Qualitative properties and asymptotic solutions of a class of Volterra-Stieltjes functional integral equations

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Propriedades qualitativas e soluções assintóticas de uma classe de equações integrais funcionais de Volterra-Stieltjes

Tese apresentada ao Programa de Pós-Graduação em Matemática, como parte dos requisitos para obtenção do título de Doutora em Matemática – Equações Diferenciais Funcionais. *VERSÃO REVISADA*

Orientadora: Profa. Dra. Jaqueline Godoy Mesquita
Coorientador: Prof. Dr. Rogelio Grau Acuña

Universidade de Brasília – Brasília
Janeiro de 2026

UNIVERSIDADE DE BRASÍLIA

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Ata Nº: 17

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"Of all the places, I always preferred to be last. The things of this world do not fill our emptiness; this emptiness can only be filled by eternity–by infinity."

(The author)

RESUMO

Nesta tese, estudamos as equações integrais funcionais de Volterra-Stieltjes dadas por

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s)\,\mathrm{d}g(s), \quad t \geqslant \tau_0 \\ x_{\tau_0} = \phi, \end{cases} \tag{1}$$

onde a integral do lado direito é considerada no sentido de Perron-Stieltjes. Provamos critérios de limitação e instabilidade para soluções da equação funcional integral de Volterra-Stieltjes com retardo finito via funcionais de Lyapunov. Também provamos resultados de estabilidade para a equação funcional de Volterra-Stieltjes com retardo infinito via funcionais de Lyapunov. Considerando que as equações dadas por (1) generalizam equações fracionárias, impulsivas e em escalas de tempo, apresentamos os resultados obtidos nesse contexto, bem como trazemos as aplicações a diferentes modelos.

Os resultados desta tese deram origem a 3 artigos científicos descritos abaixo:

1. Instability results for functional Volterra equations driven by Stieltjes measures with application to a mechanical mass-spring-damper system with delay. (submetido, 2025)
2. Well–posedness of impulsive functional Volterra equations with infinite delay driven by Stieltjes measures. (submetido, 2025)
3. Boundedness of the solutions of functional Volterra-Stieltjes integral equations. (submetido, 2025)

Palavras-chave: Equações integrais funcionais; equações integrais impulsivas; equações Δ-integrais em escalas de tempo; equações funcionais fracionárias; critérios de instabilidade e estabilidade; integral de Perron-Stieltjes.

ABSTRACT

In this thesis, we study the functional Volterra–Stieltjes integral equations given by

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s)\,\mathrm{d}g(s), \quad t \geqslant \tau_0 \\ x_{\tau_0} = \phi, \end{cases} \tag{1}$$

where the integral on the right-hand side is considered in the Perron-Stieltjes sense. We proved boundedness and instability criteria for solutions of the Volterra-Stieltjes integral functional equation with finite delay via Lyapunov functionals. We also proved stability results for the Volterra-Stieltjes functional equation with infinite delay via Lyapunov functionals. Considering that the equations given by (1) generalize fractional, impulsive and dynamic equations on time scales, we present the results obtained in this context, as well as bring the applications to models.

The results of this thesis gave rise to three scientific articles, described below:

1. Instability results for functional Volterra equations driven by Stieltjes measures with application to a mechanical mass-spring-damper system with delay. (submitted, 2025)
2. Well–posedness of impulsive functional Volterra equations with infinite delay driven by Stieltjes measures. (submitted, 2025)
3. Boundedness of the solutions of functional Volterra-Stieltjes integral equations. (submitted, 2025)

Keywords: Functional integral equations; Volterra–Stieltjes equations; impulsive integral equations; Δ-integral equations on time scales; fractional functional equations; stability and instability criteria; Perron-Stieltjes integral.

LIST OF FIGURES

LIST OF SYMBOLS

$\mathbb{N}$ — set of positive integers $\{1, 2, 3, \ldots\}$

$\mathbb{R}$ — set of real numbers

$\mathbb{R}^n$ — n-dimensional Euclidean space

$\|\cdot\|$ — norm of Euclidean space

X — the Banach space

$\|\cdot\|_X$ — norm of Banach space

$[\alpha,\beta],[a,b]$ — closed intervals of real numbers

$G([\alpha,\beta],\mathbb{R}^n)$ — the space of all regulated functions $f\colon[\alpha,\beta]\to\mathbb{R}^n$

$\Delta^+g(t)$ — jumps to the right

$\Delta^-g(t)$ — jumps to the left

$\mathcal{D}[\alpha,\beta]$ — The set of all partitions of $[\alpha,\beta]$

D — partition of the interval

$\rightrightarrows$ — for convergence uniformly

$\mathrm{var}_\alpha^\beta(f)$ — variation of f over $[\alpha,\beta]$

$V(f,D)$ — total variation of a function over a partition D

$\mathrm{BV}([\alpha,\beta],X)$ — the set of all the functions of bounded variation

$\|\cdot\|_{BV}$ — the variation norm

$C([\alpha,\beta],\mathbb{R}^n)$ — the space of all continuous functions $f\colon[\alpha,\beta]\to\mathbb{R}^n$

$B(x_i,\varepsilon)$ — open ball with center x_i and radius ε

$S(f,D)$ — Riemann sum

$(P)\int,(PS)\int$ — Perron and Perron-Stieltjes integrals, used only when working with multiple senses of integrals

$\int_\alpha^\beta f(s)\mathrm{d}s$ — definite integral in the classical sense

$\int_\alpha^\beta f(s)\mathrm{d}g(s)$ — definite integral in the stieljtes sense

$\mathbb{T}$ — a time scale

$\int_{\alpha}^{\beta} f(s)\Delta s$ — Riemann $\Delta -$ integral

CONTENTS

INTRODUCTION

The principles and laws governing the behavior of the physical world are expressed as propositions and relationships that describe the rates at which phenomena occur. In mathematical terms, the relations are equations and the rates are derivatives; hence, equations containing derivatives are differential equations. To understand and analyze problems involving fluid motion, electric current in circuits, heat dissipation in solids, seismic wave propagation, population dynamics, and many other phenomena, it is essential to investigate the theory of differential equations, including **ordinary differential equations** (ODEs) and **partial differential equations** (PDEs).

However, many phenomena in mathematical physics cannot be adequately modeled by differential equations alone, due to the complexity of the variables involved and the special properties of the materials. Problems such as viscoelasticity, thermoviscoelasticity, dynamics of isotropic materials, heat conduction in materials with memory, and memory electrodynamics require alternative approaches. Volterra-type equations, including integral equations and integro-differential equations, were developed to address these challenges. Studying the behavior of their solutions allows us to gain insight into the dynamics of the associated models, see (Volterra 1959, Prüss 1993) for examples and models.

Vito Volterra (Figure 1) is a pioneer in the theory of integral equations. He first encountered an integral equation in a paper in 1884 on the distribution of electric charge on a segment of a sphere. He demonstrated that the problem depends on the solution of what today would be called an integral equation of the first kind with a symmetric

Figure 1 – Vito Volterra (1860–1940)

Source: Rai Cultura

kernel. However, it was only in 1896 that Volterra seriously dedicated himself to this field, applying his theory of functionals to what was then called the inversion of definite integrals, and obtaining widely impressive results. In these works, he studied the integral equation of the second kind with a variable upper limit of integration given by

$$\phi(y) = f(y) + \int_a^y K(x,y) f(x)\, dx \tag{1}$$

where f is the unknown function, which is now generally called the *integral equation of the Volterra type*, as the limiting case of a system of linear algebraic equations; the n-th equation of this system contains only the first n variables, so the system can be solved recursively, see (Volterra 1907), leading to the formulation of what is now known as an integral equation of the second type. His approach allowed the transformation of certain problems into solvable forms using resolvent kernels.

Further developments, including contributions by Fredholm, Hilbert and Schmidt, explored the differentiation of integral equations of the first kind to obtain equations of the second kind. An example of such a formulation is given by

$$\theta(y) - \theta(a) = \int_a^y H(x,y) f(x)\, dx \tag{2}$$

where $H(x,y)$ is a kernel function and $\theta(y)$ is derived from the original problem formulation.

