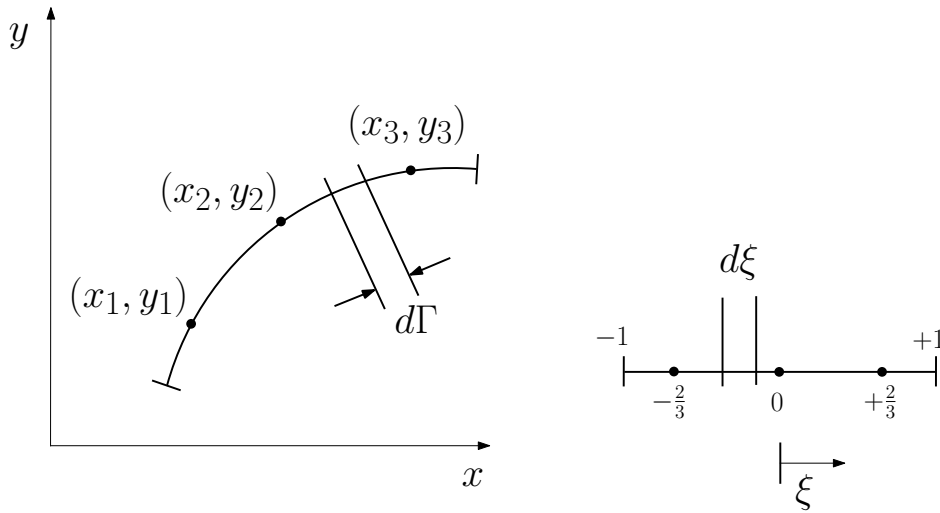


Figure 4.2: Transformation of coordinates $x_1 - x_2$ to ξ .

The formulation adopted in this work is isoparametric. Therefore, in addition to the physical variables on the boundary (displacements and surface tractions), the geometry is also approximated using discontinuous quadratic elements. This choice is made to maintain consistency with the adopted boundary element formulation, in which both the geometry and the field variables are interpolated using the same discontinuous quadratic elements (ALBU-

QUERQUE, 2001). The geometry interpolation is given by:

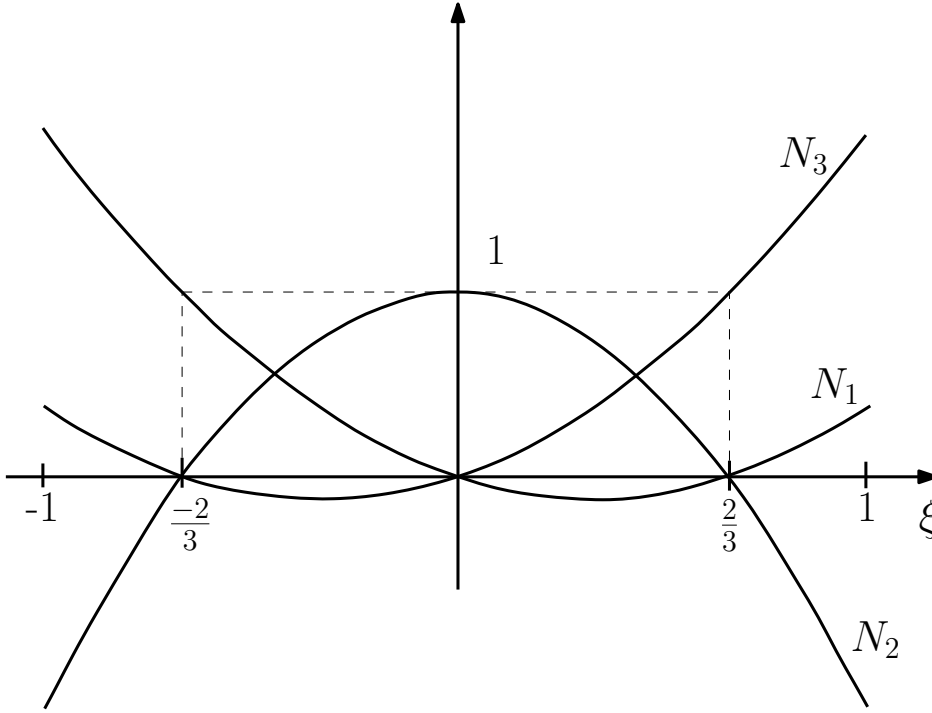


Figure 4.3: Discontinuous quadratic shape functions.

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} & 0 \\ 0 & \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} \end{bmatrix} \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_1^{(2)} \\ x_2^{(2)} \\ x_1^{(3)} \\ x_2^{(3)} \end{Bmatrix} = \phi \mathbf{x}^{(n)} \quad (4.54)$$

In this way, the boundary integrals can be written as:

$$H^{(j)} = \oint_{\Gamma_j} T_{lk} \phi^{(j)} d\Gamma = \int_{-1}^1 T_{lk} \phi^{(j)} |J| d\xi \quad (4.55)$$

$$G^{(j)} = \int_{\Gamma_j} U_{lk} \phi^{(j)} d\Gamma = \int_{-1}^1 U_{lk} \phi^{(j)} |J| d\xi \quad (4.56)$$

where $|J|$ represents the Jacobian modulus of the transformation $(x_1, x_2) \rightarrow \xi$, and is by:

$$|J| = \frac{d\Gamma}{d\xi} = \left\{ \left(\frac{dx_1}{d\xi} \right)^2 + \left(\frac{dx_2}{d\xi} \right)^2 \right\}^{1/2} \quad (4.57)$$

where $dx_1/d\xi$ and $dx_2/d\xi$ are obtained by deriving Eqs. (4.54) with respect to ξ .

It has been very common in the literature to use discontinuous quadratic shape functions for the physical variables of the problem, for example, temperature and flow in heat conduction problems and displacements and tractions in structural analysis problems. However, the use of discontinuous functions to interpolate the geometry was always overlooked. The justification for not using it was that the nodes are located inside the element. Thus, in the case of curved elements, there is a risk of a discontinuity between the end of one element and the beginning of another, as shown in Figure 4.4 and in detail in Figure 4.5. This problem does not exist when the elements are continuous, as there are nodes at the ends of the elements where they are shared between neighboring elements, as shown in Figure 4.6.

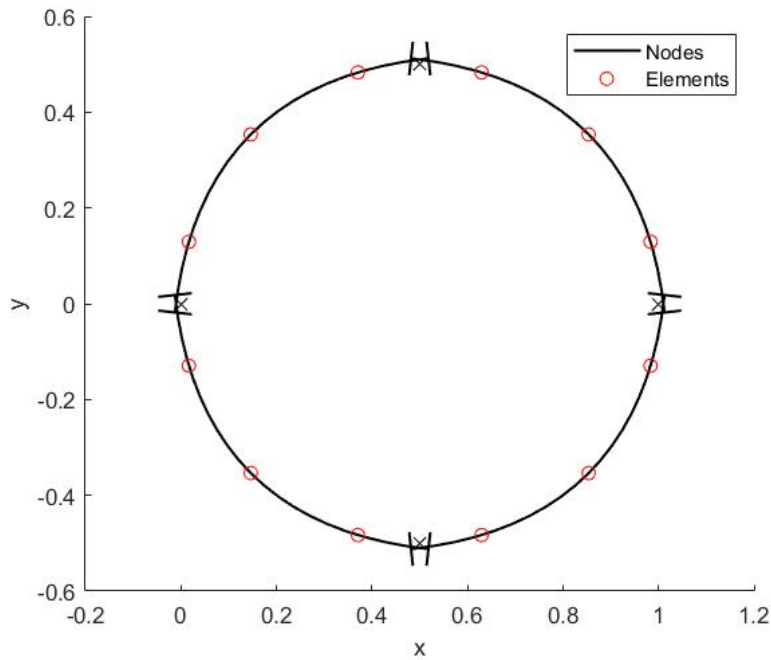


Figure 4.4: Discontinuous curved elements.

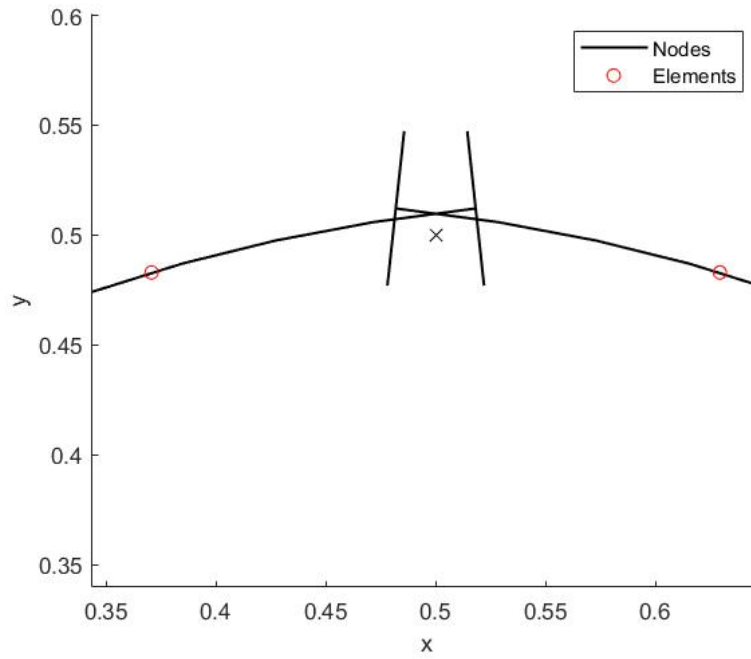


Figure 4.5: Overlapping curved discontinuous elements.

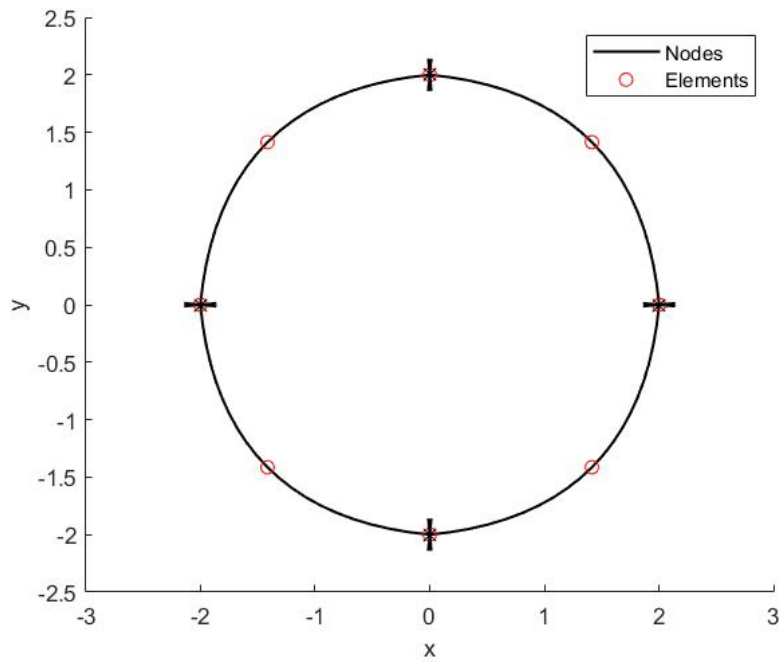


Figure 4.6: Curved continuous elements.

The shape functions for the continuous elements with nodes in $\xi = -2/3$, $\xi = 0$ and $\xi = 2/3$ are given by:

$$\phi^{(1)} = \frac{1}{2}\xi(\xi - 1) \quad (4.58)$$

$$\phi^{(2)} = 1 - \xi^2 \quad (4.59)$$

$$\phi^{(3)} = \frac{1}{2}\xi(\xi + 1) \quad (4.60)$$

These shape functions are parabolas, as shown in Figure 4.7.

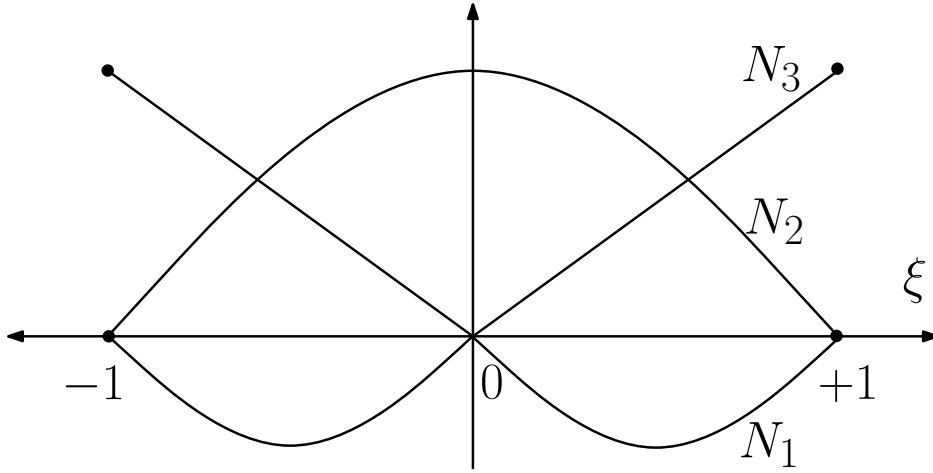


Figure 4.7: Continuous quadratic form functions.

To ensure that there is no overlap between two neighboring elements, in this work, continuous shape functions are used to define the node positions of discontinuous elements. Proceed as follows:

- The elements in the parabola are generated using the continuous shape functions, given by Equations (4.58)–(4.60) and shown in Figure 4.7.
- The positions of the nodes of discontinuous elements are calculated by evaluating the functions at $\xi = -2/3$, $\xi = 0$ and $\xi = 2/3$.

The use of discontinuous quadratic elements for the geometric interpolation is adopted in the present work to maintain consistency with the boundary element formulation proposed by Albuquerque (2001), in which both the geometry and the field variables are interpolated using the same discontinuous approximation. Therefore, this choice is motivated by the adopted formulation rather than by an intrinsic improvement in the geometric representation.

Following this procedure, since three points uniquely define a parabola, discontinuous elements can be used to interpolate the geometry without overlap at the ends of adjacent elements.

4.7 Local reference system

In the present implementation, the governing equations are expressed in a local coordinate system (n, t) , defined by the normal and tangential directions at each boundary node, rather than in the global Cartesian system (x_1, x_2) . This choice facilitates the imposition of boundary conditions involving prescribed normal and tangential displacements and provides a convenient framework for the implementation of contact mechanics formulations.

To change the reference system (x_1, x_2) to the system (n, t) , the vectors of displacements and forces of the nodal surfaces, given respectively by $\mathbf{u}^{(i)}$ and $\mathbf{t}^{(i)}$ written in the frame, (x_1, x_2) must be multiplied by a transformation matrix, which in this case is a matrix of 2×2 whose elements are given as a function of the components of the unit vector normal to the boundary $\mathbf{n}$. Thus, we have the following.

$$\begin{Bmatrix} u_1^{(i)} \\ u_2^{(i)} \end{Bmatrix} = \begin{bmatrix} n_1 & n_2 \\ -n_2 & n_1 \end{bmatrix} \begin{Bmatrix} u_n^{(i)} \\ u_t^{(i)} \end{Bmatrix} \quad (4.61)$$

Thus, the vectors $\mathbf{u}$ and $\mathbf{t}$ representing the displacements and surface forces, respectively, at any point on the boundary, are given, respectively, by:

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} n_1 & n_2 \\ -n_2 & n_1 \end{bmatrix} \begin{bmatrix} \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} & 0 \\ 0 & \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} \end{bmatrix} \begin{Bmatrix} u_n^{(1)} \\ u_t^{(1)} \\ u_n^{(2)} \\ u_t^{(2)} \\ u_n^{(3)} \\ u_t^{(3)} \end{Bmatrix} = \phi \hat{\mathbf{u}}^{(n)} \quad (4.62)$$

and

$$\mathbf{t} = \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = \begin{bmatrix} n_1 & n_2 \\ -n_2 & n_1 \end{bmatrix} \begin{bmatrix} \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} & 0 \\ 0 & \phi^{(1)} & 0 & \phi^{(2)} & 0 & \phi^{(3)} \end{bmatrix} \begin{Bmatrix} t_n^{(1)} \\ t_t^{(1)} \\ t_n^{(2)} \\ t_t^{(2)} \\ t_n^{(3)} \\ t_t^{(3)} \end{Bmatrix} = \phi \hat{\mathbf{t}}^{(n)} \quad (4.63)$$

Note that the vectors $\mathbf{u}$ and $\mathbf{t}$ are still represented in the reference system (x_1, x_2) .