However, in many of the questions of mathematical physics, mechanics, and analysis studied by Volterra, it was necessary to consider analytical relations that simultaneously possessed the properties of integral and differential equations. Volterra called these *équations intégro-différentielles*. The physical nature of the problems modeled by this type of equation led Volterra to observe that his theory fit with the ideas of Charles E. Picard—the physics of heredity and non-heredity, see (Picard 1908). While integro-differential equations modeled problems of heredity, which were characterized by M. Painlevé by the property that the current state of the system depended not only on the current state of the forces acting on it, but also on previous states. Non-hereditary mechanics, on the other hand, was represented by ordinary differential equations, in which the current state depended only on the current instant, see (Volterra 1907) for details.

The contributions of Fredholm's theory, published in swedish in 1900 and in french (Fredholm 1903, in Acta Mathematica), led Volterra to point out a connection between Fredholm's theory and some problems in his own theory of functionals: the solution of an integral equation is in fact a simple case of the solution of a functional equation. See (Lauricella 1910) for more details.

The works of Volterra (Volterra 1907), (Volterra 1909) and (Volterra 1928), which discussed how integro-differential differential equations modeled hereditary effects, spurred the development of the theory, particularly in the Soviet Union after the 1940s. Before that, the effect had been observed but ignored due to the lack of sufficient theory to discuss such models in detail.

In many applications, one assumes the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. However, the principle of causality is often only a first approximation to the true situation and that a more realistic model would include some of the past states of the system. In his research on predator-prey models and viscoelasticity, Volterra in (Volterra 1928) and (Volterra 1931) formulated some rather general differential equations incorporating the past states of the system.

Over the past 70 years, the theory of functional differential equations has been extensively developed and has become part of the vocabulary of researchers dealing with specific applications, such as viscoelasticity, mechanics, nuclear reactors, distributed networks, heat flow, neural networks, combustion, species interactions, microbiology, learning models, epidemiology, physiology, among many others; see (Kolmanovskii and Myshkis 1999). During the 1950s, there was considerable activity on the subject, which led to important publications by (Myshkis 1955), (Krasovskii 1963), (Bellman and Cooke 1963), and (Halanay 1966). These books provide a clear overview of the subject up to the early 1960s.

Thus, it is estimated that delay differential equations, integral differential equations, and functional differential equations have been studied for at least 200 years, see (Hale 1977) and (Schmidt 1911) for references and some properties of linear equations. Most research on functional differential equations (FDEs) dealt primarily with linear equations and the preservation of the stability or instability of equilibria under small nonlinear perturbations when the linearization was stable (or unstable). For linear equations with constant coefficients, it was natural to use the Laplace transform. This led to expansions of solutions in terms of the eigenfunctions and the convergence properties of these expansions. For the stability of equilibria, it was important to understand the extent to which Lyapunov's second method (Lyapunov 1891) could be applied.

A rich source of Volterra equations is continuum mechanics for memory materials, namely the theory of viscoelastic materials. Viscoelasticity refers to the property of materials that exhibit both viscous and elastic characteristics when subjected to deformation, such as rubber and polymers. These materials exhibit properties that, in some respects, resemble an elastic solid that obeys Hooke's law of elasticity; and, in others, Newton's law of viscosity, see (Prüss 1993, Chapter 5).

Viscoelasticity materials have been widely studied in fields such as materials engineering and medicine for example, especially since the 1990s, due to advances in **nanotechnology**. In (Seitz, Warnecke and Dürselen 2022) the authors described the model on cartilage biomechanics, this gives us an example of viscoelasticity material in the human body. **Articular cartilage** (AC) is defined as a highly specialized connective tissue that covers the long bones of load-bearing joints, providing a smooth surface for low-friction movement and facilitating load transmission while reducing stress on the underlying bone.

Another example in the human body is the skin and underlying soft tissues, described in section 7.3.3 in (Cheung and Zhang 2006). In this text, the authors describe the mechanics of human skin and underlying tissues exposed to finite deformations, highlighting that their viscoelasticity response is nearly linear, where the deformation considered is the stretching of the body. Thus, we have a surprising source of applications for the Volterra equations, which motivates us to study them.

Linear viscoelasticity models describe the material response to deformation via convolution-type integral relations, reflecting the material's memory effects. These models are valid only for small strains, where linearization is appropriate, e.g. (Prüss 1993, Chapter 5). A given strain-history of the body causes stress in a way to be specified, expressing the properties of the material the body is made of, let us denote the stress tensor by $\mathcal{S}(t,x)$ and the strain tensor by $\mathcal{E}(t,x)$ for the three-dimensional case. The Boltzmann-Volterra constitutive relations or stress-strain relations in the compressible[1] case is given by

$$\mathcal{S}(t,x) = \int_0^\infty \dot{\mathcal{E}}(t-\tau,x)dA(\tau,x) \quad t \in \mathbb{R}, \quad x \in \Omega$$

and

$$\mathcal{E}(t,x) = \int_0^\infty \mathcal{S}(t-\tau,x)dK(\tau,x) \quad t \in \mathbb{R}, \quad x \in \Omega$$

where $\Omega \subset \mathbb{R}^3$ is a given three-dimensional body with boundary $\partial\Omega$ of class C^1, the points x in Ω are material points, A, K are the **relaxation** and **creep kernels** respectively, representing the memory of the material, defined by $A, K\colon \mathbb{R}_+ \times \Omega \to \mathcal{B}(\mathrm{Sym}\{3\})$ such that they are locally of bounded variation with respect to t, here $B(\mathrm{Sym}\{3\})$ denotes the space of 3-dimensional real symmetric matrices bounded. Thus, convolution integrals make explicit the hereditary nature of linear viscoelasticity, connecting stress and deformation through history-dependent kernels.

The kernels A and K are important functions in the system, as they quantify the memory. For example, the relaxation kernel $A(\tau,x)$ describes how the stress dissipates (relaxes) over time if the strain is kept constant. The creep kernel $K(\tau,x)$ describes the opposite, how the strain increases (flows) over time under a constant stress. Material behavior is quantified by the kernels A and K. For example, for the case homogeneous[2] isotropic[3] linear materials, the properties are described by only two main modules, the **creep modulus** $\mathrm{d}k$ and the **shear modulus** $\mathrm{d}a$, which is given by the following relation

$$(\mathrm{d}a * k)(t) = (a * \mathrm{d}k)(t) = t, \quad t > 0.$$

There are two ways to classify materials, according to the behavior of $a(t)$ and $k(t)$ at $t = 0$ or at $t = \infty$. Thus, the fundamental distinction between materials is made

[1] A material is called compressible, if there are changes of volume in the body Ω during a deformation.

[2] A material is called homogeneous if A, K and the mass density do not depend on the material points $x \in \Omega$.

[3] The material is called isotropic if the constitutive law is invariant under the rotation group.

by examining the asymptotic behavior of the kernels at $t = \infty$ and $t = 0$. A material is considered a **solid** if the relaxation modulus is positive at infinity $a_\infty > 0$, a_∞ is the elasticity modulus. It is considered a **fluid** if the creep modulus is positive at infinity $(k_\infty > 0)$, $\frac{1}{k_\infty}$ is termed static viscosity. A material is called **rigid** if $k_0 > 0$; k_0 is the rigidity. It is called **viscous** if $a_0 > 0$; a_0 is termed dynamic viscosity. This diversity is well illustrated by classical models, often visualized through mechanical models, which use basic elements in series or parallel.