Thus, equation (4.48) is rewritten as:

$$\hat{\mathbf{H}}\hat{\mathbf{u}} = \hat{\mathbf{G}}\hat{\mathbf{t}} + \mathbf{p} \quad (4.64)$$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{t}}$ are vectors that contain displacements and surface forces, respectively, at all nodes, written in the local reference system (n, t) . The matrices $\hat{\mathbf{H}}$ and $\hat{\mathbf{G}}$ are the influence matrices $\mathbf{H}$ and $\mathbf{G}$, companions taking into account that the shape functions were multiplied by the transformation matrices. Since the vector $\mathbf{p}$ does not depend on the values of displacements and surface forces, it remains unchanged.

To calculate the unknown variables, an exchange of columns is performed between the matrices $\hat{\mathbf{H}}$ and $\hat{\mathbf{G}}$, according to the boundary conditions, generating the matrices $\mathbf{A}$ and $\mathbf{B}$. In this way, all unknown variables on the boundary, be they displacements or surface forces, pass to a vector $\mathbf{x}$ and the known variables pass to a vector $\mathbf{y}$, obtaining:

$$\mathbf{Ax} = \mathbf{By} + \mathbf{p} \quad (4.65)$$

which can be rewritten as a linear system:

$$\mathbf{Ax} = \mathbf{b} \quad (4.66)$$

Once the linear system has been solved, the unknown variables contained in $\mathbf{x}$ are combined with the prescribed boundary values to recover the complete displacement and traction vectors, $\mathbf{u}$ and $\mathbf{t}$.

4.8 Treatment of singular integrals

Singular integrals of order $O(\ln r)$ can be efficiently evaluated by Gaussian squaring with a cubic variable transformation, as proposed by Telles (1987), which exactly cancels the logarithmic singularity. Using the Gaussian logarithmic quadrature of by (STROUD; SECREST, 1966), terms containing logarithmic singularities can be integrated as follows:

$$I = \int_0^1 \ln\left(\frac{1}{\xi}\right) f(\xi) d\xi \cong \sum_{i=1}^N w_i f(\xi_i) \quad (4.67)$$

where N is the number of Gauss points. The coordinate of the integration point ξ_i and the weight factor w_i can be found in the literature by Stroud e Secrest (1966), Brebbia e Dominguez (1996).

In this work, the non-singular terms of the matrices $\hat{\mathbf{H}}$ and $\hat{\mathbf{G}}$ are integrated using standard Gaussian quadrature with 10 integration points. The singular terms of $\mathbf{G}$ are of type $\log(r)$ and are integrated using Gaussian logarithmic quadrature. The singular terms of $\hat{\mathbf{H}}$ are of type $1/r$ and need to be evaluated in the sense of Cauchy's principal value. A very simple way to handle this singularity is by considering rigid body considerations (BREBBIA; DOMINGUEZ, 1996).

Assuming that a rigid body has all its boundary points displaced by a unit value in a given direction of the system (x_1, x_2) and that there are no body forces ($p_i = 0$) in the direction of one of the coordinates of the axes, the surface forces at any point on the boundary of this body must be zero. Note that in this case, the system must be (x_1, x_2) to represent the rigid body displacement and not the system (n, t) . In this way, Eq. (4.48) becomes:

$$\mathbf{H}\mathbf{v}^q = 0 \quad (4.68)$$

where $\mathbf{v}^q$ is a vector that for all nodes has unit displacements along the q direction and zero in the other direction. To satisfy Eq. (4.68) we have the following.

$$H_{ii} = - \sum_{j=1}^N H_{ij} \quad j \neq i \quad (4.69)$$

where j is either even or odd, depending on the degree of freedom (direction) under consideration.

The diagonal term of the $\mathbf{H}$ matrix is equal to the sum of all other non-diagonal terms that correspond to the degree of freedom (direction) under consideration.

4.9 Boundary stresses

To calculate the stress tensor at a given boundary node, consider a node where the boundary normal and tangent directions do not coincide with the global coordinate axes (Fig. 4.8). A local coordinate system (n, t) , aligned with the normal and tangent directions, is adopted. Expressing the displacements and surface tractions in this local system yields:

$$\begin{aligned} u'_i &= l_{ij}u_j \\ t'_i &= l_{ij}t_j \end{aligned} \quad (4.70)$$

where l_{ij} are the direct cosines.

The local system has the following relationship:

$$\begin{aligned} \sigma'_{22} &= t'_2 \\ \sigma'_{12} &= t'_1 \end{aligned} \quad (4.71)$$

The deformation ε'_{11} can be calculated, knowing that:

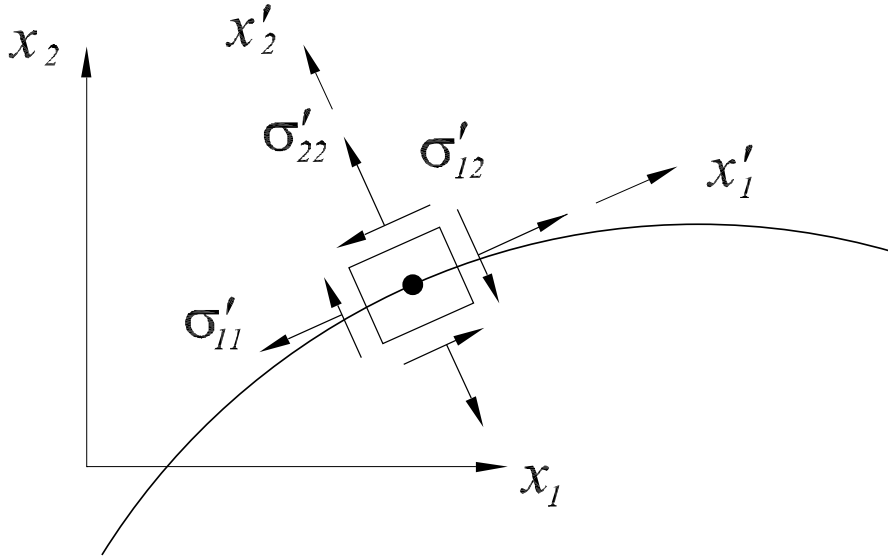


Figure 4.8: Boundary stresses.

$$\begin{aligned}\varepsilon'_{11} &= \frac{1}{2}(u'_{1,1} + u'_{1,1}) = u'_{1,1} \\ u'_{1,1} &= \frac{du'_1}{dx'_1} = \frac{du'_1}{d\xi} \frac{d\xi}{dx'_1}\end{aligned}\tag{4.72}$$

Using differential geometry in Eq. (4.72), it can be seen that the local direction x'_1 is tangent to the infinitesimal arc length ds given by:

$$\begin{aligned}ds &= \sqrt{dx'^2_1 + dx'^2_2} = \sqrt{\left(\frac{dx'_1}{d\xi}\right)^2 + \left(\frac{dx'_2}{d\xi}\right)^2} d\xi \\ \frac{ds}{d\xi} &= J\end{aligned}\tag{4.73}$$

A small move along s corresponds to a small move along x'_1 . This allows x'_1 in Eq. (4.72) to be replaced by Eq. (4.73), that is,

$$\begin{aligned}\varepsilon'_{11} &= \frac{du'_1}{d\xi} \frac{d\xi}{ds} \\ \varepsilon'_{11} &= \frac{du'_1}{d\xi} J^{-1}\end{aligned}\tag{4.74}$$

Substituting the shape functions, we obtain:

$$\begin{aligned}
u_1 &= \sum_{i=1}^3 \phi^{(i)} u_1^{(i)} \\
\frac{du_1}{d\xi} &= \sum_{i=1}^3 \frac{d\phi^{(i)}}{d\xi} u_1^{(i)}
\end{aligned} \tag{4.75}$$

where $\phi^{(i)}$ are the shape functions. The deformation can then be obtained by:

$$\varepsilon'_{11} = \sum_{i=1}^3 \frac{d\phi^{(i)}}{d\xi} u_1^{(i)} J^{-1} \tag{4.76}$$

From the stress-strain relationship, we have the following.

$$\begin{Bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{12} \end{Bmatrix} = \begin{bmatrix} Q'_{11} & Q'_{12} & Q'_{16} \\ Q'_{12} & Q'_{22} & Q'_{26} \\ Q'_{16} & Q'_{26} & 2Q'_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon'_{11} \\ \varepsilon'_{22} \\ \varepsilon'_{12} \end{Bmatrix} \tag{4.77}$$

where Q'_{ij} represents the stiffness tensor components written in the local frame.

In Eq. (4.77) there are three unknowns σ'_{11} , ε'_{22} , ε'_{12} , which can now be calculated.

Finally, the stresses must be written in the global reference frame x_1x_2 , that is:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \mathbf{T}^{-1} \begin{Bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{12} \end{Bmatrix} \tag{4.78}$$

where $\mathbf{T}$ is the coordinate transformation matrix.

4.10 Internal Displacements and Stresses

The displacement at an internal point $\mathbf{z}' \in \Omega$ is obtained from the boundary integral representation as:

$$u_i(\mathbf{z}') = \int_{\Gamma} U_{ij}(\mathbf{z}', \mathbf{z}) t_j(\mathbf{z}) d\Gamma(\mathbf{z}) - \int_{\Gamma} T_{ij}(\mathbf{z}', \mathbf{z}) u_j(\mathbf{z}) d\Gamma(\mathbf{z}) + \int_{\Omega} U_{ij}(\mathbf{z}', \mathbf{z}) p_j(\mathbf{z}) d\Omega(\mathbf{z}). \tag{4.79}$$

Since the source point $\mathbf{z}'$ is located inside the domain, the kernels in Eq. (4.79) are regular and no singular integration treatment is required.

The stress tensor at the internal point $\mathbf{z}'$ is obtained by differentiating Eq. (4.79) with respect to the source-point coordinates and applying the constitutive relation. Thus, the

internal stresses can be written as:

$$\sigma_{ik} + \int_{\Gamma} S_{jik} u_j d\Gamma = \int_{\Gamma} D_{jik} t_j d\Gamma - \int_{\Omega} D_{jik} p_j d\Omega \quad (4.80)$$

where S_{kij} and D_{kij} are linear combinations of the derivatives of T_{ij} and U_{ij} , respectively.

The S_{kij} tensor is given by:

$$\begin{Bmatrix} S_{11j} \\ S_{22j} \\ S_{21j} \end{Bmatrix} = - \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} T_{1j,1} \\ T_{2j,2} \\ \frac{1}{2} (T_{1j,2} + T_{2j,1}) \end{Bmatrix} \quad (4.81)$$

where $j = 1, 2$. The derivatives of T_{ij} are obtained by the equation:

$$T_{ij,k} = -2\text{Re} \left[\frac{1}{(z_1 - z'_1)^2} R_{k1} q_{j1} (\mu_1 n_1 - n_2) A_{i1} + \frac{1}{(z_2 - z'_2)^2} R_{k2} q_{j2} (\mu_2 n_1 - n_2) A_{i2} \right] \quad (4.82)$$

where

$$R_{kl} = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \quad (4.83)$$

The fundamental stress tensor D_{kij} is given by:

$$\begin{Bmatrix} D_{11j} \\ D_{22j} \\ D_{21j} \end{Bmatrix} = - \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} U_{1j,1} \\ U_{2j,2} \\ \frac{1}{2} (U_{1j,2} + U_{2j,1}) \end{Bmatrix} \quad (4.84)$$

where the derivatives of U_{ij} are given by:

$$U_{ij,k} = 2\text{Re} \left[\frac{1}{z_1 - z'_1} R_{k1} q_{j1} A_{i1} + \frac{1}{z_2 - z'_2} R_{k2} q_{j2} A_{i2} \right] \quad (4.85)$$

The domain integral in Eq. (4.79) is transformed into boundary integrals using the radial integration method described previously. Following the same procedure used to obtain Eq. (4.41), it can be written as:

$$\int_{\Omega} D_{kij} p_j d\Omega = \int_{\Gamma} K_{kj} \frac{n_i r_i}{r} d\Gamma \quad (4.86)$$

$$K_{kj} = \int_0^r D_{kij} p_j [x(\rho, \theta), y(\rho, \theta)] \rho d\rho \quad (4.87)$$

The radial integral in Eq. (4.87) is evaluated numerically using an N-point Gaussian quadrature.

Substituting Eq. (4.86) into Eq. (4.80), we have:

$$\sigma_{ij} + \int_{\Gamma} S_{kij} u_k d\Gamma - \int_{\Gamma} D_{kij} t_k d\Gamma + \int_{\Gamma} K_{kj} \frac{n_i \cdot r_i}{r} d\Gamma \quad (4.88)$$

The integral equations for displacements (4.30) and stresses (4.88) were written for a source point that belongs to the interior of the domain Ω .

The integral equations of displacement (4.42) and stress (4.88) can be used to determine the displacement and stress components, respectively, at any number of previously selected points within the domain.

4.11 Centrifugal body forces

The proposed strategy constitutes one of the main contributions of this work, since centrifugal loads can be incorporated without introducing internal cells or radial basis function interpolation.

In rotating systems, body forces arise from centrifugal effects, leading to a distributed load proportional to the radial distance from the axis of rotation. Under these conditions, the equation of motion for an anisotropic elastic medium can be written as

$$\sigma_{ij,j} + \rho\omega^2 x_i = 0 \quad (4.89)$$

where ω is the angular velocity and x_i is the coordinate of the point, considering that the rotation axis is perpendicular to the origin of the coordinate system.

This equation introduces a body force term that must be properly accounted for in the boundary integral formulation.

So, the body force b_i is given by:

$$b_i = \rho\omega^2 x_i \quad (4.90)$$

The centrifugal body force defined in Eq. (4.90) is incorporated into the proposed boundary element formulation through the radial integration method. This approach transforms the corresponding domain integrals into equivalent boundary integrals, preserving the boundary-only character of the BEM without requiring internal discretization. The resulting formulation is employed in the frictional contact analyses presented in the next chapter.

Chapter 5

Contact Formulation and Numerical Implementation

The formulation developed in Chapter 4 is now extended to contact problems involving anisotropic elastic bodies. In this context, the Boundary Element Method is combined with a contact algorithm to account for nonlinear interactions between bodies, including frictional effects and possible separation.

The numerical implementation of the proposed formulation is presented, including the discretization of the contacting boundaries, the enforcement of the contact constraints, and the nonlinear solution algorithm.

5.1 Discrete Numerical Model

The contact formulation adopted in this work is based on the classical contact theory presented by (KIKUCHI; ODEN, 1988). The two bodies in contact are bounded by their respective boundaries, which are divided into two regions: the potential contact region Γ_c and the free region Γ_{nc} , as shown in Fig. 5.1.

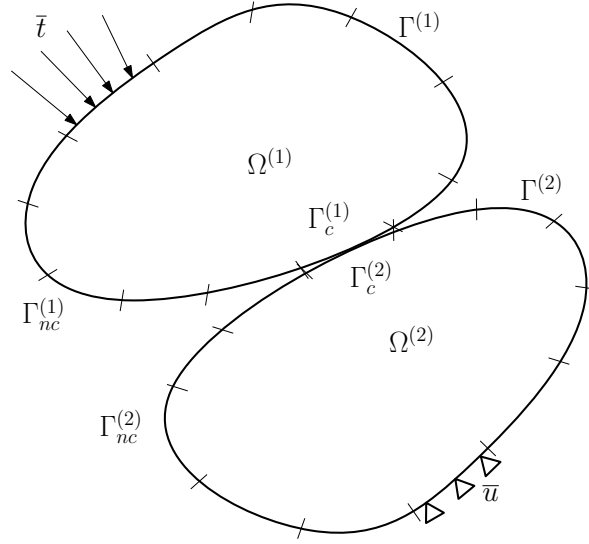


Figure 5.1: Two bodies in contact.

The contact region Γ_c is not known a priori and constitutes part of the solution of the problem, introducing boundary condition (contact) nonlinearity. In contrast, the non-contact region Γ_{nc} is defined where no interaction occurs. The objective of the formulation is to determine, for each pair of nodes along Γ_c , the corresponding contact mode and associated constraints.

Within the region of possible contact, any contact mode may be present depending on the type of loading. The boundary element method is then applied to determine the modes and a quantitative value of the contact variables. The elastic behavior of the two bodies, $\gamma = 1, 2$, is still governed by the theory of elasticity, and a boundary integral equation can be derived for each. As a function of an external load, the surface forces and displacements of the bodies are described as follows:

$$C_{ij}^\gamma u_j^\gamma + \int_{\Gamma_{nc}^\gamma} T_{ij}^\gamma u_j^\gamma d\Gamma^\gamma + \int_{\Gamma_c^\gamma} T_{ij}^\gamma u_j^\gamma d\Gamma^\gamma = \int_{\Gamma_{nc}^\gamma} U_{ij}^\gamma t_j^\gamma d\Gamma^\gamma + \int_{\Gamma_c^\gamma} U_{ij}^\gamma t_j^\gamma d\Gamma^\gamma \quad \forall, \gamma \in \Gamma. \quad (5.1)$$

Equation (5.1) can be taken to the discrete for in the same way as Eq. (4.42):

$$c_{ij}^\gamma u_j^\gamma + \sum_{n=1}^N H_{ij}^{n\gamma} u_j^{n\gamma} = \sum_{n=1}^N G_{ij}^{m\gamma} t_j^{n\gamma} \quad (5.2)$$

where n is the number of nodes in $\gamma = 1, 2$. These equations are represented in matrix form by:

$$\mathbf{H}^\gamma \mathbf{u}^\gamma = \mathbf{G}^\gamma \mathbf{t}^\gamma, \quad (5.3)$$

The vectors $\mathbf{u}^\gamma$ and $\mathbf{t}^\gamma$ represent the displacement and surface forces of the bodies, which could be easily discovered in multi-region problems. However, contact problems have contact nonlinearity resulting from the unknown extent of the contact zone. The size of the contact area becomes part of the problem, along with the surface forces and displacements in the entire region of possible contact Γ_c .

After applying the problem's boundary conditions, the system becomes unbalanced, with more variables than equations. Typically, in BEM problems, there is a known boundary condition and the generation of a mixed solution vector, which is not the case in this formulation. The other region Γ_{nc} behaves similarly to the boundary method matrices, exchanging the influence matrices of each body, forming $\mathbf{A}_{nc}^\gamma$, which do not change during the solution process (ABASCAL, 1995) and relates all nodal values external to the region of probable contact, $\mathbf{x}_{nc}^\gamma$.

The resulting system is obtained by coupling the boundary integral equations of the two bodies through the contact constraints. The unknowns comprise the vectors $\mathbf{X}_{nc}^1$ and $\mathbf{X}_{nc}^2$ associated with the non-contact regions, the displacement and traction variables on the contact regions, and the auxiliary variables introduced to enforce the contact constraints.

The contact variables, $\mathbf{u}_c^\gamma$ and $\mathbf{t}_c^\gamma$, are related to their corresponding influence matrices, $\mathbf{H}_c^\gamma$ and $\mathbf{G}_c^\gamma$, respectively. The resulting unbalanced system is represented by:

$$\begin{bmatrix} \mathbf{A}_{nc}^1 & 0 & \mathbf{H}_c^1 & 0 & -\mathbf{G}_c^1 & 0 \\ 0 & \mathbf{A}_{nc}^2 & 0 & \mathbf{H}_c^2 & 0 & -\mathbf{G}_c^2 \end{bmatrix} \begin{Bmatrix} \mathbf{x}_{nc}^1 \\ \mathbf{x}_{nc}^2 \\ \mathbf{u}_c^1 \\ \mathbf{u}_c^2 \\ \mathbf{t}_c^1 \\ \mathbf{t}_c^2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \end{Bmatrix} \quad (5.4)$$

where the vectors $\mathbf{b}^\gamma$ are the result of applying the boundary conditions. For the system to have a solution, it is necessary to allocate extra equations that relate the contact variables of each body. Table 3.1 presents a series of equations that relate surface forces and displacements of pairs of nodes that can be represented in matrix form and complete the system, characterizing the method of direct constraints. They vary for each contact mode the pairs are in: sliding, separation, and adhesion.

Each contact pair may exhibit different mechanical behaviors depending on the loading conditions. Three possible contact modes are considered: adhesion (stick), sliding, and separation. These modes are defined by specific constraints that relate displacements and tractions at the contact interface.

Sliding:

$$\mathbf{c}_u = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_n^a \\ u_t^a \\ u_n^b \\ u_t^b \end{Bmatrix} \quad \mathbf{c}_t = \begin{bmatrix} 1 & 0 & -1 & 0 \\ \pm f & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \pm f & 1 \end{bmatrix} \begin{Bmatrix} t_n^a \\ t_t^a \\ t_n^b \\ t_t^b \end{Bmatrix}$$

Adhesion:

$$\mathbf{c}_u = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_n^a \\ u_t^a \\ u_n^b \\ u_t^b \end{Bmatrix} \quad \mathbf{c}_t = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} t_n^a \\ t_t^a \\ t_n^b \\ t_t^b \end{Bmatrix}$$

Separation:

$$\mathbf{c}_u = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_n^a \\ u_t^a \\ u_n^b \\ u_t^b \end{Bmatrix} \quad \mathbf{c}_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} t_n^a \\ t_t^a \\ t_n^b \\ t_t^b \end{Bmatrix}$$

After evaluating the contact mode of each pair, a set of constraints for displacement and traction is formed, $\mathbf{C}_u^\gamma$ and $\mathbf{C}_t^\gamma$. In addition to the contact mode, the *gap* between the points is measured ($\mathbf{v}^\gamma$) and placed in the degree of freedom corresponding to the evaluated node pair. Equation (5.4) is redefined so that:

$$\begin{bmatrix} \mathbf{A}_{nc}^1 & 0 & \mathbf{H}_c^1 & 0 & -\mathbf{G}_c^1 & 0 \\ 0 & \mathbf{A}_{nc}^2 & 0 & \mathbf{H}_c^2 & 0 & -\mathbf{G}_c^2 \\ 0 & 0 & \mathbf{C}_u^1 & \mathbf{C}_u^2 & \mathbf{C}_t^1 & \mathbf{C}_t^2 \end{bmatrix} \begin{Bmatrix} \mathbf{x}_{nc}^1 \\ \mathbf{x}_{nc}^2 \\ \mathbf{u}_c^1 \\ \mathbf{u}_c^2 \\ \mathbf{t}_c^1 \\ \mathbf{t}_c^2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \\ \mathbf{v}^{1,2} \end{Bmatrix}. \quad (5.5)$$

Therefore, Eq. (5.5) results in a nonlinear algebraic system, which can be written in compact form as:

$$\mathbf{Ax} = \mathbf{b}. \quad (5.6)$$

One advantage of this formulation is its simple computational implementation and another is the fact that, for the same configuration, varying the boundary conditions does not change the entire system, but only the constraint-applied portion. This strategy is particularly effective in frictional contact problems, where convergence strongly depends on the gradual evolution of the contact state, where boundary conditions are applied gradually. It is important to

emphasize that in these cases, the likely contact region should be chosen carefully and kept as small as possible, as its size increases, resulting in a greater number of terms being checked with each boundary condition application, leading to longer computational processing time.

However, node-to-node mesh coupling is limited to problems with small deformations and little slip. When the tangential displacement of the node is large, it begins to move away from the original contact pair, and the application of contact constraints becomes inadequate. This drawback is not an issue for fretting problems, where small displacements are part of the phenomenon. The fact that mesh generation must be conformal is more important because much time is wasted preparing the mesh in some problems compared to other coupling methods.

5.2 Solution of Nonlinear System

The nonlinear system given by Eq. (5.6) is solved using the standard Newton method. This method is obtained from the first-order Taylor expansion of the nonlinear function around the current iterate, yielding:

$$\mathcal{F}(x_k) + (x^* - x_k)\mathcal{F}'(x_k) = 0 \quad (5.7)$$

The value of x^* is found iteratively, where the value x_k is found from an increment to the result obtained in the previous step:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - [\mathbf{F}'(\mathbf{x}_k)]^{-1} \mathbf{F}(\mathbf{x}_k) = \mathbf{x}_k + \delta_k, \quad (5.8)$$

the process is repeated until one of the stopping criteria is met, which are:

$$|\mathcal{F}(x_{k+1})| < \varepsilon_1 \quad (5.9)$$

$$|x_{k+1} - x_k| < \varepsilon_2 \quad (5.10)$$

in the first case, the norm of the residual satisfies $|\mathcal{F}(\mathbf{x}_{k+1})| < \varepsilon_1$, indicating that the current approximation is sufficiently close to the solution. In the second case, the difference between two successive iterations satisfies $|\mathbf{x}_{k+1} - \mathbf{x}_k| < \varepsilon_2$, indicating that the iterative process has converged since no significant changes occur between consecutive iterations.

This is one of the most common ways to find roots of nonlinear functions because it converges quickly and is easy to implement. It requires an initial guess, x_0 , and the derivative of the function. If the function is not differentiable or the finding of the derivative is a complex task, it is convenient to use the secant method.

For multidimensional systems, the same principle applies:

$$\mathbf{J}_k \Delta \mathbf{x}_k = -\mathbf{r}_k \quad (5.11)$$

The solution increment $\Delta \mathbf{x}_k$ obtained from Eq. (5.11) is then used to update the solution according to Eq. (5.12).

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta \mathbf{x}_k. \quad (5.12)$$

where $\mathbf{J}_k$ is the Jacobian matrix and $\mathbf{r}_k$ is the residual vector.

In the present formulation, the tangent Jacobian at iteration k corresponds to the current system matrix $\mathbf{A}_k$. Therefore, the Newton correction is obtained by solving

$$\mathbf{A}_k \Delta \mathbf{x}_k = -\mathbf{r}_k,$$

where the residual vector is defined as:

$$\mathbf{r}_k = \mathbf{A}_k \mathbf{x}_k - \mathbf{b}.$$

The matrix $\mathbf{A}_k$ is updated at each iteration because changes in the contact conditions modify the tangent system.

$$\varepsilon = \sqrt{(\mathbf{x}_{k+1} - \mathbf{x}_k)^T (\mathbf{x}_{k+1} - \mathbf{x}_k)} \quad (5.13)$$

The iterative procedure is repeated until the prescribed convergence criteria are satisfied.

The algorithm 1 presents the iterative process used to solve the nonlinear system of this work:

Algorithm 1 Implementation of Newton's method.

- 1: Checks the contact mode of each node
 - 2: Restrictions for each contact mode apply
 - 3: An initial value is set for $\mathbf{x}_0 = \mathbf{x}$
 - 4: Initializes the error, err , and the stopping criterion ε
 - 5: **while** $err < \varepsilon$ **do**
 - 6: $\mathbf{dy} = \mathbf{A}^{-1}$
 - 7: $\mathbf{y} = \mathbf{Ax} - \mathbf{b}$
 - 8: $\mathbf{d} = \mathbf{dy} * \mathbf{d}$
 - 9: $\mathbf{x} = \mathbf{x} - \mathbf{d}$
 - 10: $err = \sqrt{(\mathbf{x} - \mathbf{x}_0)'(\mathbf{x} - \mathbf{x}_0)}$
 - 11: $\mathbf{x}_0 = \mathbf{x}$
 - 12: Checks the contact mode of each node
 - 13: Restrictions for each contact mode apply
-

5.2.1 Incremental Load Application

In contact problems, the final solution is obtained only when the maximum external load or total contact of the contact region is reached. However, since the contact region is a function of the applied force and the coefficient of friction, the solution must be obtained using an iterative numerical solution. Especially if the contact is between nonconforming boundaries, the approximate solution is achieved by gradually applying the load in several steps.

It is known that there are several methods to apply load to contact problems. Instantaneous application of an external load in a single step is acceptable if there is node-to-node contact in the adhesion region. In this case, the iterative process of solving the system is used until the slip and adhesion regions are known.

Another, more recommended, approach is the incremental load application. In this case, the total load is applied gradually, in several increments. The external load is applied incrementally because the Newton method exhibits local convergence. By dividing the total load into several increments, the converged solution of the current increment provides an initial guess sufficiently close to the solution of the next increment. Consequently, the robustness and convergence of the nonlinear solution procedure are improved. Because it is an iterative process, the history of contact stress, which is important in specific contact cases such as fretting, must be recorded.

In the incremental approach, the boundary conditions and the contact state are updated in each iteration of each increment. The number of increments is predefined by the user, and the sum of the increments is equal to the total external load, that is:

$$P_{max} = \sum_{m=1}^M \Delta P^m \quad (5.14)$$

where ΔP is the load applied in each increment, P_{max} is the total external load applied, and M is the number of increments. The total applied force is calculated by Equation (5.15).

$$P_j^m = P_j^{m-1} + \Delta P_j^m \quad (5.15)$$

where P_j^m is the total external force up to increment m . With the application of each load increment (ΔP), the system will respond, initially in equilibrium. The system's response results in small variations of surface forces and displacements at the system boundary until a new equilibrium is reached. Variations in displacement and surface forces are defined as follows:

$$\begin{aligned} u_j^m &= u_j^{m-1} + \Delta u_j^m \\ t_j^m &= t_j^{m-1} + \Delta t_j^m \end{aligned} \quad (5.16)$$

where Δu_j^m and Δt_j^m are incremental variations in displacement and surface force due to incremental load ΔP_j^m . Rewriting the boundary integral equation, we have the following:

$$C_{ij}(u_j^{m-1} + \Delta u_j^m) + \int_{\Gamma} T_{ij}(u_j^{m-1} + \Delta u_j^m) d\Gamma = \int_{\Gamma} U_{ij}(t_j^{m-1} + \Delta t_j^m) d\Gamma$$

The boundary integral for each increment is obtained using the following equation.

$$C_{ij}\Delta u_j^m + \int_{\Gamma} T_{ij}\Delta u_j^m d\Gamma = \int_{\Gamma} U_{ij}\Delta t_j^m d\Gamma$$

The nodes of the discretized boundary must be arranged in the contact region and form pairs of nodes, creating the contact surface. All node pairs must be resolved as an independent contact system, satisfying incremental surface forces $\Delta \mathbf{t}$ and incremental displacements $\Delta \mathbf{u}$. Furthermore, this procedure must be applied to both the equilibrium equation and the compatibility equation. From this it can be seen that for each body there will be a system of equations, shown previously in Eq. (5.3). Reformulating for the incremental case, we have:

$$\mathbf{H}^1 \Delta \mathbf{u}^1 = \mathbf{G}^1 \Delta \mathbf{t}^1 \quad \text{and} \quad \mathbf{H}^2 \Delta \mathbf{u}^2 = \mathbf{G}^2 \Delta \mathbf{t}^2 \quad (5.17)$$

where the vectors $\Delta \mathbf{u}$ and $\Delta \mathbf{t}$ contain, respectively, the displacements and the values of the surface forces for each increment. To obtain a solution of the problem, the boundary conditions outside the contact region and contact constraints within the contact region must be applied.

5.2.2 Contact Mode Decision

As mentioned earlier, the contact mode is checked at each load increment step to decide which nodes are in contact and which are not. At the beginning of each load increment, an initial estimate of the contact region is adopted. After solving the nonlinear system, the contact conditions are verified. If any contact constraint is violated, the contact region is updated, the system matrix is rebuilt, and the iterative procedure is repeated until a consistent contact configuration and a converged solution are obtained. Considering the contact pair a and b , the nodes are classified as separated at load step m when:

$$(\Delta u_n^a + \Delta u_n^b)^m > 0. \quad (5.18)$$

Otherwise, the nodes are considered to be in contact when:

$$(\Delta u_n^a + \Delta u_n^b)^m \leq 0. \quad (5.19)$$

nodes a and b are considered to be separated. The normal gap is positive, indicating that no

contact exists between the two bodies.