1) **Ideal Elastic Solids (Hookean Solid)**: In these materials, stress is directly proportional to strain
$$a(t) = \mu t, \quad k(t) = \frac{1}{\mu}, \quad t > 0.$$

2) **Ideal Viscous Fluids (Newtonian Fluids)**: Stress is proportional to the rate of deformation
$$a(t) = \nu, \quad k(t) = \frac{t}{\nu}, \quad t > 0.$$

3) **Kelvin-Voigt Solid**: The material has an elastic and viscous response in parallel.
$$a(t) = \nu + \mu t, \quad k(t) = \frac{1}{\mu}(1 - e^{-\frac{\mu t}{\nu}}), \quad t > 0.$$

4) **Maxwell Fluid**: The material has a series elastic and viscous response.
$$a(t) = \nu(1 - e^{\frac{-\mu t}{\nu}}), \quad k(t) = \frac{1}{\mu} + \frac{t}{\nu}, \quad t > 0.$$

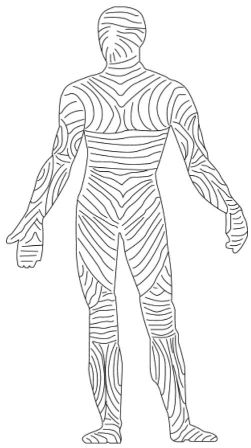
5) **Poynting-Thompson Solid**: The Poynting-Thompson solid is the most complete model in the standard classification, acting as the prototype of the behavior of viscoelastic solids, as it combines Maxwell's fluidity with Hooke's elastic stability.
$$a(t) = \mu_0 t + \nu\Big(1 - e^{\frac{-\mu t}{\nu}}\Big), \quad k(t) = \mu_0^{-1}\Big[1 - \mu(\mu_0 + \mu)^{-1} e^{\frac{-\mu\mu_0 t}{\nu(\mu+\mu_0)}}\Big].$$

The combination and behavior of these functions define the material.

It is worth noting that the use of Stieltjes integration, $\mathrm{d}a(t)$ and $\mathrm{d}k(t)$, instead of the simple Riemann integral $\mathrm{d}t$, makes it possible to incorporate discontinuities and instantaneous jumps in the material's behavior. This feature is crucial, for instance, when modeling an instantaneous (impulse) response at time zero, which cannot be adequately represented by a standard time integral. Such generalization is essential to describe a wide variety of materials —from purely elastic to viscous —within a unified framework.

An interesting example is the mechanics of human skin and underlying soft tissues, described in (Cheung and Zhang 2006). This is a case of finite deformation (see Figure 2), since the nonlinear stress-strain characteristics of soft tissue must be taken into account. Thus, the relaxation, creep, and stress characteristics can be condensed into the mathematical formulation called the quasi-linear viscoelasticity theory. In this theory, the instantaneous stress-strain response is nonlinear. The stress at any given time is related to the entire past history of the elastic stress with a linear relationship. The present stress is obtained by multiplying the past elastic stress increment with a relaxation function and integrating the product in a convolution from the beginning of the time to the present. The relaxation function has a continuous relaxation spectrum. Here, considering a cylindrical

Figure 2 – Mechanics of human skin



Source: Adapted from (Cheung and Zhang 2006)

specimen subjected to tensile load, if a step increase in elongation (from $\lambda = 1$ to λ) is imposed on the specimen, the stress developed will be a function of time as well as of the stretch λ. The history of the stress response, called the relaxation function, $K(\lambda, t)$, is assumed to be of the form

$$\begin{cases} K(\lambda, t) = G(t)T^{(e)}(\lambda) \\ G(0) = 1 \end{cases}$$

where $G(t)$, a normalized function of time, is called the reduced relaxation function, and $T^{(e)}(\lambda)$, a function of λ alone, is called the elastic response. Assuming that the stress response to an infinitesimal change in stretch, $\delta\lambda(\tau)$ superposed on a specimen in a state of stretch λ at an instant of time τ, is, for $t > \tau$:

$$G(t-\tau)\frac{\partial T^{(e)}[\lambda(\tau)]}{\partial \lambda}\delta\lambda(\tau)$$

and with superposition principles, the stress response is written as

$$T(t) = \int_{-\infty}^{t} G(t-\tau)\frac{\partial T^{(e)}[\lambda(\tau)]}{\partial \lambda}\delta\lambda(\tau)\mathrm{d}\tau \tag{3}$$

where the tensile stress at time t is the sum of contributions of all the past changes, each governed by the same reduced relaxation function. From (3), the stress response is described by a linear law relating the stress T with the elastic response $T^{(e)}$. The function

$T^{(e)}(\lambda)$ plays the role assumed by the strain $\mathcal{E}$ in the conventional theory of viscoelasticity. The inverse of (3) may be written as

$$\mathcal{E}(T,t) = \int_{-\infty}^{t} J(t-\tau)\frac{\mathrm{d}\mathcal{E}^{(e)}[T(\tau)]}{\mathrm{d}\tau}\mathrm{d}\tau,$$

which defines the reduced creep function $J(t)$. The elastic response, $T^{(e)}(\lambda)$, we obtain assuming $T = \mathcal{E} = 0$ for $t < 0$ and from (3):

$$T(t) = T^{(e)}[\lambda(t)] + \int_{0}^{t} T^{(e)}[\lambda(t-\tau)]\frac{\partial G(\tau)}{\partial \tau}\mathrm{d}\tau.$$

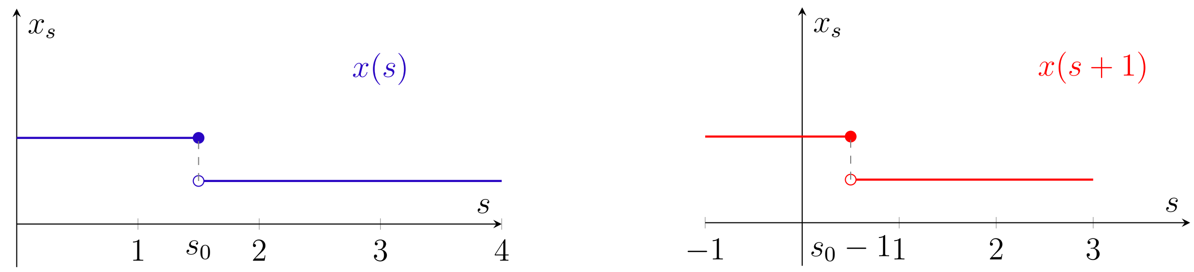
Motivated by our interest in the theory of Volterra equations, we study a class of Volterra-Stieltjes functional integral equations that generalize the convolution relations found in linear viscoelastic models. In this thesis, we study the qualitative and asymptotic properties of the solutions of the class of equations of the form

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s)f(x_s,s)\,\mathrm{d}g(s), & t \geqslant \tau_0 \\ x_{\tau_0} = \phi, \end{cases} \tag{4}$$

where the integral on the right-hand side is considered in the Perron–Stieltjes sense. More precisely, we are interested in establishing: uniform boundedness and instability criteria in the sense of Lyapunov and Chetaev, for equations with finite delay; stability criteria in the case of infinite delay and applications of these results to fractional, impulsive and dynamic equations on time scale and and to models found in viscoelasticity theory.

Equation (4) is more general than those described by Vito Volterra and, as demonstrated in (Gallegos, Grau and Mesquita 2021), (Grau, Lafetá and Mesquita 2024) and (Alvarez *et al.* 2021), it can encompass many types of equations as special cases. In the current literature, many authors have studied this type of equation, considering the integral that appears on the right-hand side in the sense of Riemann, Riemann–Stieltjes, Lebesgue, and Lebesgue–Stieltjes. However, in a generalized form, that is, considering the Henstock–Kurzweil–Stieltjes or Perron–Stieltjes integral because such integrals are more general than the Lebesgue–Stieltjes integral, see (Bonotto, Federson and Mesquita 2021), (Grau 2025), (Monteiro, Slavík and Tvdý 2019) and (Slavík 2012), it was investigated only recently in (Alvarez *et al.* 2021), (Grau, Lafetá and Mesquita 2024) and (Lafetá 2022). This generalized form provides a greater number of applications to models that describe impulsive systems with discontinuous solutions and, therefore, can be used to study evolutionary processes such as optimal control models in economics, biological or physical phenomena, control theory, among others.