Once contact occurs, a second check is made regarding the tangential motion of the contact nodes. The pair evaluation, for adhesion mode, is given by

$$|t_t^m| = |t_t^{m-1} + \Delta t_t^m| \leq f |t_n^{m-1} + \Delta t_n^m| < f |t_n^m| \quad (5.20)$$

If this condition holds, the pair is in adhesion mode. Conversely:

$$|t_t^{m-1} + \Delta t_t^m| \geq f |t_n^{m-1} + \Delta t_n^m| \quad (5.21)$$

the sliding condition is met. Coulomb's law of friction defines two coefficients of friction, one static and one dynamic. However, in problems where sliding and adhesion exist simultaneously, f can be approximated by an average value (NOWELL; HILLS, 1987). The algorithm 2 encompasses Eq. (5.18), (5.19), (5.20) and the decision logic of the contact modes.

Algorithm 2 Contact mode decision procedure.

```

for  $m = m + 1, m = m_f$ 
2:   if  $(\Delta u_n^a + \Delta u_n^b)^m \geq g_0^{m-1}$  and  $t_n^{m-1} + \Delta t_n^m < 0$  then
      Contact detected between  $a$  and  $b$ 
4:   if  $(|t_t^m| = |t_t^{m-1} + \Delta t_t^m| \leq f |t_n^{m-1} + \Delta t_n^m| = f |t_n^m|)$  then
      Contact nodes in adhesion,
6:   else
      Contact nodes in sliding,
8:   end if
   else
10:  No contact:  $(\Delta u_n^a + \Delta u_n^b)^m < g_0^{m-1}$ 
   end if
12: continue
end for

```

An schematic algorithm for the main code is presented in Algorithm 3.

Algorithm 3 Main code

Input Geometry, load, material properties

```
1: for  $i = 1, 2$  ▷ Assemble the matrices for bodies 1 and 2
2:   Assemble  $G_i$  and  $H_i$ 
3:   Apply BC in the non-contact region
4:   Insert matrices  $A_i$  into  $A^{NL}$ 
5: end for
6: for  $i = \text{number of load steps}$ 
7:   Calculate the gap  $g_0$  between node pairs in the contact zone.
8:   Apply Newton's method to solve  $\mathbf{R} = \mathbf{A}\mathbf{x} - \mathbf{b}$ 
9:   Update vectors  $\Delta\mathbf{u}$  and  $\Delta\mathbf{t}$  based on BCs and unknowns.
10:   $\mathbf{u}_i = \mathbf{u}_{i-1} + \Delta\mathbf{u}$ 
11:   $\mathbf{t}_i = \mathbf{t}_{i-1} + \Delta\mathbf{t}$ 
12: end for
```

5.3 Final remarks

The formulation and numerical strategy presented in this chapter provide a consistent framework for the analysis of contact problems within the Boundary Element Method. By combining the boundary-only formulation with an efficient iterative solution procedure and a robust contact mode decision algorithm, the proposed approach is capable of accurately capturing nonlinear contact behavior, including frictional effects and separation.

This framework serves as the basis for the numerical applications presented in the next chapter, where the performance and accuracy of the proposed methodology are evaluated.

Chapter 6

Numerical Results and Validation

This chapter presents and discusses the results obtained from the formulation and numerical model developed in the previous chapters. The objective is to assess the performance of the proposed method and verify its accuracy in the analysis of different cases of anisotropic elasticity and contact mechanics.

The first case examines an orthotropic rotating disk, with the aim of verifying the correct application of the centrifugal load and to compare the numerical results with the available analytical solutions. This analysis also serves to validate the implementation of the subregion technique within the boundary element formulation for the analysis of centrifugal load problems. Next, a flat-ended punch contact problem involving orthotropic materials is analyzed to evaluate the behavior of the model under contact conditions. The method is then validated through a fretting fatigue case, comparing the obtained results with reference data from the literature. Finally, the proposed formulation is applied to the analysis of a turbine blade dovetail joint, highlighting its capability to handle complex geometries and realistic loading conditions.

6.1 Orthotropic rotating disk

Consider an orthotropic rotating disk of radius R , rotating with angular velocity ω , around its center, and an origin coordinate system at the center of the disk and whose axes directions coincide with the directions of the axes of the greater and lower stiffness of the material, according to Fig. 6.1.

Because of the symmetry, only a quarter of the disk was analyzed with the boundary conditions shown in Figure 6.1. The geometry and material properties can be seen in Table 6.1.

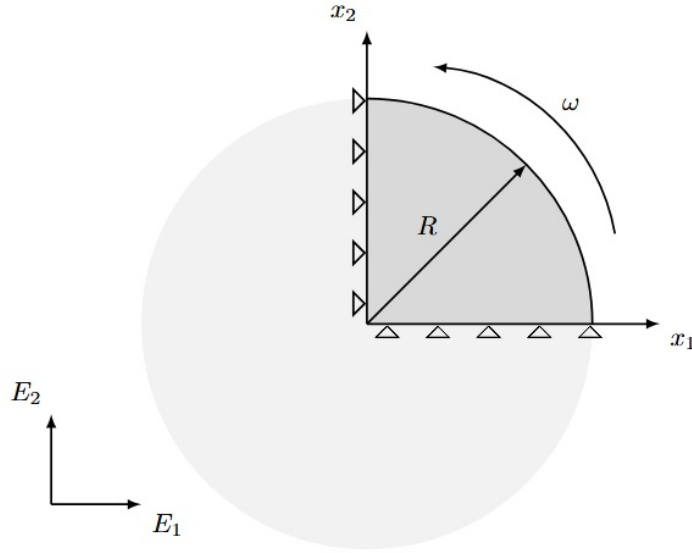


Figure 6.1: Representation of a quarter of the orthotropic rotating disk.

Table 6.1: Geometry and material properties - rotating disk.

Properties	Symbol	Unit
Radius	R	1 m
Density	ρ	1 kg/m ³
Constant Angular Velocity	ω	20 rad/s
Young's Modulus	E_1	17.24 GPa
Young's Modulus	E_2	48.26 GPa
Transverse Elasticity	G_{12}	6.89 GPa
Poisson's ratio	ν_{12}	0.29

The body forces p_i acting on a point with coordinates x_i in the (x_1, x_2) plane, induced by the constant angular velocity ω , are given by Eq. (6.1):

$$p_i = \rho\omega^2 x_i \quad (6.1)$$

According to (CHANG, 1974), the analytical stress components developed in the rotating disk are then expressed as:

$$\begin{aligned}
\sigma_1 &= \frac{1}{2}\rho\omega^2 \{ K(x_1^2 + 3x_2^2 - R^2) + [R^2 - (x_1^2 + x_2^2)] \} \\
\sigma_2 &= \frac{1}{2}\rho\omega^2 \{ K(3x_1^2 + x_2^2 - R^2) + [R^2 - (x_1^2 + x_2^2)] \} \\
\sigma_6 &= \rho\omega^2 K x_1 x_2 \\
\sigma_4 &= \sigma_5 = 0
\end{aligned} \quad (6.2)$$

where K is written in terms of the coefficients of the compliance matrix (inverse of the stiffness matrix), in the form:

$$K = \frac{a_{11} + 2a_{12} + a_{22}}{3a_{11} + 2a_{12} + a_{66} + 3a_{22}} \quad (6.3)$$

In the case of plane strain state, we have:

$$\sigma_3 = -\frac{1}{a_{33}(a_{13}\sigma_1 + a_{23}\sigma_2)} \quad (6.4)$$

and in case of plane stress state:

$$\sigma_3 = 0 \quad (6.5)$$

The components of the displacement vector are given by the following:

$$\begin{aligned} u_1 &= \frac{a_{11}\rho\omega^2 x_1}{2} \left\{ K \left(\frac{x_1^2}{3} + 3x_2^2 - R^2 \right) + \left[R^2 - \left(\frac{x_1^2}{3} + x_2^2 \right) \right] \right\} + \\ &\quad \frac{a_{12}\rho\omega^2 x_1}{2} \left\{ K (x_1^2 + x_2^2 - R^2) + \left[R^2 - \left(\frac{x_1^2}{3} + x_2^2 \right) \right] \right\} \\ u_2 &= \frac{a_{12}\rho\omega^2 x_2}{2} \left\{ K (x_1^2 + x_2^2 - R^2) + \left[R^2 - \left(x_1^2 + \frac{x_2^2}{3} \right) \right] \right\} + \\ &\quad \frac{a_{22}\rho\omega^2 x_2}{2} \left\{ K \left(3x_1^2 + \frac{x_2^2}{3} - R^2 \right) + \left[R^2 - \left(x_1^2 + \frac{x_2^2}{3} \right) \right] \right\} \end{aligned} \quad (6.6)$$

the radial displacement being given by:

$$u_r = u_1^2 + u_2^2 \quad (6.7)$$

If the results are transformed into polar coordinates, the components of the stress tensor corresponding to the rotating disk are given by Chang (1974):

$$\begin{aligned} \sigma_r &= \frac{1}{2}\rho\omega^2(1 - K)(R^2 - r^2) \\ \sigma_t &= \frac{1}{2}\rho\omega^2[K(3r^2 - R^2) + (R^2 - r^2)] \\ \sigma_{rt} &= 0 \end{aligned} \quad (6.8)$$

where σ_r is the radial component of the stress tensor, σ_t is the tangential component, and σ_{rt} is the shear component.

Based on the symmetry that the disk in Figure 6.1 presents, the problem can then be simulated considering a quarter of the disk, as shown in Figure 6.2.

Fig. 6.2 shows the discretized mesh. The single region case a) has 108 discontinuous quadratic elements (36 elements per segment). The subregion case b) contains 126 discontinuous quadratic elements, with 18 elements per segment. This figure also shows the boundary

conditions that are divided into restricted displacements in the x direction on the vertical edge, restricted displacements in the y direction on the horizontal edge and a free segment ($t_x = t_y = 0$) on the circular arc.

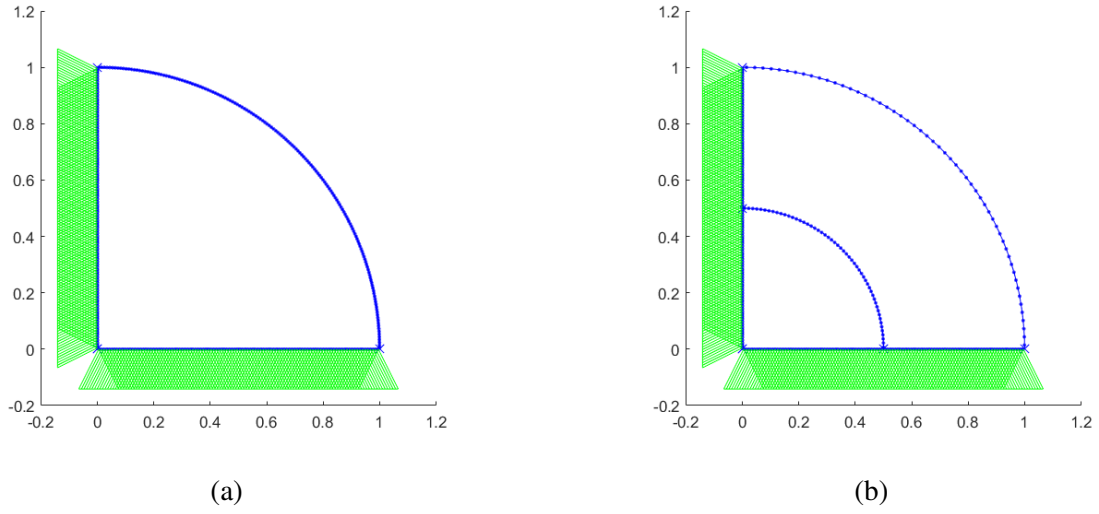


Figure 6.2: Boundary conditions and discretization employed in the analysis: a) BEM mesh; b) BEM mesh subregion.

Figure 6.3 shows the total displacement field obtained using the Boundary Element Method (BEM) for the analyzed semicircular domain. The results correspond to the model without subregions, in which the entire domain is treated as a single continuous body.

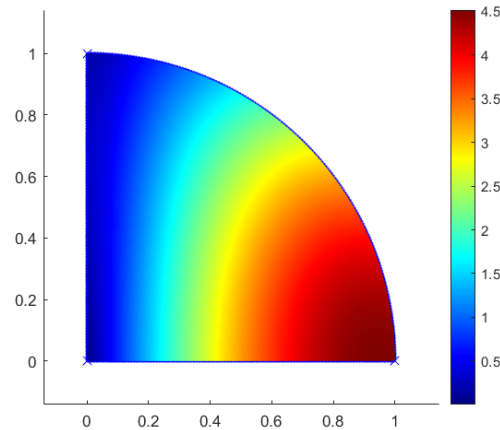
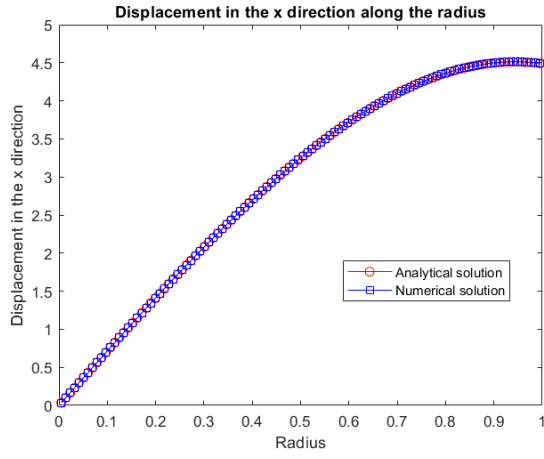
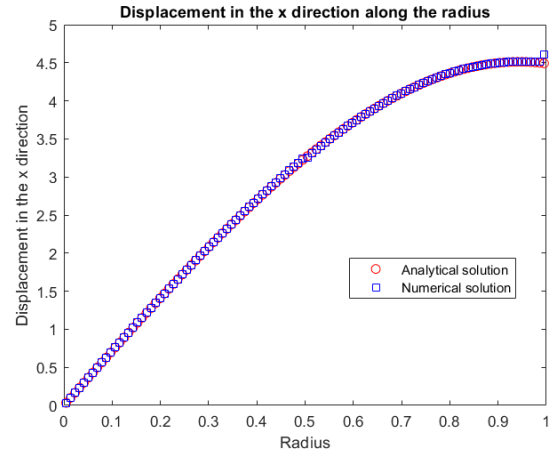


Figure 6.3: Total displacement distribution from the BEM analysis for the model without subregions. The color map indicates the total displacement magnitude in millimeters.

Figure 6.4a presents the numerical results obtained using the orthotropic formulation for displacements in the direction x along the horizontal boundary, compared to the corresponding analytical solution. Similarly, Figure 6.4b shows the same comparison, but considering the simulation with subregions. In both cases, excellent agreement is observed between the analytical and numerical results.



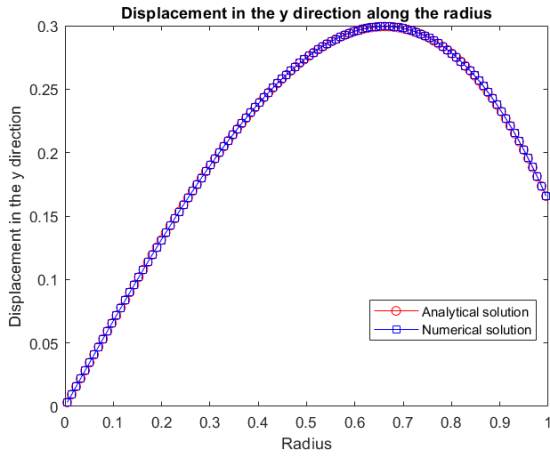
(a)



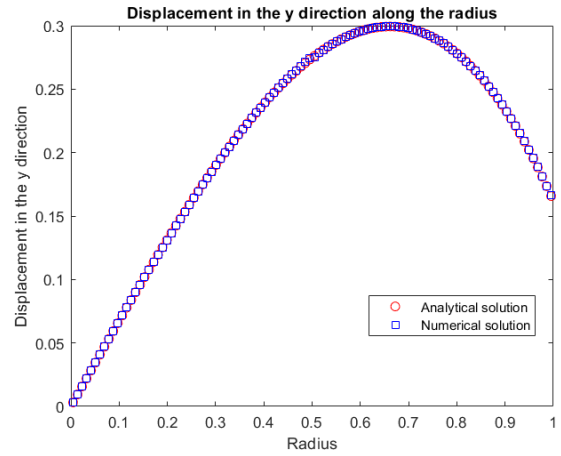
(b)

Figure 6.4: Displacement of the horizontal edge in the x direction in millimeters: a) without subregion; b) with subregion.