The problem (4) is, in fact, a **retarded Volterra-Stieltjes integral equation**. Its structure incorporates the delayed term $f(x_s,s)$, where x_s represents the history (delay) of the unknown function x at time s. We generally describe the delay of a function by considering that, for each s, x_s lies in a predetermined function space, which we call phase space, which is characterized by some axioms depending on the problem. Consider $r(s)$ to be the delay such that all arguments of the delay of x at s lie in the interval $[s-r(s), s]$. Setting $r = \sup r(s)$, we have that $0 \leq r \leq \infty$. Then, for each s, this situation is reflected in terms of x_s as long as such a function has domain $[-r, 0]$ and is defined by $x_s(\theta) = x(s+\theta)$, with $-r \leq \theta \leq 0$, called the history of x to s. If r is finite, the equation is said to be of finite delay; otherwise, it is said to be of infinite delay. As example, see Figures 3a and 3b for finite delays. See (Hino, Murakami and Naito 2025) and (Hale and Kato 1978) for more details.



(a) Graph of $x(s)$, $\theta = 0$ (b) Graph of $x(s+1)$, $\theta = 1$

The main objective of this thesis is to advance the understanding of the qualitative behavior of solutions to functional integral equations and to contribute to both the theoretical framework and the modeling of applied problems involving Perron-Stieltjes integrals. In particular, we focus on results regarding uniform boundedness and instability in the sense of Lyapunov and Chetaev for equation (4)with finite delay, as well as stabilization criteria for the case of infinite delay.

This thesis is organized as follows. Chapter 1 presents the necessary preliminaries on regulated functions, integral theory, and theory of time scales, which form the basis for subsequent analysis. Chapter 2 studies the Volterra-Stieltjes functional integral equation with finite delay, including existence, uniqueness, solution prolongation, and its correspondence with other classes of equations.

Chapter 4 establishes criteria for the uniform boundedness of solutions, including a reverse Lyapunov theorem and applications to concrete models. Chapter 5 addresses instability, presenting the Lyapunov and Chetaev theorems adapted to functional integral equations. Chapter 6 extends the analysis to infinite delay, introducing an appropriate phase space and deriving stability results. Chapter 7 concludes the thesis and discusses open problems.

As a direct application of this unified approach, we illustrate the results through correspondences with time-scale, fractional, and impulsive equations whenever possible. We also apply the theory to physical models such as the Poynting-Thomson solid model in viscoelasticity and the Maxwell fluid model, showing how the qualitative and asymptotic behavior of solutions can be effectively analyzed within the Volterra-Stieltjes framework.

Overall, the results generalize and unify previous approaches and open new perspectives for the study of systems with memory, discontinuities, and dynamics in continuous and discrete time. By combining modern and classical analytical tools, this work provides a solid foundation for future investigations of integral equations and their applications.

CHAPTER

1

PRELIMINARIES

In this chapter, we present fundamental results on regulated functions, integrals theory, and time scale theory, which are important for the development of this work. We divide this chapter into three sections. The first one addresses the theory of regulated functions, presenting results such as Franková's Theorem, which is a version of the Arzelá-Ascoli Theorem for this class of functions. The second section focuses on generalized integrals, bringing together important results that are generalizations of those on Riemann integrals, such as the Fundamental Theorem of Calculus, as well as showing the equivalence between different definitions of the integral concept, such as the Perron and Henstock-Kurzweil integrals. Finally, in the third section, we introduce the theory of time scales and present results on this concept. The basic references for this chapter are (Mesquita 2012, Fraňková 1991, Kurtz and Swartz 2004, Monteiro, Slavík and Tvdý 2019, Kurzweil 1957, Kurzweil 2000).

1.1 Regulated Functions

Throughout this work, let $\mathbb{N}$ be the set of positive integers, given $n \in \mathbb{N}$, $\mathbb{R}^n$ is the n-dimensional Euclidean space with norm $\|\cdot\|$, if $n = 1$ then we denote the norm in $\mathbb{R}^1 = \mathbb{R}$ by $\|\cdot\| = |\cdot|$. The set X we denote the Banach space over $[\alpha, \beta]$, whose norm is given by $\|\cdot\|_X$. The main references here are (Fraňková 1991) and (Monteiro, Slavík and Tvdý 2019).

Definition 1.1.1. A function $f\colon [\alpha, \beta] \to \mathbb{R}^n$ is **regulated** if the limit $f(t^-) = \lim_{s\to t^-} f(s)$, exists for all $t \in (\alpha, \beta]$, and the limit $f(t^+) = \lim_{s\to t^+} f(s)$, exists for all $t \in [\alpha, \beta)$.

The space of all regulated functions $f\colon [\alpha, \beta] \to \mathbb{R}^n$ will be denoted by $G([\alpha, \beta], \mathbb{R}^n)$. If a sequence of regulated functions $(f_n) \in G([\alpha, \beta], \mathbb{R}^n)$ converges uniformly to a function f, we write $f_n \rightrightarrows f$.

The space $G([\alpha,\beta],\mathbb{R}^n)$ is endowed with the sup-norm defined below

$$\|f\|_\infty = \sup_{s\in[\alpha,\beta]} \|f(s)\| \tag{1.1.1}$$

is a Banach space, where $\|f(s)\|$ is norm on Euclidean space and a non-negative real number if f is bounded and $\|f\| = \infty$ otherwise. See the result below.

Theorem 1.1.2. (Monteiro, Slavík and Tvdý 2019, Theorem 4.2.1). The space $G([\alpha,\beta],\mathbb{R}^n)$ equipped with the standard norm (1.1.1) is a Banach space.

Let $I \in \mathbb{R}$ be an interval. We denote by $G(I,\mathbb{R}^n)$ the space of all locally regulated functions $x\colon I \to \mathbb{R}^n$, that is, for each compact interval $[\alpha,\beta] \subset I$ the restriction of x to $[\alpha,\beta]$ belongs to the space $G([\alpha,\beta],\mathbb{R}^n)$. Let $g\colon [\alpha,\beta] \to \mathbb{R}^n$ be a regulated function, we will denote by $\Delta^+g(t)$ and $\Delta^-g(t)$ the jumps to the right $g(t^+) - g(t)$ and the jumps to the left $g(t) - g(t^-)$, respectively.

Remark 1.1.3. If $f \in G(I,\mathbb{R}^n)$ and $[\alpha,\beta] \subset I$, where I is interval of $\mathbb{R}$, we will use the notation

$$\|f\|_{\infty,[\alpha,\beta]} := \sup_{s\in[\alpha,\beta]} \|f(s)\|$$

to denote the norm of the function f restricted to the interval $[\alpha,\beta]$.

Let us also define the variation of a function $f\colon [\alpha,\beta] \to \mathbb{R}^n$ over $[\alpha,\beta]$. Let $D = \{t_0, t_1, \dots, t_m\}$ be a finite set of points in the closed interval $[\alpha,\beta]$, such that

$$\alpha = t_0 < t_1 < \cdots < t_m = \beta.$$

This set is called a **partition of** $[\alpha,\beta]$. Given a partition of $[\alpha,\beta]$, its elements are usually denoted by $t_0, t_1, \dots, t_m$, where $|D| = m+1$. The set of all partitions of $[\alpha,\beta]$ is denoted by $\mathcal{D}[\alpha,\beta]$. Moreover, we call a pair $(\tau_i, [t_{i-1}, t_i])$ a **tagged interval**, where $\tau_i \in [t_{i-1}, t_i]$ is called a *tag* of $[t_{i-1}, t_i]$. Let $\delta > 0$. We say that a partition of $[\alpha,\beta]$ is **δ-fine** if, for every $i = 1, 2, \dots, |D|$, we have:

$$t_i - t_{i-1} < \delta.$$

Definition 1.1.4. Let X be Banach space. A function $f : [\alpha,\beta] \to X$ is called a **finite step function** if there exists a finite partition

$$\alpha = t_0 < t_1 < \cdots < t_m = \beta$$

such that in every open interval (t_{i-1}, t_i), for $i = 1, \dots, m$, the function is identically equal to an element $c_i \in X$.

Let us now present a crucial result. Hönig's Theorem is an important tool because it characterizes regulated functions in $\mathbb{R}^n$ with respect to a partition. The proof can be found in (Fraňková 1991).

Theorem 1.1.5. (Hönig) The following two statements are equivalent:

(i) $f \in G([\alpha, \beta], \mathbb{R}^n)$;

(ii) For every $\varepsilon > 0$, there exists a δ-fine partition D of $[\alpha, \beta]$ such that

$$\|f(t) - f(s)\| < \varepsilon \tag{1.1.2}$$

holds for every $j \in \{1, \dots, |D|\}$ and each pair $t, s \in (s_{j-1}, s_j)$.

Proof. **(i) ⇒ (ii):**

Without loss of generality, consider that $s_{j-1} < t < s < s_j$ is a partition D, where $j = 1, 2, \dots, |D|$. Let $\varepsilon > 0$ be given and denote by S the set of all $\tau \in (\alpha, \beta]$ such that there is a δ-fine partition on $[\alpha, \tau]$, $\alpha = s_0 < s_1 < \dots < s_k = \tau$, satisfying (1.1.2) with k instead of $|D|$. Our goal is to prove that $\beta \in S$. First, we show that S is nonempty.

By hypothesis, f is regulated, that is, the limit $\lim_{t \to \alpha^+} f(t) = f(\alpha^+)$ exists, and this guarantees the existence of $\rho > 0$ such that

$$\|f(t) - f(\alpha^+)\| < \frac{\varepsilon}{2}, \quad \text{for } t \in (\alpha, \rho).$$

Then, for every $\alpha < t < s < \rho$, we get

$$\|f(s) - f(t)\| \leq \|f(s) - f(\alpha^+)\| + \|f(t) - f(\alpha^+)\| < \varepsilon.$$

Since S satisfies the above statement, it follows that $\rho \in S$.

Note that by the definition of S, we have that $c = \sup S > \alpha$. Consequently, there is the limit $f(c^-) = \lim_{t \to c^-} f(t)$, which implies the existence of a $\delta > 0$ such that $\|f(t) - f(c^-)\| < \frac{\varepsilon}{2}$ for every $t \in (c - \delta, c)$. Taking $\gamma \in S$ such that $\gamma \in (c - \delta, c)$. Since $\gamma \in S$, there is a finite sequence $\alpha = s_0 < s_1 < \dots < s_k = \gamma$ such that (1.1.2) holds with k instead of $|D|$.

Without loss of generality, we can take $s_{k+1} > c$. Then

$$\|f(s) - f(t)\| \leq \|f(s) - f(c^-)\| + \|f(t) - f(c^-)\| < \varepsilon,$$

for $s_k = \gamma < t < s < c < s_{k+1}$, which implies that (1.1.2) holds also for $|D| = k + 1$. It implies that $c \in S$.

Assume by contradiction that $c < \beta$. Now, take $s_{k+1} \in (c, \beta]$ such that $s_{k+1} - s_k < \delta$. For any $t, s \in (s_k, s_{k+1})$, we have:

$$\|f(s) - f(t)\| \leq \|f(s) - f(c^-)\| + \|f(t) - f(c^-)\| < \varepsilon.$$

Thus, adding the interval (s_k, s_{k+1}) extends the partition from α to $s_{k+1} > c$ and preserves the property required to be in S, which contradicts the assumption that $c = \sup S$. Therefore, the assumption $c < \beta$ must be false, and we conclude that $c = \beta$.

Now, let $\varepsilon > 0$ be given. By assumption, there exists a δ-fine partition D such that

$$\alpha = s_0 < s_1 < \cdots < s_{|D|} = \beta,$$

for all $t, s \in [s_{i-1}, s_i]$, $i = 1, \ldots, |D|$, we have:

$$\|f(s) - f(t)\| < \varepsilon.$$

Let $n \in \mathbb{N}$ be given and let $\hat{D}$ be a partition of $[\alpha, \beta]$ such that $\|f(t) - f(s)\| \leq \frac{1}{n}$ for all $t, s \in (s_{j-1}, s_j)$ and $j \in \{1, \ldots, |\hat{D}|\}$.

For every $j \in \{1, \ldots, |\hat{D}|\}$, choose an arbitrary $\tau_j \in (s_{j-1}, s_j)$ and define

$$f_n(t) = \begin{cases} f(t), & \text{if } t \in \hat{D}, \\ f(\tau_j), & \text{if } t \in (s_{j-1}, s_j). \end{cases}$$

Obviously, $\{f_n\}$ is bounded and $\|f - f_n\| < \frac{1}{n}$ for every $n \in \mathbb{N}$, i.e., $f_n \rightrightarrows f$ on $[\alpha, \beta]$ as $n \to \infty$. We conclude that f is regulated. □

As a consequence of Hönig's theorem, we have the following corollaries.

Corollary 1.1.6. (Monteiro, Slavík and Tvdý 2019, Corollary 4.1.6) Every regulated function $f \colon [\alpha, \beta] \to \mathbb{R}^n$ is bounded.

Corollary 1.1.7. (Monteiro, Slavík and Tvdý 2019, Corollary 4.1.7) For every regulated function $f \colon [\alpha, \beta] \to \mathbb{R}^n$ and every $\varepsilon > 0$, there are at most finitely many points $t \in [\alpha, \beta]$ such that

$$t \in [\alpha, \beta) \text{ and } \|\Delta^+ f(t)\| > \varepsilon \text{ or } t \in (\alpha, \beta] \text{ and } \|\Delta^- f(t)\| > \varepsilon$$

Definition 1.1.8. Let X a Banach space and $f \colon [\alpha, \beta] \to X$, we define the variation of f over $[\alpha, \beta]$ as

$$\operatorname{var}_\alpha^\beta(f) = \sup_{D \in \mathcal{D}[\alpha,\beta]} \sum_{j=1}^{|D|} \|f(\alpha_j) - f(\alpha_{j-1})\|_X.$$

We define the total variation of a function f over a partition D is defined as:

$$V(f, D) = \sum_{j=1}^{|D|} \|f(\alpha_j) - f(\alpha_{j-1})\|_X.$$

Thus

$$\operatorname{var}_\alpha^\beta f = \sup_{D \in D[\alpha,\beta]} V(f, D).$$

If $\alpha = \beta$, we define $\operatorname{var}_\alpha^\beta(f) = 0$. If $\operatorname{var}_\alpha^\beta(f) < \infty$, then f is said to be a function of bounded variation on $[\alpha, \beta]$. We will denote the set of all the functions $f \colon [\alpha, \beta] \to X$ of bounded variation by $\mathrm{BV}([\alpha, \beta], X)$. The space provided with the variation norm

$$\|f\|_{BV} = \|f(\alpha)\|_X + \operatorname{var}_\alpha^\beta(f)$$

is a Banach space. See (Monteiro, Slavík and Tvdý 2019, Chapter 2)

The following example shows that a regulated function may not have bounded variation. See (Monteiro, Slavík and Tvdý 2019, Example 2.1.10)

Example 1.1.9. Let $f\colon [0,2] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ x \sin\left(\frac{\pi}{x}\right), & \text{if } x \in (0,2]. \end{cases}$$

Notice that $f(x) = 0$ if and only if $x = 0$ or $x = \frac{1}{k}$ for some $k \in \mathbb{N}$. Futhermore, for $x \in (0,2]$, we have

$$f(x) = \begin{cases} x, & \text{if and only if } x = y_k = \frac{2}{4k+1} \text{ for some } k \in \mathbb{N} \cup \{0\}, \\ -x, & \text{if and only if } x = z_k = \frac{2}{4k-1} \text{ for some } k \in \mathbb{N}. \end{cases}$$

Thus, for a given $n \in \mathbb{N}$ and for $D_n = \{0, y_n, z_n, y_{n-1}, z_{n-1}, \dots, y_2, z_2, y_1, z_1, 2\}$ in which $0 = \alpha_0 < y_n = \alpha_1 < z_n = \alpha_2 < \cdots < \alpha_{n+1} = 2$, we have

$$\begin{aligned} V(f, D_n) &= \sum_{j=1}^{|D|} |f(\alpha_j) - f(\alpha_{j-1})| \\ &= |f(y_n) - f(0)| + |f(y_{n-1}) - f(z_n)| + \cdots + |f(y_1) - f(z_2)| + |f(2) - f(z_1)| \\ &= |y_n| + |y_{n-1} + z_n| + |y_{n-2} + z_{n-1}| + \cdots + |y_1 + z_2| + |-2 + z_1| \\ &= y_n + (y_{n-1} + z_n) + (y_{n-2} + z_{n-1}) + \cdots + (y_1 + z_2) + \frac{4}{3} \end{aligned}$$

for $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} V(f, D_n) &= \frac{4}{3} + y_n + \sum_{k=2}^{n} (y_{k-1} + z_k) \\ &= \frac{4}{3} + \sum_{k=1}^{n} (y_k + z_k) = \frac{4}{3} + \sum_{k=1}^{n} \frac{16k}{16k^2 - 1} \\ &= 4\left(\frac{1}{3} + \sum_{k=1}^{n} \frac{4k}{16k^2 - 1}\right) \geq 4\left(\frac{1}{3} + \sum_{k=1}^{n} \frac{1}{k}\right). \end{aligned}$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

we see that

$$\lim_{n \to \infty} V(f, D_n) = \infty,$$

and, consequently,

$$\operatorname{var}_0^2(f) = \infty.$$

Thus the function is regulated, since it has only isolated discontinuities, but it does not have bounded variation.