Figure 6.5a presents the numerical results for the displacements of direction y along the vertical boundary, calculated using the orthotropic formulation, compared to the analytical solution. Figure 6.5b provides the corresponding results obtained from the subregion based simulation. In both cases, the numerical simulation shows excellent agreement with the analytical reference, confirming the accuracy of the proposed approach.



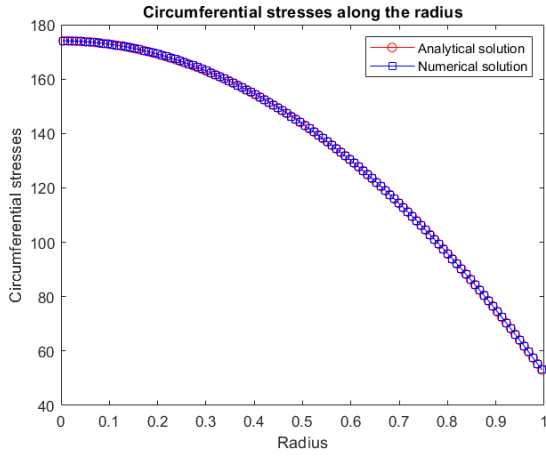
(a)



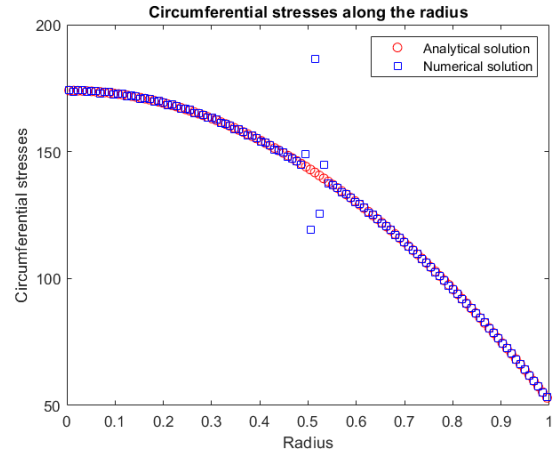
(b)

Figure 6.5: Displacement of the vertical edge in the y direction in millimeters: a) without subregion; b) with subregion.

Figures 6.6a and 6.6b present a comparison between the analytical and numerical results for the circumferential stress components, without and with subregion decomposition, respectively, evaluated along the vertical edge. In both cases, excellent agreement is observed between the analytical and numerical results.



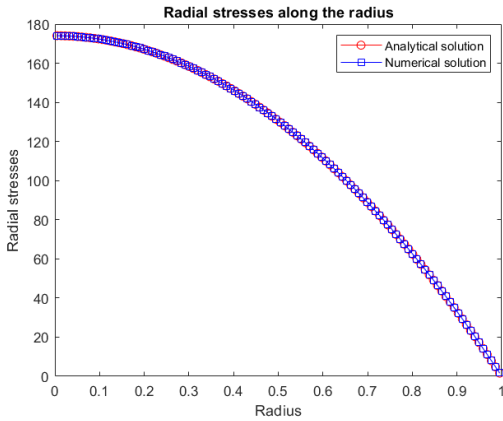
(a)



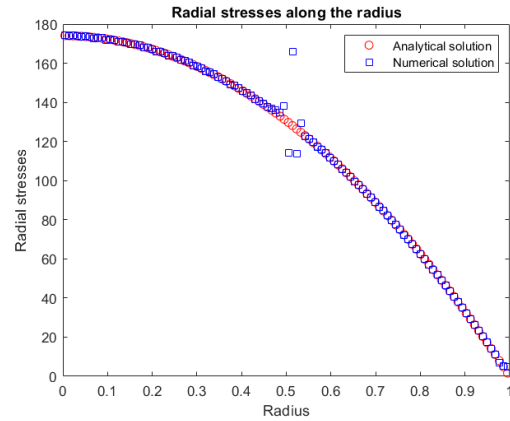
(b)

Figure 6.6: Circumferential stresses along the vertical edge in MPa: a) without subregion; b) with subregion.

Figures 6.7a and 6.7b present, respectively, the comparison between analytical and numerical results for radial stresses without and with subregion decomposition, evaluated along the horizontal edge. In the analyzes without subregions, the agreement is better than with subregion. In both cases, the agreement with analytical results can be enhanced with the increasing in the number of boundary elements (mesh refining) and with the increase in the number of integration points. This is shown in Figure 6.8 where the number of elements increases to 126 boundary elements and the number of integration points increases to 20, both are the double of the amount used to obtain Figure 6.7b.



(a)



(b)

Figure 6.7: Radial stresses along the horizontal edge in MPa: a) without subregion; b) with subregion.

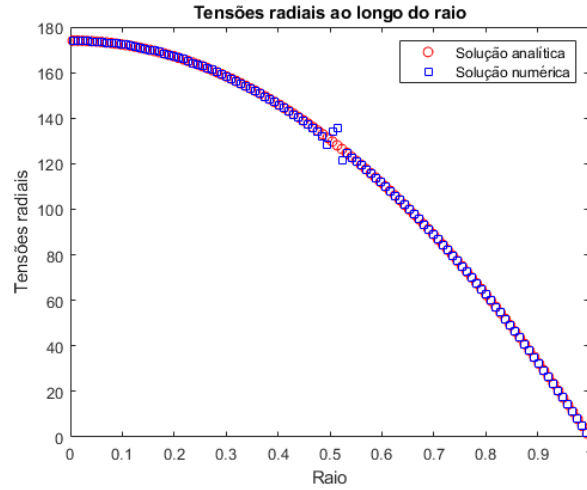


Figure 6.8: Radial stresses along the horizontal edge in MPa with subregion (126 discontinuous quadratic elements and 20 integration points).

To demonstrate how the results converge to the analytical solutions, an analysis was performed with different meshes and the results for displacements in the x and y directions, radial and circumferential stress components, were compared with the numerical results.

In order to see the influence of orthotropy in the results, the same problem is analyzed considering an isotropic material with the following properties: $E_1 = E_2 = 48.26$ Pa, $G_{12} = E_1/[2(1 + \nu_{12})]$. All other properties and dimensions are the same as the orthotropic disk that has been analyzed already. The results for displacements are shown in Figure 6.9 and for stresses are shown in Figure 6.10.

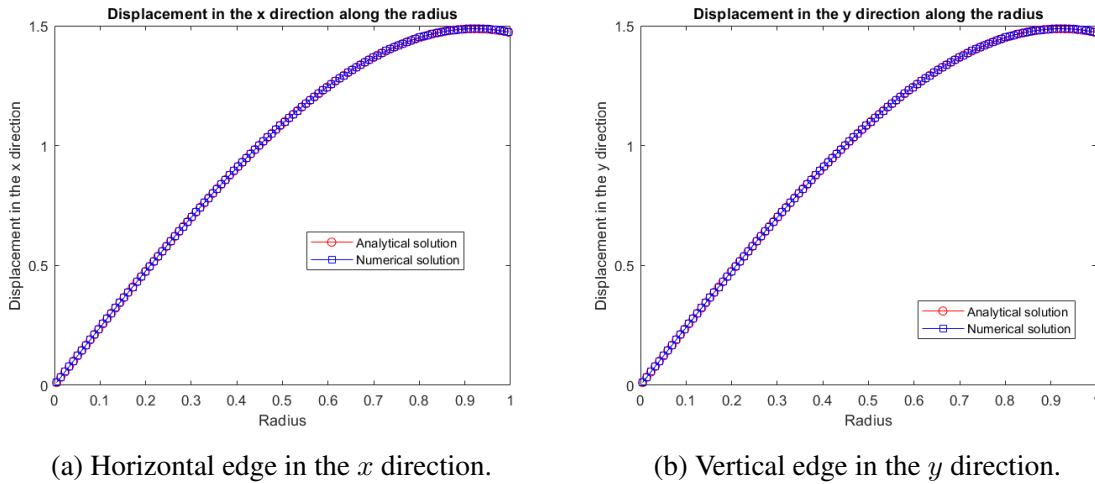


Figure 6.9: Numerical and analytical displacement curves obtained for the limiting isotropic material case (in millimeters).

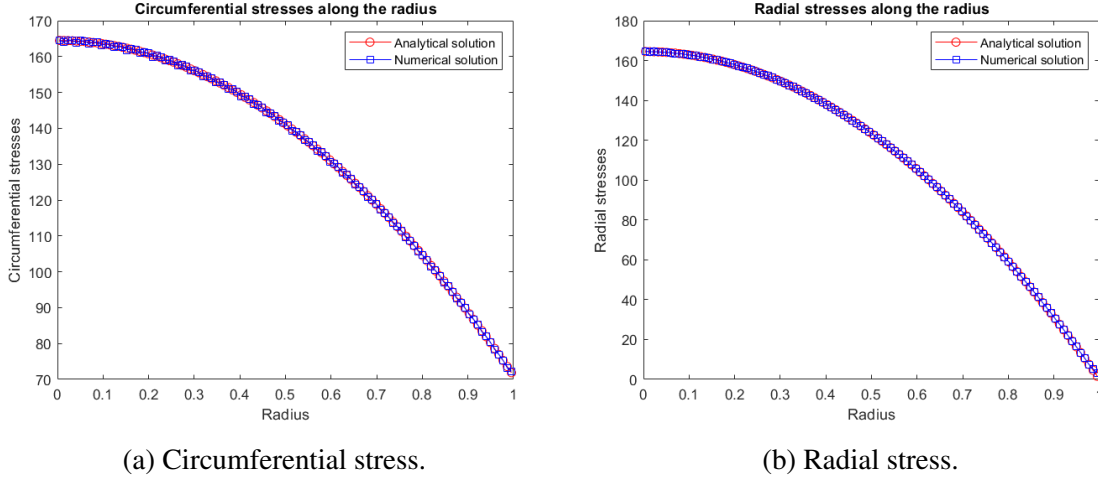


Figure 6.10: Comparison of circumferential and radial stress distributions along the radius for the isotropic disk (in MPa).

As illustrated in Figures 6.9 and 6.10, the numerical results calculated by the proposed formulation of the boundary element are in excellent agreement with the analytical solutions for the isotropic case. Due to the inherent material symmetry present in isotropic media ($E_1 = E_2$), the displacement profiles along both the horizontal and vertical radii exhibit identical maximum magnitudes at the outer edge ($r = 1$ m). This contrasts significantly with the orthotropic case (Figure 6.4a), where the higher stiffness along the x_2 -direction suppresses displacements on the vertical edge compared to the more compliant horizontal direction.

Furthermore, the stress distributions depicted in Figure 6.10 also reflect this physical consistency, where the circumferential and radial stress curves behave symmetrically and perfectly match the classical analytical solutions for isotropic rotating structures. This behavior reveals a major influence of the material configuration compared to the orthotropic scenario shown in Figure 6.6a and Figure 6.7a. In the orthotropic disk (Figures 6.6a and 6.7a), structural anisotropy leads to a distinct mismatch between the stress components, particularly where the circumferential stress levels are altered by the directional elastic constants. In contrast, the isotropic curves (Figure 6.10) demonstrate a perfectly balanced and identical decay profile for both radial and circumferential components toward the outer boundary.

To conclude this analysis, the successful capture of both orthotropic structural mismatch and isotropic limiting behavior demonstrates the high accuracy, versatility, and robustness of the developed BEM framework. The formulation proves to be completely stable and invariant under constitutional material transitions, making it highly reliable for handling contact mechanics coupled with domain inertial fields under complex material descriptions.

The percentage errors (Mean Relative Error - MRE) at each node on the specified edges were calculated according to the following equation:

$$MRE = \left| \frac{u_a - u_n}{u_a} \right| \times 100 \quad (6.9)$$

where MRE is the percentage error, u_a is the analytical value of the variable, and u_n is the numerical value of the variable.

Figures 6.11 and Figure 6.12 show the percentage error as a function of the radius for different boundary discretizations and the number of Gauss integration points in the boundary element method, in the x -direction. Figure 6.11 corresponds to the standard formulation, while Figure 6.12 includes the use of a subregion.

In both cases, the error is higher near the origin and decreases with increasing radius due to the singular behavior of the fundamental solution. Refinement of the boundary mesh from 56 to 112 elements produces a moderate reduction in the error. In contrast, increasing the number of Gauss integration points from $n_{gauss} = 10$ to 20 leads to a more pronounced improvement in accuracy.

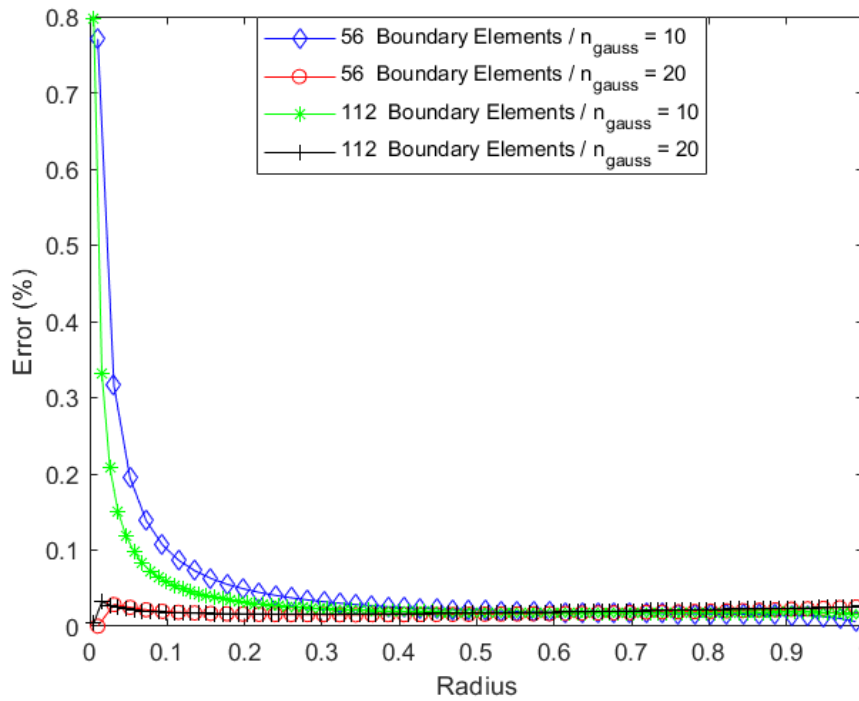


Figure 6.11: Percentage error along the x -direction for different boundary discretizations and Gauss integration points without subregion.

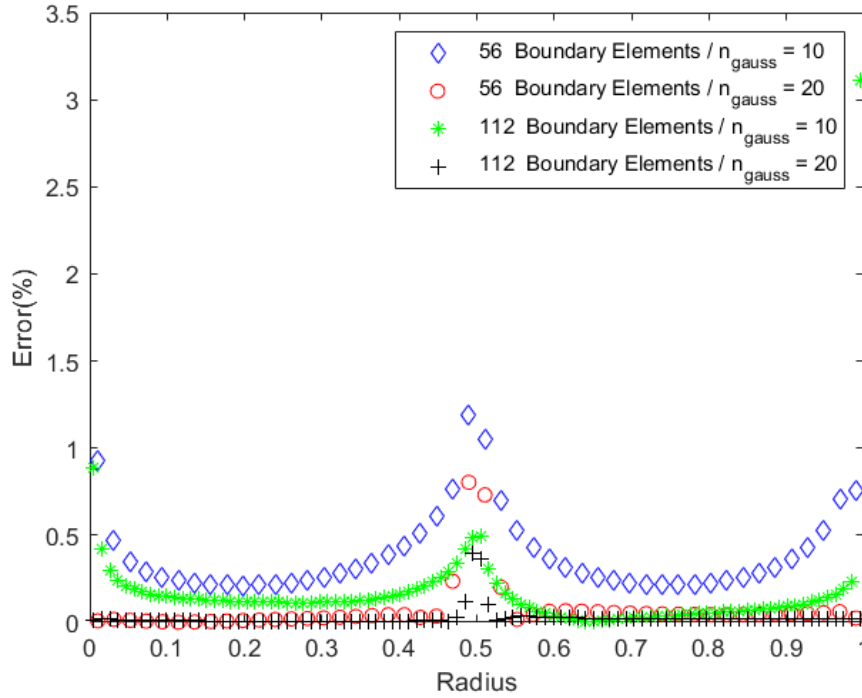


Figure 6.12: Percentage error along the x -direction for different boundary discretizations and Gauss integration points with subregion.