Remark 1.1.10. Evidently, the following relations hold:

$$BV([\alpha,\beta],\mathbb{R}^n)\cup C([\alpha,\beta],\mathbb{R}^n)\subset G([\alpha,\beta],\mathbb{R}^n),$$

$$G([\alpha,\beta],\mathbb{R}^n)\setminus C([\alpha,\beta],\mathbb{R}^n)\neq\emptyset,\quad\text{and}\quad G([\alpha,\beta],\mathbb{R}^n)\setminus BV([\alpha,\beta],\mathbb{R}^n)\neq\emptyset.$$

One of the most important characteristics of regulated functions lies in the fact that such functions may have several discontinuities. This is helpful to describe in a more precise way the real world models. On the other hand, the discontinuities of a regulated functions defined in a compact interval are a most countable, as ensured by the next result which can be found in (Monteiro, Slavík and Tvdý 2019).

Definition 1.1.11 (Discontinuity of a Vector-Valued Function)**.** Let $f:[\alpha,\beta]\to\mathbb{R}^n$ be a vector-valued function, where each component $f_i:[\alpha,\beta]\to\mathbb{R}$ is a scalar function. We say that f is **discontinuous** at a point $t_0\in[\alpha,\beta]$ if at least one of the following conditions fails:

1. The limit $\lim_{t\to t_0} f(t)$ exists in $\mathbb{R}^n$.
2. The limit equals the function value: $\lim_{t\to t_0} f(t)=f(t_0)$.

Equivalently, f is discontinuous at t_0 if there exists an index $i\in\{1,\dots,n\}$ such that the scalar function f_i is discontinuous at t_0.

Theorem 1.1.12. (Monteiro, Slavík and Tvdý 2019, Theorem 4.1.8) Every regulated function $f\colon[\alpha,\beta]\to\mathbb{R}^n$ has at most countably many discontinuities.

Proof. For each $k\in\mathbb{N}$, denote:

$$E_k^+=\left\{t\in[\alpha,\beta):\|\Delta^+f(t)\|>\frac{1}{k}\right\},$$

$$E_k^-=\left\{t\in(\alpha,\beta]:\|\Delta^-f(t)\|>\frac{1}{k}\right\}.$$

Then,

$$E^+=\bigcup_{k\in\mathbb{N}}E_k^+=\{t\in[\alpha,\beta):\|\Delta^+f(t)\|>0\}$$

is the set of all points where the function f is discontinuous from the right, and

$$E^-=\bigcup_{k\in\mathbb{N}}E_k^-=\{t\in(\alpha,\beta]:\|\Delta^-f(t)\|>0\}$$

Obviously,

$$E=E^+\cup E^-$$

is the set of all discontinuity points of f on $[\alpha,\beta]$. By Corollary 1.1.7 every set E_k^+, E_k^-, with $k\in\mathbb{N}$, are finite. As a result, E is at most countable. □

We know from the classical Arzelá-Ascoli theorem that a subset of $C([\alpha,\beta],\mathbb{R}^n)$ is relatively compact if and only if it is uniformly bounded and equicontinuous. We will derive an analogous criterion for subsets of the space $G([\alpha,\beta],\mathbb{R}^n)$ with the notion of equicontinuity replaced by the analogue notion of equiregulated in the next section. The definition of this notion resembles the definition of equicontinuity with ordinary limits replaced by one-sided limits.

Definition 1.1.13. A subset M of $G([\alpha,\beta],\mathbb{R}^n)$ is called an **equiregulated set** if the following conditions hold:

- For each $\varepsilon > 0$ and $\tau \in (\alpha,\beta]$, there exists a $\delta_1(\tau) \in (0, \tau-\alpha)$ such that
$$\|f(\tau^-) - f(t)\| < \varepsilon \quad \text{for all } t \in (\tau - \delta_1(\tau), \tau) \text{ and } f \in M.$$
- For each $\varepsilon > 0$ and $\tau \in [\alpha,\beta)$, there exists a $\delta_2(\tau) \in (0, \beta-\tau)$ such that
$$\|f(\tau^+) - f(t)\| < \varepsilon \quad \text{for all } t \in (\tau, \tau + \delta_2(\tau)) \text{ and } f \in M.$$

The following result generalizes and characterizes regulated functions in Euclidean space $\mathbb{R}^n$, that is, the range $f([\alpha,\beta]) \subset \mathbb{R}^n$. The proof can be found in (Fraňková 1991, Lemma 2.1) and (Mesquita 2012).

Theorem 1.1.14. A set $M \subset G([\alpha,\beta],\mathbb{R}^n)$ is said to be equiregulated if, and only if, for every $\varepsilon > 0$, there exists a δ-fine partition D
$$\alpha = s_0 < s_1 < \cdots < s_n = \beta$$
such that
$$\|f(s) - f(t)\| \leq \varepsilon, \tag{1.1.3}$$
for all $f \in M$ and all $s, t \in [s_{i-1}, s_i]$, $i = 1, \ldots, |D|$.

Proof. $(\Rightarrow)$ Assume without loss of generality, that $s_i < t < s < s_{i-1}$.

Let $\varepsilon > 0$ and F be the set of all points $\eta \in (\alpha,\beta]$ such that there exists a $\delta-$fine partition D
$$\alpha = s_0 < s_1 < \cdots < s_k = \eta$$
such that (1.1.3) holds with k instead of $|D|$. Our goal is to prove that $\beta \in F$. First, we show that F is nonempty. Indeed, since M is equiregulated, there exists $\gamma_1 \in (0, \beta-\alpha]$ such that
$$\|f(t) - f(\alpha^+)\| \leq \frac{\varepsilon}{2}, \quad \text{for all } f \in M \text{ and } t \in (\alpha, \alpha+\gamma_1).$$
Thus, for arbitrary $t, s \in (\alpha, \alpha+\gamma_1)$, we get
$$\|f(s) - f(t)\| \leq \|f(s) - f(\alpha^+)\| + \|f(t) - f(\alpha^+)\| \leq \varepsilon.$$

Denote $\tau = \alpha + \gamma_1$ and define the partition $\alpha = s_0 < s_1 = \tau$ of $[\alpha, \alpha + \gamma_1]$. Then for any $t, s \in (\alpha, \tau)$, we get:

$$\|f(s) - f(t)\| \leq \varepsilon,$$

so $\tau \in F$, this means that the set F is nonempty and $\hat{\tau} := \sup F \subset (\alpha, \beta]$.