The introduction of a subregion Figure 6.12 further enhances accuracy, particularly close to singularities, leading to a more uniform and lower overall error distribution throughout the domain.

Figures 6.13 and 6.14 present the percentage error along the y -direction. Compared to the x -direction, the error along y is generally smaller and more evenly distributed. This behavior occurs because, in this problem, the variation of the solution along the y -axis is less affected by the singular behavior of the fundamental solution and by the boundary conditions, which act predominantly in the x -direction. As a result, numerical integration becomes more stable and the influence of local singularities is reduced.

Without subregions (Figure 6.13), the curves still exhibit small peaks near the boundaries, mainly for coarser meshes and lower values of n_{gauss} , but their magnitudes are noticeably smaller than those observed along the x -axis. When the subregion technique is applied (Figure 6.14), these peaks are practically eliminated, and the error remains below 0.5% throughout the domain.

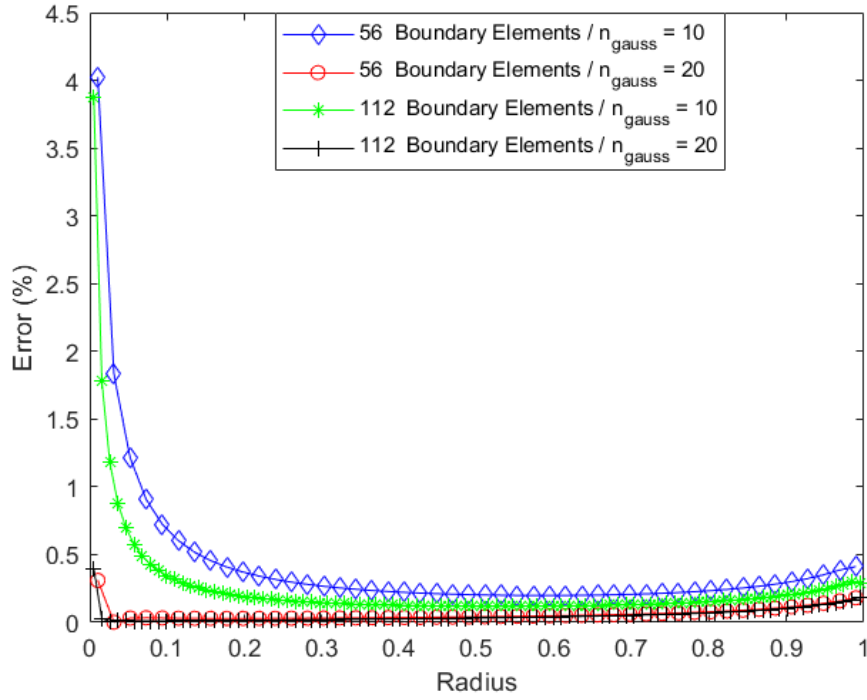


Figure 6.13: Percentage error along the y -direction for different boundary discretizations and Gauss integration points without subregion.

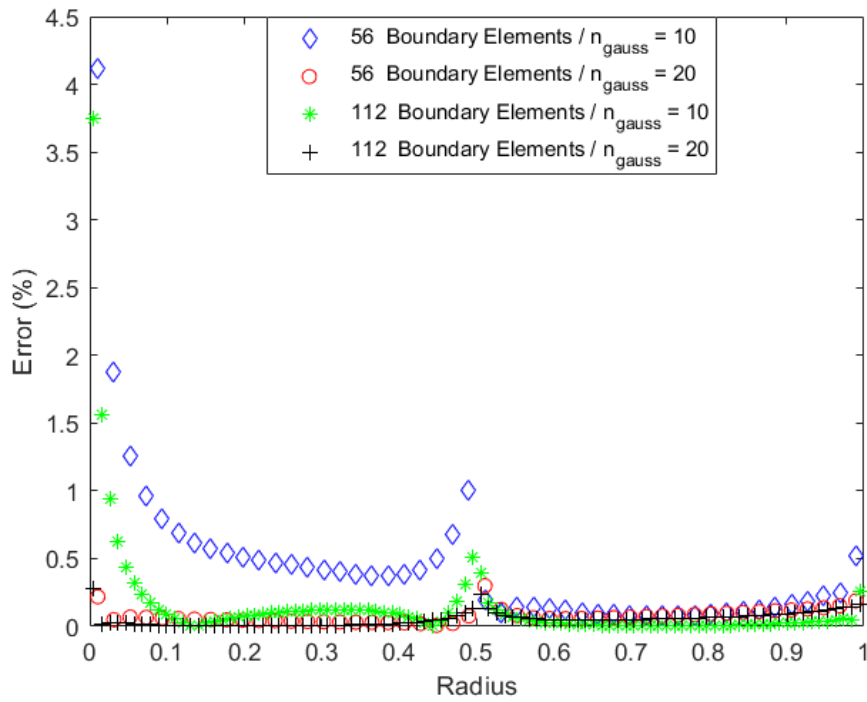


Figure 6.14: Percentage error along the y -direction for different boundary discretizations and Gauss integration points with subregion.

In summary, the results along the y -direction confirm the overall stability of the formulation, indicating that the most significant numerical effects occur along the x -axis, where the

influence of the singularity is stronger.

6.2 Flat-ended punch contact problem using orthotropic material

In this second example, an orthotropic flat-ended punch problem is analyzed to verify the implementation of the boundary element formulation for a contact interface whose contact region is known *a priori*. Therefore, this example is intended to assess the accuracy of the numerical implementation rather than to validate the contact detection algorithm. The case considered involves a flat-ended punch in contact with an elastic foundation, following the geometry and loading conditions illustrated in Fig. 6.15. This problem was originally presented by (NGUYEN; HWU, 2019).

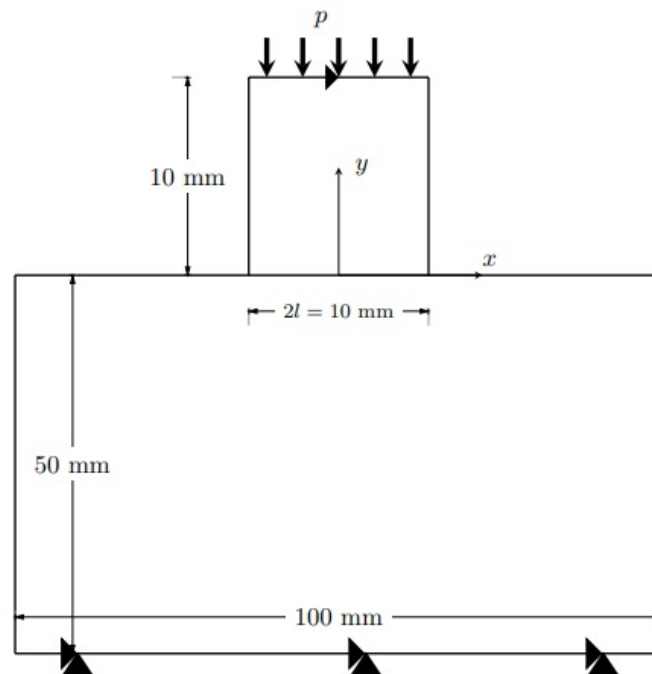


Figure 6.15: A flat-ended punch in contact with an elastic foundation (NGUYEN; HWU, 2019).

The uniform vertical load $p = 1 \text{ N/mm}^2$ is applied to the top of the punch. Both geometries are made up of an orthotropic material whose properties, expressed by elastic stiffness C_{ij} , are defined as: $C_{11} = 147.34 \text{ GPa}$, $C_{22} = C_{33} = 10.78 \text{ GPa}$, $C_{23} = 3.32 \text{ GPa}$, $C_{12} = C_{13} = 4.23 \text{ GPa}$, $C_{44} = C_{55} = C_{66} = 4.1 \text{ GPa}$.

Figure 6.16 shows the boundary element mesh and boundary conditions used in the discretization of the flat-end punch in contact with an elastic foundation. The mesh used consisted of 60 discontinuous quadratic boundary elements on each edge of the contact region (180 nodes for each edge).

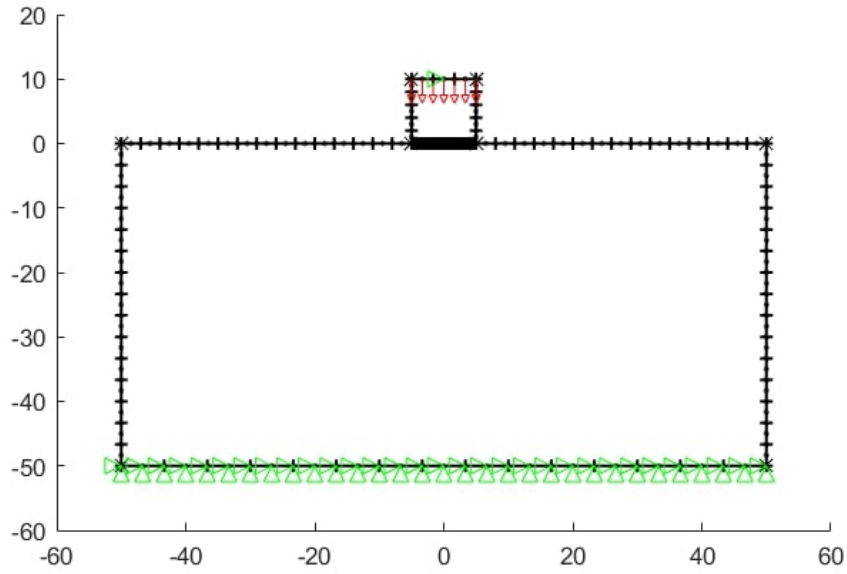


Figure 6.16: Detailed representation of geometry with boundary conditions.

In Figure 6.17, we compare the normalized tangential tension τ_{nt} and the normalized normal tension σ_n for different friction coefficients f with the results obtained by (NGUYEN; HWU, 2019).

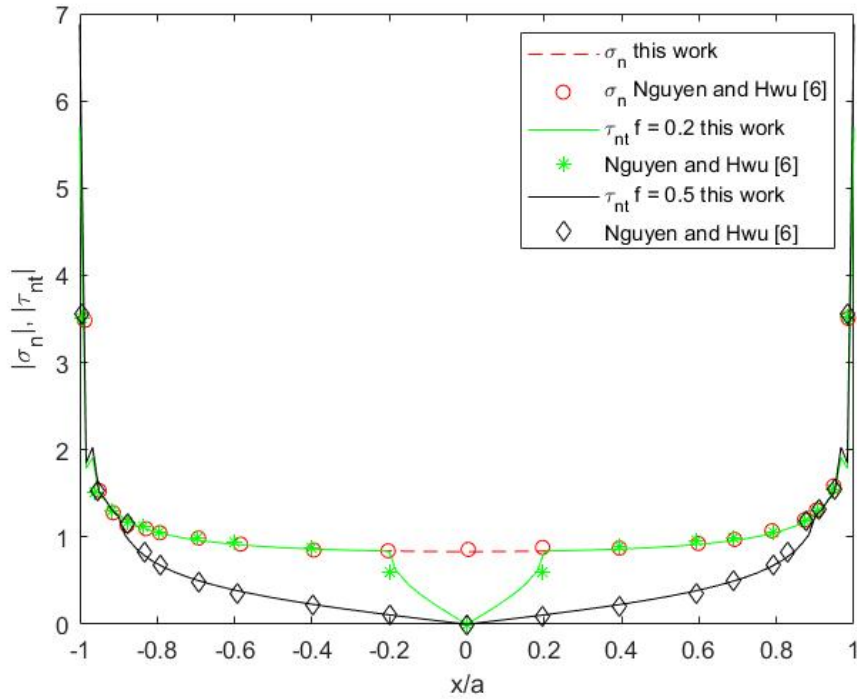


Figure 6.17: Contact tractions in the flat-ended elastic punch for different friction coefficient.

It can be seen that when the friction coefficient increases, the stick zone increases and the

slip zone decreases, and that the results obtained showed good agreement with (NGUYEN; HWU, 2019).

6.3 Validation for a case of fretting fatigue problem

In this example, an isotropic contact problem with fretting conditions is analyzed in order to assess the capacity of the code to treat load-varying conditions. The problem has been chosen because it is commonly used for fretting fatigue experiments, as shown by (LOYOLA, 2022). Furthermore, although it is an isotropic problem, there are analytical solutions for normal and tangential stresses in the contact zone that can be used to validate the proposed orthotropic formulation, considering the isotropic problem as a special case of the orthotropic one.

The geometry of the problem is shown in Figure 6.18. In this problem, the tangential Q and bulk B loads are in phase with each other. Geometry, material and load properties are presented in Table 6.2.

Table 6.2: Bulk stress problem - geometric and material properties.

Property	Symbol	Value
Radius	R	70 mm
Width	w	6.5 mm
Young's Modulus	E	73.4 GPa
Poisson's ratio	ν	0.33
Pressure	P	100 N/mm
Friction coefficient	f	0.3
Bulk load	B	15 N/mm
Tangential load	Q	15 N/mm

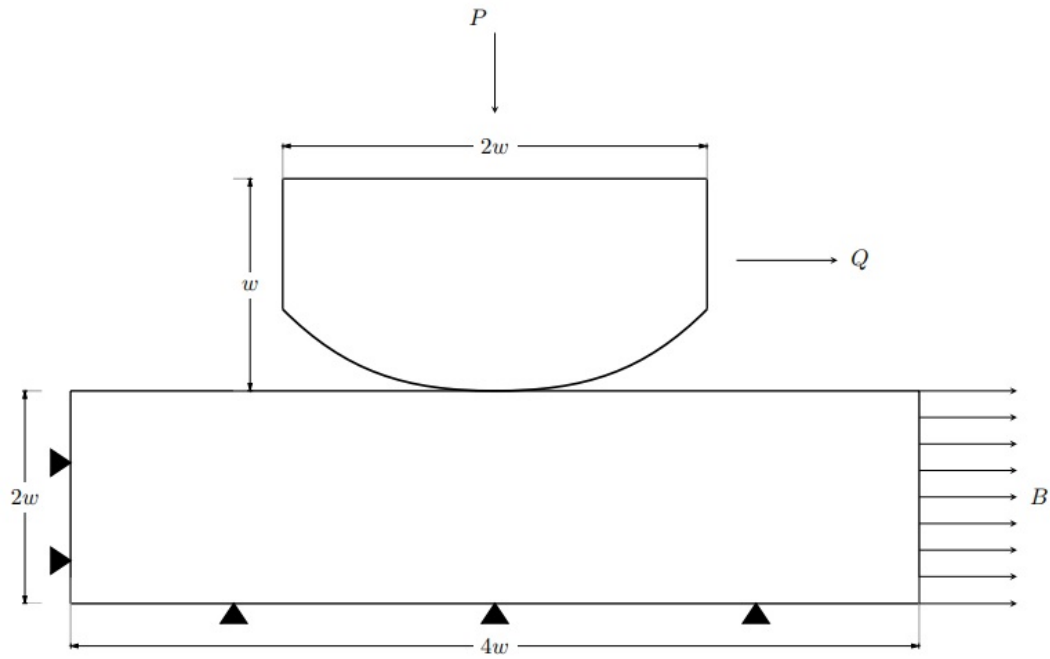


Figure 6.18: Bulk stress problem geometry (LOYOLA, 2022).

A depiction of the loading scheme is shown in Figure 6.19. Firstly, a normal load is applied. Then, this load is maintained constant, while tangential loads are applied cyclically. In Figure 6.19, point *A* is when the normal load P reaches its final value, while the point *B* is the moment when the maximum tangential loads are applied. Afterward, tangential loads begin to decrease, passing through zero at the point *C* and further decreasing. The point *D* is when the tangential loads are in their minimum values and, finally, the point *E* is when the tangential loads reach zero again. A complete description of this problem with results for all points *A*, *B*, *C*, *D*, and *E* can be found in Loyola, Doca, Campos, Trevelyan e Albuquerque (2022), Cavalcante, Shaterzadeh, Sollero, Albuquerque e Doca (2016).

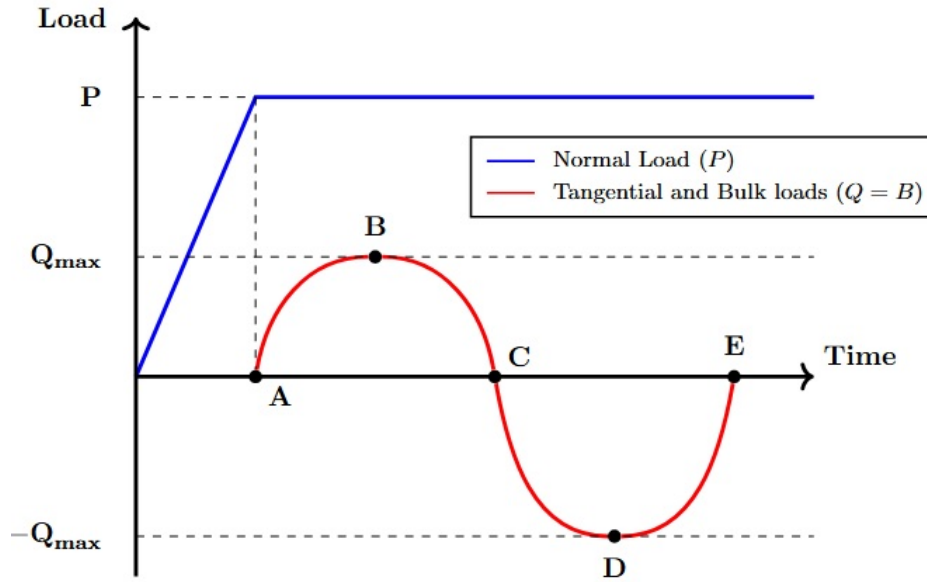


Figure 6.19: Loading scheme containing five steps (A–E). The maximum tangential load is applied at point B, where the tangential load Q is equal to the prescribed bulk load B . (LOYOLA, 2022).

To access the ability of the proposed formulation to model iterative loading problems, the results are shown at the second loading point B (Figure 6.19), where the maximum tangential loads are applied. The material is considered isotropic and the results are compared with the analytical solution available in the literature and the finite element results presented by Loyola, Doca, Campos, Trevelyan e Albuquerque (2022).

The geometry has a finer mesh in the segments that may be in contact as shown in Figures 6.20, 6.21, and 6.22. Discretization was performed using discontinuous quadratic elements, resulting in a boundary element mesh with 60 elements in the contact region (180 nodes).

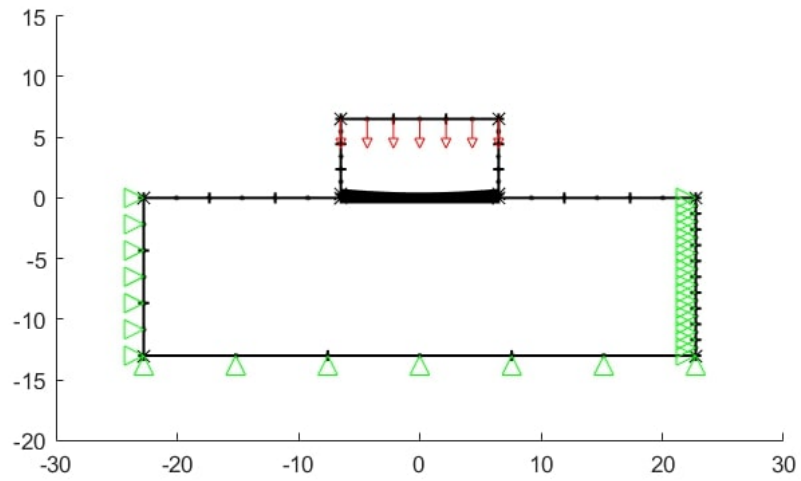


Figure 6.20: Detailed representation of boundary conditions (vertical load application, before point *A*).

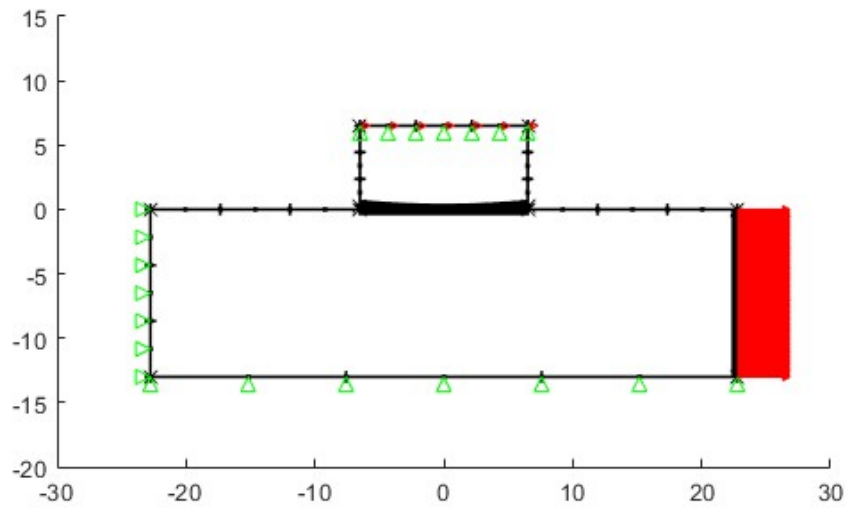


Figure 6.21: Detailed representation of geometry with boundary conditions (horizontal load application, between points *A* and *B*, and *D* and *E*).

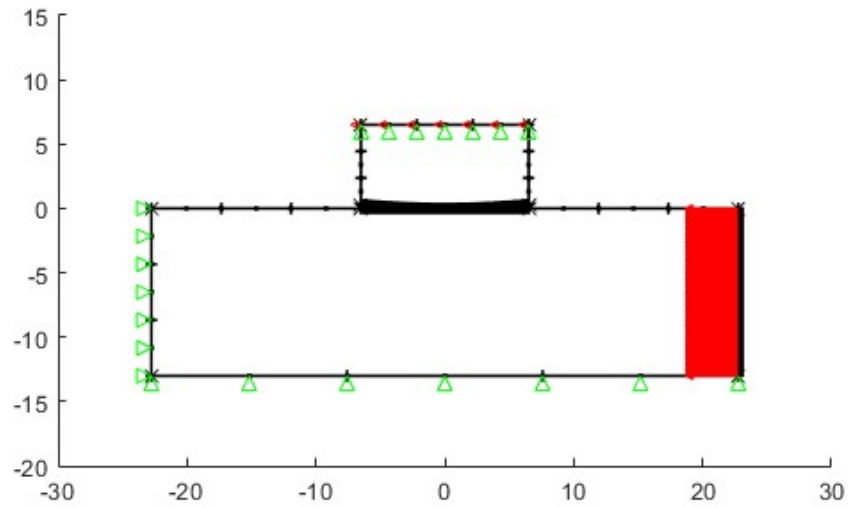


Figure 6.22: Detailed representation of geometry with boundary conditions (horizontal load application, between points *B* and *D*).

Figure 6.23 shows the normal stress distribution on the contact surface compared to the analytical solution.

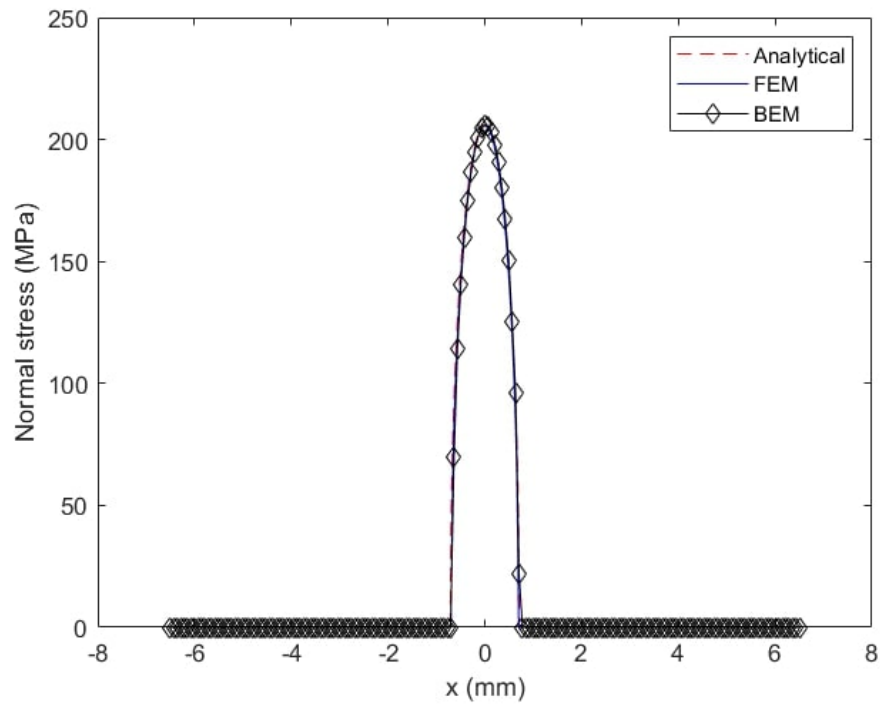


Figure 6.23: Contact pressure distribution.

In Figure 6.24 presents the result of a shear stress distribution on the contact surface compared to the analytical solution and the finite element results presented by Loyola, Doca, Campos, Trevelyan e Albuquerque (2022).

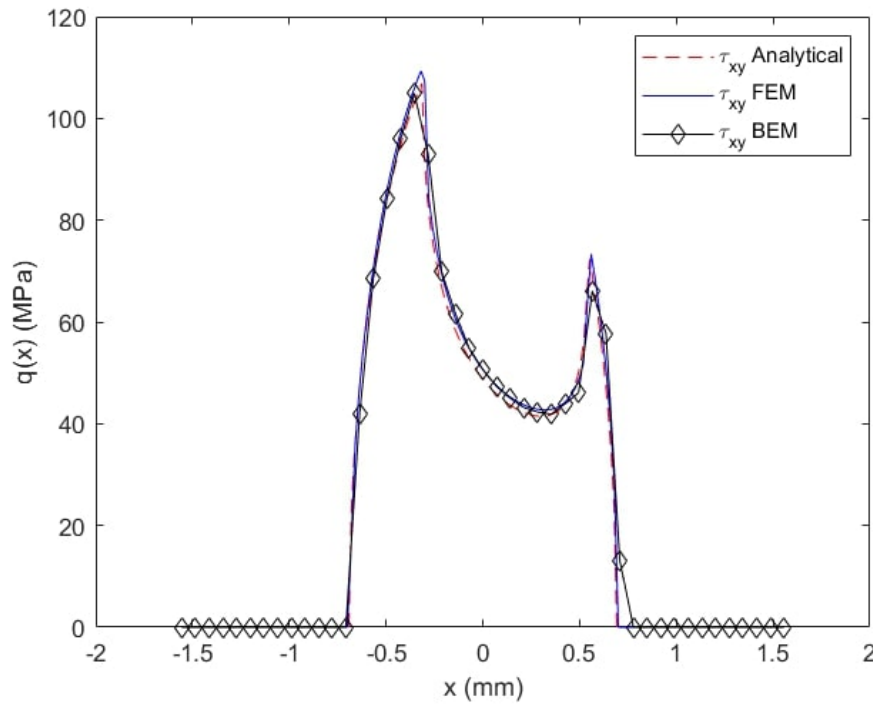


Figure 6.24: Shear traction distribution in presence of the Bulk load.

6.4 Analysis of a dovetail joint in turbine blades

In this example a contact analysis of a dovetail joint on a disk blade junction of an aeronautical engine compressor is carried out. Figure 6.25 presents the disk geometries considered in this study, as well as a representative configuration of the blade–disk assembly interface. The dovetail slots were arranged equidistantly around the perimeter of the disk, with an angular spacing of 30° between each slot. The assembly results in a total of 12 evenly spaced slots ($360^\circ / 30^\circ$), ensuring symmetrical distribution of the blades.

The blade and disk are composed of a single-crystal nickel-based superalloy, MD2 (ZHANG, 2020), a recently developed material that is free of the chemical element Re (Rhenium). As a result, one of its main advantages is the significantly lower cost compared to typical turbine blade alloys. The properties of this alloy can be expressed by the three anisotropic stiffness constants C_{ij} , defined as: $C_{11} = 174.03$ GPa, $C_{12} = 77.38$ GPa, and $C_{44} = 118.52$ GPa.

An additional parameter of significant influence in the dovetail region is the friction coefficient at the blade-disk interface. The friction coefficient used was $f = 0.3$. Furthermore, two other variables were considered: the specific mass, $\rho = 9000$ kg/m³ and the angular

velocity of the disk, $\omega = 104 \text{ rad/s}$.

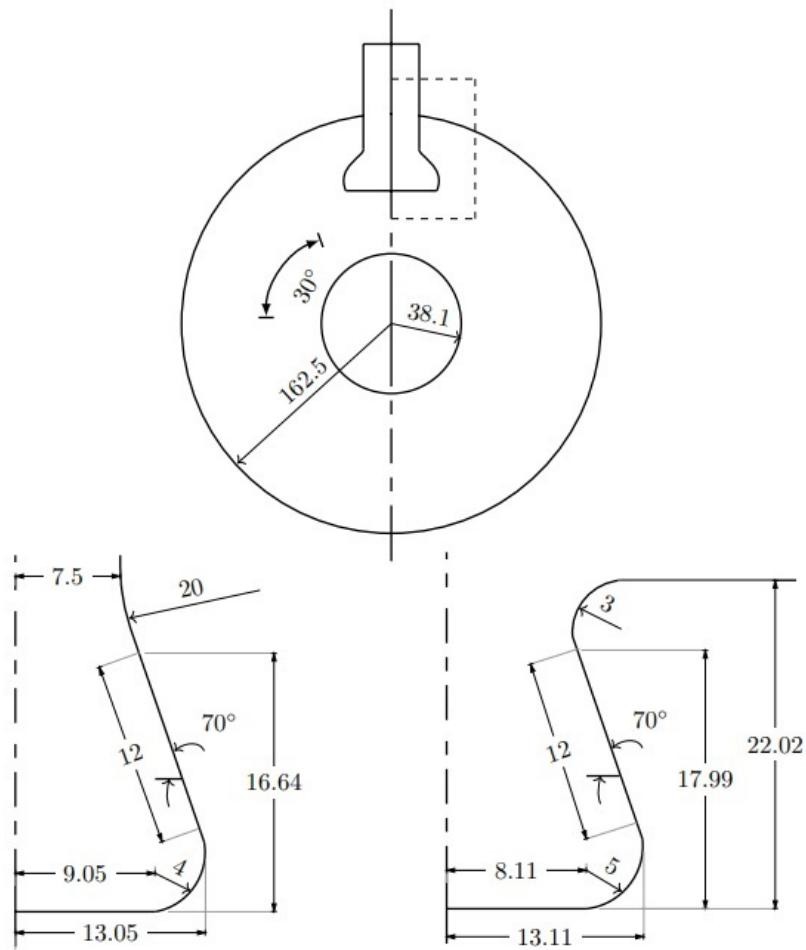


Figure 6.25: Disk and blade dovetail geometry (PAPANIKOS, 1998), dimensions in millimeters.

The cyclic symmetry condition was employed to significantly reduce computational time, in contrast to a full-disk/blade model, which would entail substantially higher computational demands.

Figure 6.26 shows the boundary conditions, with 105 discontinuous quadratic boundary elements (315 nodes). Green triangles represent displacement restriction in the normal direction.

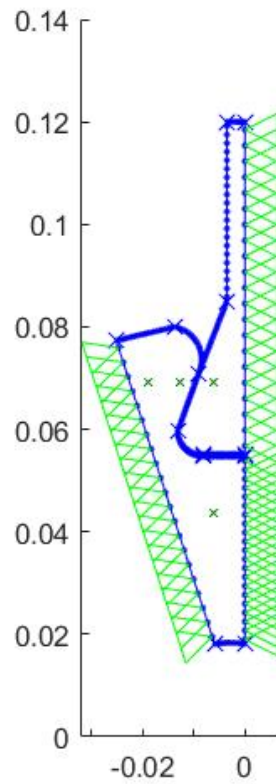


Figure 6.26: Detailed representation of symmetric dovetail with boundary conditions (dimensions in meters).