Next, we will show that $\hat{\tau} \in F$. Since $f \in M$, we can choose a $\delta_1 \in (0, \hat{\tau} - \alpha)$ such that

$$\|f(t) - f(\hat{\tau}^-)\| \leq \frac{\varepsilon}{2}, \quad \text{holds for all } f \in M \text{ and } t \in (\hat{\tau} - \delta_1, \hat{\tau}) \cap [\alpha, \beta].$$

Hence, for arbitrary $t, s \in (\hat{\tau} - \delta_1, \hat{\tau})$, we have

$$\|f(s) - f(t)\| \leq \|f(s) - f(\hat{\tau}^-)\| + \|f(t) - f(\hat{\tau}^-)\| \leq \varepsilon. \tag{1.1.4}$$

Moreover, by the definition of the supremum, there exists $\tau \in F \cap (\hat{\tau} - \delta_1, \hat{\tau})$. Let D_1 be a partition of $[\alpha, \tau]$ such that (1.1.3) holds, and let $\tilde{D_1} = D_1 \cup \{\hat{\tau}\}$, then $\tilde{D_1} = \{s_0, s_1, \ldots, \tau, \hat{\tau}\}$ is a partition of $[\alpha, \hat{\tau}]$ with $|\tilde{D_1}| = |D_1| + 1$, whose partition points are:

$$\tilde{s}_j = \begin{cases} s_j, & \text{if } j \in \{1, \ldots, |D_1|\}, \\ \hat{\tau}, & \text{if } j = |\tilde{D_1}|. \end{cases}$$

Using (1.1.3) and (1.1.4), we obtain that

$$\|f(t) - f(s)\| \leq \varepsilon \quad \text{for all } t, s \in (\tilde{s}_{j-1}, \tilde{s}_j),$$

where $j \in \{1, \ldots, |\tilde{D_1}|\}$. This implies that $\hat{\tau} \in F$.

Finally, we prove that $\hat{\tau} = \beta$. Suppose, on the contrary, that $\hat{\tau} < \beta$. By definition regulated function, we can choose $\delta_2 \in (0, \beta - \hat{\tau})$ such that

$$\|f(t) - f(\hat{\tau}^+)\| \leq \frac{\varepsilon}{2} \quad \text{for all} \quad t \in (\hat{\tau}, \hat{\tau} + \delta_2).$$

Similarly as before, we can deduce that the inequality below

$$\|f(t) - f(s)\| \leq \|f(t) - f(\tau^+)\| + \|f(s) - f(\tau^+)\| \leq \varepsilon \tag{1.1.5}$$

holds for arbitrary $t, s \in (\hat{\tau}, \hat{\tau} + \delta_2)$. Let D be a partition of the interval $[\alpha, \hat{\tau}]$ such that (1.1.3) holds. Set $\tau = \hat{\tau} + \delta_2$ and $\tilde{D} = D \cup \{\tau\}$. Then $\tilde{D} = \{s_0, s_1, \ldots, \hat{\tau}, \tau\}$ is a partition of $[\alpha, \tau]$ with $|\tilde{D}| = |D| + 1$, whose partition points are

$$\tilde{s}_j = \begin{cases} s_j & \text{if } j \in \{1, \ldots, |D|\}, \\ \tau & \text{if } j = |\tilde{D}|. \end{cases}$$

Using (1.1.3) and (1.1.5), we have

$$\|f(t) - f(s)\| \leq \varepsilon \quad \text{for all } t, s \in (s_{j-1}, s_j) \text{ and } j \in \{1, \ldots, |\tilde{D}|\},$$

which implies $\tau \in F$. However, since $\tau > \hat{\tau}$, this contradicts the fact that $\hat{t} = \sup F$. Thus, $\hat{t} = \beta$ and we finished the proof.

($\Leftarrow$) Conversely, given $\varepsilon > 0$, there exists a δ-fine partition D

$$\alpha = s_0 < s_1 < \cdots < s_{|D|} = \beta.$$

Choose an arbitrary $\tau \in (\alpha, \beta]$. There is a unique $j \in \{1, \ldots, |D|\}$ such that $\tau \in (s_{j-1}, s_j]$. For all $t, s \in (s_{j-1}, \tau)$ and $f \in M$, we have $\|f(s) - f(t)\| < \varepsilon$. Letting $s \to \tau^-$, we get

$$\|f(\tau^-) - f(t)\| \leq \varepsilon \quad \text{for all } t \in (\tau - \delta_1, \tau) \quad \text{and } f \in M,$$

where $\delta_1 = \tau - \alpha_{j-1}$.

Analogously, if $\tau \in [\alpha, \beta)$, there is a unique $j \in \{1, \ldots, |D|\}$ such that $\tau \in [s_{j-1}, s_j)$. Hence, $\|f(s) - f(t)\| < \varepsilon$ holds for all $t, s \in (\tau, s_j)$ and $f \in M$. Letting $s \to \tau^+$ yields

$$\|f(\tau^+) - f(t)\| \leq \varepsilon \quad \text{for all } t \in (\tau, \tau + \delta_2), \quad f \in M,$$

where $\delta_2 = \alpha_j - \tau$. This shows that $M \subseteq G([\alpha, \beta], \mathbb{R}^n)$ is equiregulated, getting the desired result. □

1.2 Compactness in $G([\alpha,\beta],\mathbb{R}^n)$

In this section, we present a sequence of results that characterize the compactness in the set of regulated functions, taking values in in $\mathbb{R}^n$. In the following, we present a criterion analogous to the Arzelá-Ascoli theorem for subsets of the space $G([\alpha, \beta], \mathbb{R}^n)$ with the notion of equicontinuity replaced by the related notion of equiregulated sets. The pioneer of this theory is (Fraňková 1991). The interested reader can find more details in (Mesquita 2012, Monteiro, Slavík and Tvdý 2019).

In sequel, some results that help to understand Fraňková's theorem, the analogue of Azelá-Ascoli theorem. The first one describes a particular property of equiregulated sets.

Theorem 1.2.1. Let $M \subset G([\alpha, \beta], \mathbb{R}^n)$ be an equiregulated set. Then for every $t \in [\alpha, \beta]$, there exists a constant $\gamma_t > 0$ such that

$$\|f(t) - f(t^-)\| \leq \gamma_t, \quad t \in (\alpha, \beta], \quad \text{and} \quad \|f(t^+) - f(t)\| \leq \gamma_t, \quad t \in [\alpha, \beta) \tag{1.2.1}$$

for all $f \in M$. Then there is a constant $K > 0$ such that

$$\|f(t) - f(\alpha)\| \leq K \quad \text{for all } f \in M \text{ and } t \in [\alpha, \beta].$$

Proof. Denote by B the set of all $t \in [\alpha, \beta]$ for which there exists $K_t > 0$ such that

$$\|f(t) - f(\alpha)\| \leq K_t \quad \text{for every } f \in M \text{ and } t \in [\alpha, \beta].$$

Since M is equiregulated, there exists $\delta > 0$ such that

$$\|f(t) - f(\alpha + \delta)\| \leq 1 \quad \text{for all } f \in M,\ t \in (\alpha, \alpha + \delta].$$

Then, for $t \in (\alpha, \alpha + \delta]$

$$\|f(t) - f(\alpha)\| \leq \|f(t) - f(\alpha + \delta)\| + \|f(\alpha + \delta) - f(\alpha)\| \leq 1 + \gamma_\alpha = K_{\alpha+\delta}.$$

This implies that $(\alpha, \alpha + \delta] \subset B$. Let $\tau_0 = \sup B$. There exists $\delta' > 0$ such that

$$\|f(t) - f(\tau_0)\| \leq 1 \quad \text{for all } f \in M,\ t \in [\tau_0 - \delta', \tau_0].$$

Pick $\tau \in B \cap [\tau_0 - \delta', \tau_0]$. Then, for all $t \in (\tau, \tau_0)$,

$$\|f(t) - f(\alpha)\| \leq \|f(t) - f(\tau_0)\| + \|f(\tau_0) - f(\tau)\| + \|f(\tau) - f(\alpha)\| \leq 1 + 1 + K_\tau.$$

Thus, we get

$$\|f(\tau_0) - f(\alpha)\| \leq 2 + K_\tau.$$

Also,

$$\|f(\tau_0) - f(\alpha)\| \leq \|f(\tau_0) - f(\tau_0 - \delta')\| + \|f(\tau_0 - \delta') - f(\alpha)\| \leq \gamma_{\tau_0} + 2 + K_\tau.$$