The deformation of the dovetail is shown in Figure 6.27. In blue is the non-deformed geometry of the dovetail, and in red is the deformation after applying the displacement restrictions and forces.

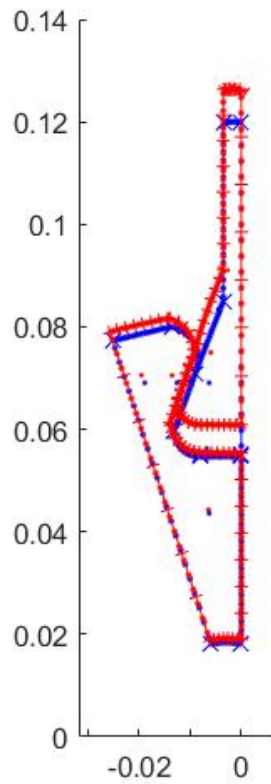


Figure 6.27: Undeformed and deformed configurations of the dovetail mesh.

Figure 6.28 illustrates normal stress and shear stresses in the contact region (shear stress is divided by the friction coefficient). The coincidence of the curves indicates that sliding contact conditions occur throughout the contact region.

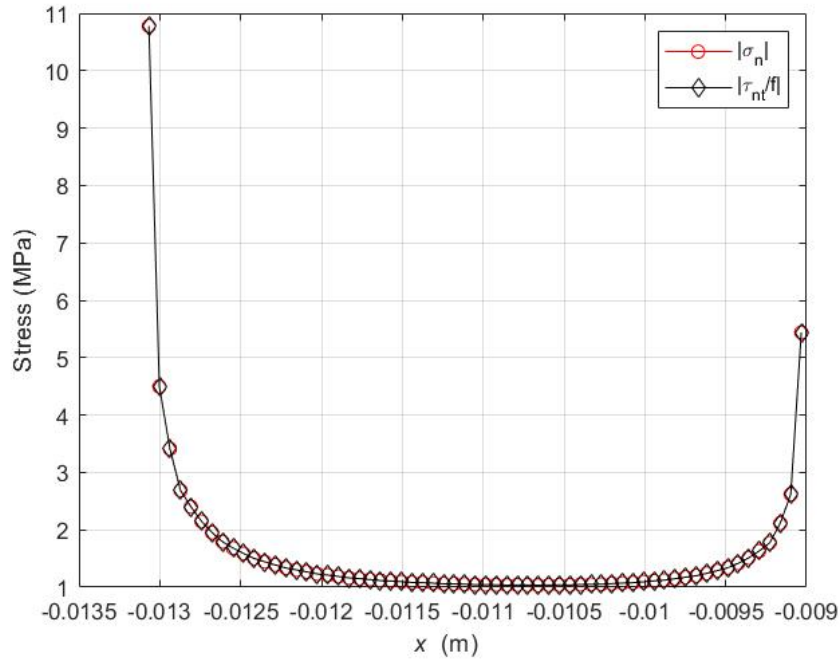


Figure 6.28: Normal stress absolute value in the contact region.

Figure 6.29 shows the comparison of the results for different friction coefficients (0.3, 0.6 and 0.9), using only one mesh (105 discontinuous boundary elements in the contact region). Even for a higher friction coefficient, sliding contact conditions still occur throughout the contact region.

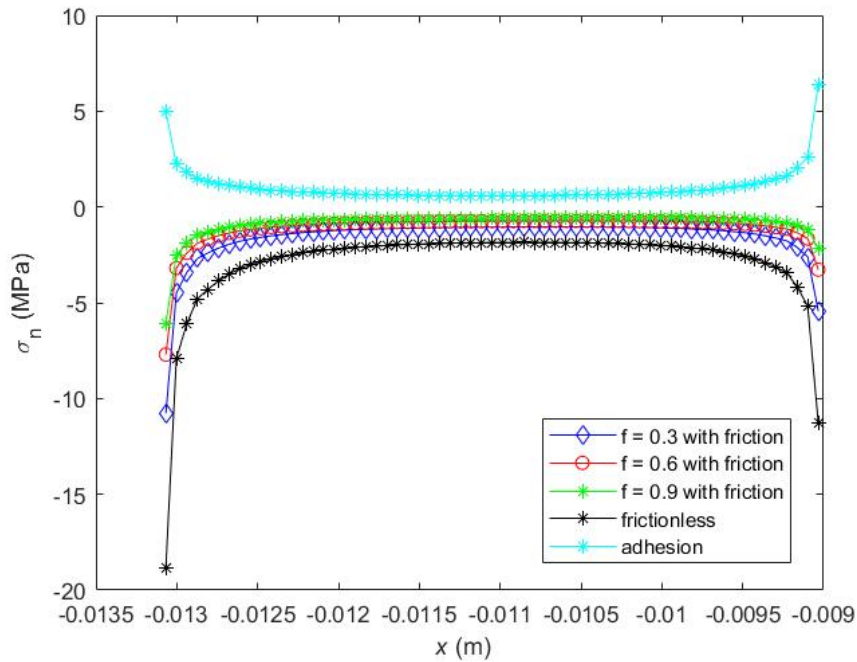


Figure 6.29: Effect of different coefficients of friction on normal stress absolute value.

With an increasing friction coefficient f , the absolute value of normal stress σ_n decreases.

This behavior was expected once the friction coefficient (f) is the ratio between the moduli of tangential and normal stresses in the contact region. This occurs because the frictional force becomes more effective in resisting the applied separation load, reducing the need for large normal forces. In systems such as dovetail joints, where the contact surfaces are inclined, the normal force required to maintain equilibrium is inversely related to the friction coefficient. Therefore, an increase in f allows the same external force to be balanced with a smaller normal force, leading to a lower normal stress in the contact area.

In frictionless conditions, a sharp increase in compressive stresses is observed near the edges of the interface, demonstrating that, in the absence of tangential resistance, the normal load tends to concentrate at the edges of the contact area. This behavior is associated with the absence of shear stress contributions that would otherwise help redistribute contact stresses longitudinally.

However, the adhesion case presents a qualitative modification of the stress field: Tensile peaks appear near the edge, while the central region remains predominantly under moderate compression. This local inversion is attributed to adhesive forces, which introduce attractive components at the edges and disrupt the classical compressive equilibrium expected in purely mechanical contact. This phenomenon can promote the nucleation of microcracks or initiate delamination processes, since the presence of tensile stresses at the edges facilitates cohesive failure and the loss of surface integrity.

Figure 6.30 shows the distribution of the normal stress σ_n along the contact interface for different crystallographic orientations, defined by the angles $\theta = 0^\circ, 30^\circ, 45^\circ$, and 60° . In all cases, a constant friction coefficient of $f = 0.3$ was assumed.

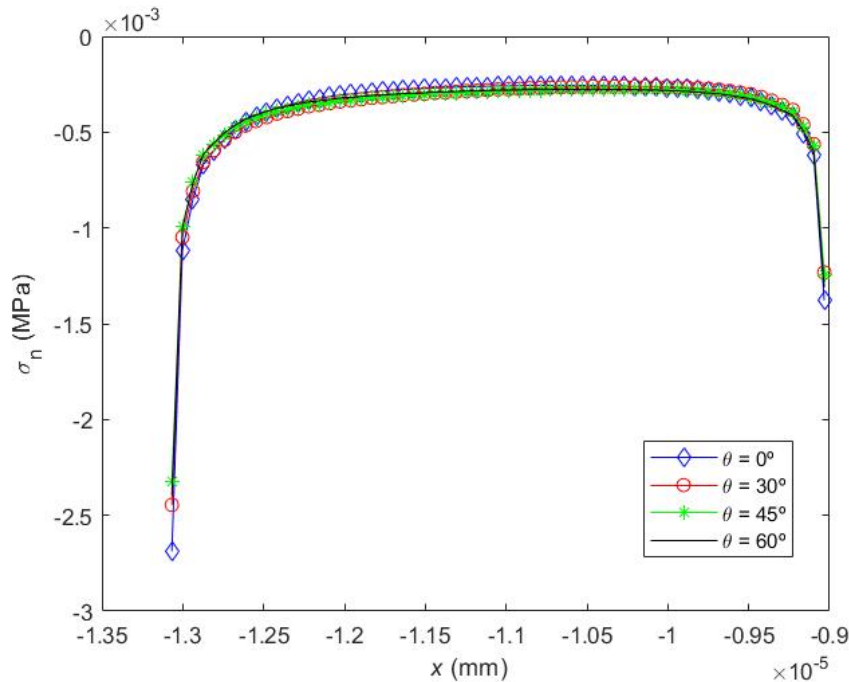


Figure 6.30: Effect of changing different angles ($\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$) on normal stress.

In all cases, the normal stress exhibits a similar pattern: it reaches peak values at the edges of the contact region and remains approximately constant in the central portion. However, small variations in σ_n are observed as the crystal orientation changes. These subtle differences are attributed to the anisotropic nature of the material, meaning that its mechanical response depends on the direction within the crystal lattice.

Figure 6.30 presents an enlarged view of Figure 6.31, which is focused on the center of the contact region. The small influence of crystal orientations on the stress distribution in the contact region was also observed in (FLEURY, 2014). As the crystal orientation changes, so do the preferred planes and directions for stress accommodation. This leads to slight differences in the local stress distribution, even under identical loading conditions. Since BEM accurately captures these local effects, it allows for the detection of such variations across different orientations.

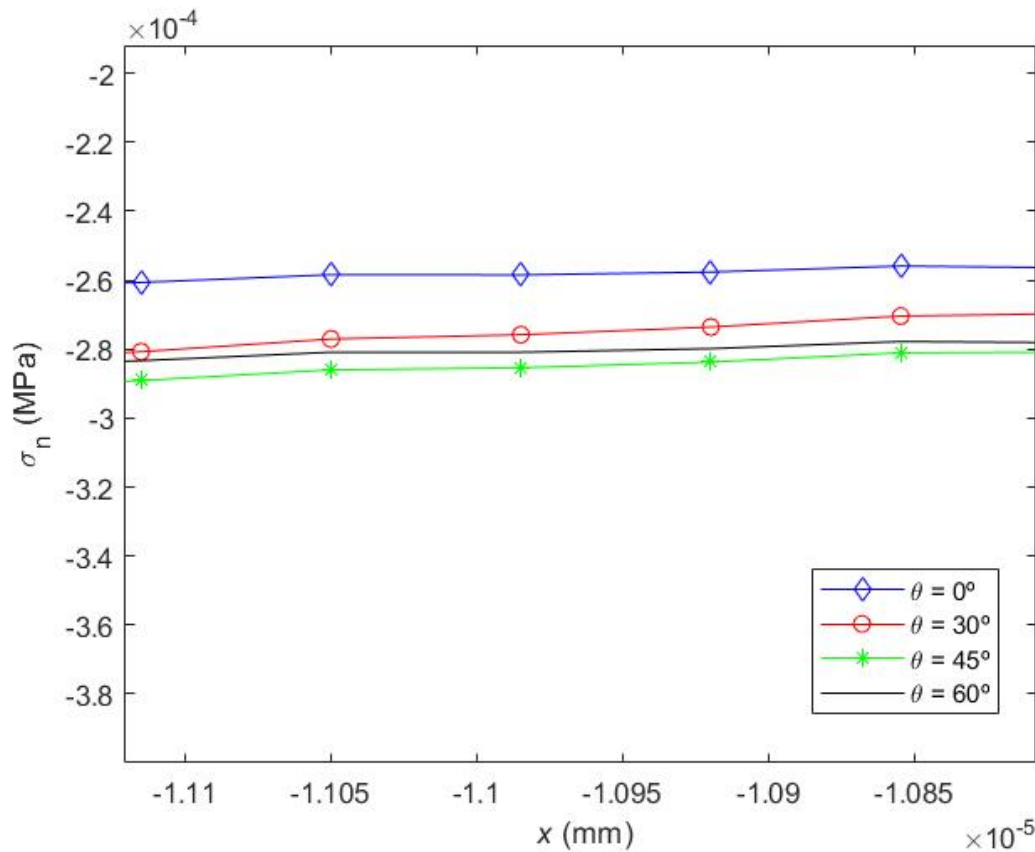


Figure 6.31: Detailed view of normal stress for different θ angles.

However, the curves remain very close to each other, suggesting that the influence of crystallographic orientation on the normal stress is relatively minor within the angular range considered. However, these differences may become more significant in analyzes involving shear stresses, frictional contact, or interfacial damage and fracture mechanisms. These analyzes are outside the scope of this work and will be proposed in future studies.

Chapter 7

Conclusions

This thesis proposed, developed, and validated a robust boundary element formulation for the analysis of frictional contact problems in orthotropic materials under fretting conditions, explicitly incorporating centrifugal body forces. By integrating domain source terms directly through the radial integration method, the present approach avoids the need for domain discretization into cells or internal points, strictly preserving the boundary-only character of the method and leading to simpler, yet highly efficient numerical models.

7.1 Major Findings

The numerical capabilities and precision of the proposed framework were rigorously assessed and confirmed through four representative engineering examples, leading to the following major findings:

- **Effective Nonlinear Coupling:** The mathematical coupling of contact constraints directly onto the boundary element influence matrices proved to be an effective strategy for capturing localized non-linear boundary interactions across diverse conditions (adhesion, sliding, and separation), providing highly accurate results even when utilizing relatively coarse boundary meshes.
- **RIM Validation for Centrifugal Loads:** The application to an orthotropic rotating disk successfully validated the implementation of the Radial Integration Method (RIM) for centrifugal loading, proving that the formulation is general and can be readily extended to other categories of domain forces simply by adjusting the corresponding domain source terms.
- **Mesh and Node Sensitivity:** A comprehensive convergence analysis revealed that variables evaluated directly through the boundary integral equations (displacements) produced exceptionally smooth convergence curves. Conversely, boundary stress components, calculated through shape function derivatives, exhibited a strong dependency

on node positioning, highlighting that central nodes in discontinuous quadratic elements consistently yield superior accuracy compared to boundary nodes.

- **Dovetail Joint Mechanics:** The single-tooth dovetail joint analysis illustrated the potential of the method to address realistic industrial problems with inclined contact interfaces. Parametric studies revealed that both the friction coefficient and the crystallographic orientation significantly influence normal stress and tangential traction distributions, markedly shifting stress concentration peaks, which is highly critical for fretting fatigue life predictions in turbine blades.

7.2 Limitations of the Work

Despite the accuracy, robustness, and numerical efficiency demonstrated by the proposed boundary element formulation, some limitations of the present work must be acknowledged:

- **Two-Dimensional Restraints:** The current framework is restricted to two-dimensional analysis (plane stress and plane strain conditions). Consequently, it cannot capture three-dimensional stress fields, complex spatial geometries, and critical edge effects that typically occur in actual industrial disk-blade attachments.
- **Elastic Material Behavior:** The constitutive modeling assumes ideal linear elastic behavior for the anisotropic medium. This formulation does not account for material nonlinearities, such as crystal plasticity or viscoplastic effects, which often arise under severe contact pressures and high-temperature environments in turbine operations.
- **Static Boundary Profile:** The contact algorithm does not progressively update the boundary profile geometry to account for material loss and surface degradation caused by debris ejection over long cyclic loading histories.

7.3 Recommendations for Future Work

Regarding suggestions for future research, several paths can be taken to expand the scope and impact of the boundary element formulation developed in this work:

- **Fracture Mechanics Integration:** Integration of fracture mechanics capability into the contact algorithm, enabling the simulation of fretting crack initiation and subsequent propagation under mixed-mode conditions in anisotropic media.
- **Thermoelastic Fields:** Incorporating coupled thermal fields into dual reciprocity or radial integration schemes to allow for the analysis of thermoelastic contact problems, capturing the significant influence of frictional heating and thermal gradients on the interface stress state.

- **Material and Surface Non-linearities:** Expanding the framework to account for the behavior of elastoplastic materials, localized surface roughness effects, and time-dependent wear models, such as Archard's law, to progressively update the boundary profile geometry due to material degradation over cyclic loading histories.
- **Three-Dimensional Extension:** Transforming the current two-dimensional formulation into a full three-dimensional boundary element framework, which would represent a major contribution, allowing for the evaluation of critical edge effects and highly complex spatial stress fields in realistic industrial components.

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