Hence $\tau_0 \in B$ with $K_{\tau_0} = \gamma_{\tau_0} + 2 + K_\tau$. If $\tau_0 < \beta$, there exists $\delta'' > 0$ such that

$$\|f(t) - f(\tau_0 + \delta'')\| \leq 1 \quad \text{for all } f \in M,\ t \in (\tau_0, \tau_0 + \delta'').$$

Then

$$\|f(\tau_0) - f(\alpha)\| \leq \|f(\tau_0) - f(\tau_0 + \delta'')\| + \|f(\tau_0 + \delta'') - f(\alpha)\| \leq 1 + \gamma_{\tau_0} + K_{\tau_0+\delta''}.$$

Therefore $\tau_0 + \delta'' \in B$, contradicting the definition of τ_0 as the supremum. Hence $\tau_0 = \beta$, and $\beta \in B$. □

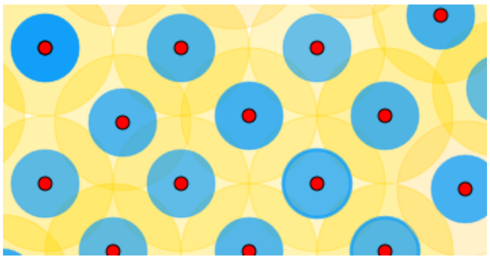
In the following result, we characterize the relatively compact sets in $G([\alpha, \beta], \mathbb{R}^n)$. But first, let us recall the concept of $\varepsilon-$net. Let A be a subset of a normed linear vector space N. We say that A is totally bounded if, for all $\varepsilon > 0$, we can find a finite number of points x_i, for $i = 1, \cdots, n$, such that

$$A \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$$

where $B(x_i, \varepsilon)$ is the ball with center x_i and radius $\varepsilon > 0$. Thus, compactness is well defined, through the following definition.

Definition 1.2.2. Let $\varepsilon > 0$ be a real number and let N be a normed linear vector space. A subset $A \subseteq N$ is called an *ε-net* for N if the family of open balls $\{B(x_i, \varepsilon) : x_i \in A\}$ covers N. That is, for every point $y \in N$, there exists $x \in N$ such that $\|x - y\| < \varepsilon$.

Figure 4 – $\varepsilon-$net for the Euclidean plane



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Example 1.2.3. The set of integer lattice points in the plane (i.e., points whose coordinates are both integers) forms a 1-net for the Euclidean plane $\mathbb{R}^2$. This means that every point in $\mathbb{R}^2$ lies within distance 1 of some lattice point.

In Figure 4, the red points are part of an $\varepsilon-$net for the Euclidean plane, where ε is the radius of the large yellow disks. The blue disks with half the radius are disjoints, and the yellow disks together cover the entire plane, satisfying the requirements in the definition to be an $\varepsilon-$net.

Definition 1.2.4. A subset $U \subseteq N$ is called *totally bounded* if for every $\varepsilon > 0$, there exists a finite set of points $W \subseteq N$ such that W is an ε-net for U. In other words, the collection of open balls $\{B(w,\varepsilon) : w \in W\}$ covers U.

Proposition 1.2.5. (Fraňková 1991, Proposition 2.3) A set $M \subseteq G([\alpha,\beta],\mathbb{R}^n)$ is relatively compact in the sup-norm topology if and only if it is equiregulated, satisfies (1.2.1), and there is a $\kappa > 0$ such that $\|f(\alpha)\| \leq \kappa$ for any $f \in M$.

Proof. It is well known that a subset A of a Banach space X is relatively compact if and only if it is totally bounded; that is, for every $\varepsilon > 0$ there exists a finite ε-net $F = \{x_1, x_2, \ldots, x_k\} \subset X$ such that for every $x \in A$ there exists $x_n \in F$ satisfying $\|x - x_n\| \leq \varepsilon$.

($\Rightarrow$) Assume M is relatively compact. Then it is bounded by a constant positive C, and condition (1.2.1) is satisfied with $\gamma_t = 2C$ for all $t \in [a,b]$.

Let $t_0 \in [\alpha,\beta]$ and $\varepsilon > 0$. Let $\{f_1, f_2, \ldots, f_k\} \subseteq G([\alpha,\beta],\mathbb{R}^n)$ be a finite $\varepsilon/3$-net for M. For each $n = 1,\ldots,k$, there exists $\delta_n > 0$ such that

$$\|f_n(t) - f_n(t_0)\| < \frac{\varepsilon}{3} \quad \text{for } t \in [t_0 - \delta_n, t_0 + \delta_n] \cap [\alpha,\beta].$$

and

$$\|f(t_0) - f_n(t_0)\| < \frac{\varepsilon}{3}.$$

Let $\delta = \min\{\delta_1,\ldots,\delta_k\}$. For any $f \in M$, choose f_n such that $\|f - f_n\| \leq \varepsilon/3$. Then for every $t \in [t_0 - \delta, t_0 + \delta] \cap [\alpha,\beta]$,

$$\|f(t) - f(t_0)\| \leq \|f(t) - f_n(t)\| + \|f_n(t) - f_n(t_0)\| + \|f_n(t_0) - f(t_0)\| < \varepsilon.$$

Therefore M is equiregulated.

($\Leftarrow$) Assume that M is equiregulated, (1.2.1) holds, and $\|f(\alpha)\| \leq \kappa$ for every $f \in M$. By Theorem 1.2.1, there exists $K > 0$ such that $\|f(t) - f(\alpha)\| \leq K$ for any $f \in M$ and $t \in [\alpha, \beta]$. Hence, $\|f(t)\| \leq \|f(t) - f(\alpha)\| + \|f(\alpha)\| \leq K + \kappa$. Letting $\gamma = K + \kappa$, we have $\|x\| \leq \gamma$ for every $f \in M$.

Let $\varepsilon > 0$ be given. By Theorem 1.1.14, there exists a partition D, given by $\alpha = s_0 < s_1 < \cdots < s_{|D|} = \beta$ such that (1.1.3) holds when ε is replaced by $\varepsilon/2$.

Let $\{\kappa_1, \kappa_2, \ldots, \kappa_m\}$ be a finite $\varepsilon/2$-net of the compact set $\{\kappa \in \mathbb{R}^n ; |\kappa| \leq \gamma\}$. Define

$$F = \{f \colon [\alpha, \beta] \to \mathbb{R}^n | f \text{ is constant on } (s_{j-1}, s_j) \text{ for each } j = 1, \ldots, |D| \\ \text{and } f(s_j) \in \{\kappa_1, \ldots, \kappa_m\}\}$$

Clearly, $F \subset G([\alpha, \beta], \mathbb{R}^n)$ is finite.

Let us show that F is an ε-net for M. Take any $f \in M$. For each $n = 0, 1, \ldots, |D|$, choose $i_n \in \{1, \ldots, m\}$ such that $\|f(t_n) - \kappa_{i_n}\| \leq \varepsilon/2$. Also, for each $n = 1, \ldots, |D|$, choose $j_n \in \{1, \ldots, m\}$ such that $\|f(t_{n-1}) - \kappa_{j_n}\| \leq \varepsilon/2$. Define a function $g \in F$ by setting $g(t_n) = \kappa_{i_n}$ for each $n = 0, \ldots, |D|$, and $g(t) = \kappa_{j_n}$ for all $t \in (t_{n-1}, t_n)$. Then for $t_n \in [\alpha, \beta]$, we have $\|g(t_n) - f(t_n)\| \leq \varepsilon$, and for $t \in (t_{n-1}, t_n)$,

$$\|g(t) - f(t)\| = \|\kappa_{j_n} - f(t)\| \leq \|\kappa_{j_n} - f(t_{n-1})\| + \|f(t_{n-1}) - f(t)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\|g - f\| \leq \varepsilon$, which shows that M is totally bounded. □

The next result is the analogue of Arzelá-Ascoli theorem for regulated functions for functions taking value in $\mathbb{R}^n$. It follows immediate as a consequence of the previous results. It can be found in (Monteiro, Slavík and Tvdý 2019, Theorem 4.3.6).

Theorem 1.2.6. (Fraňková) A set $M \subset G([\alpha, \beta], \mathbb{R}^n)$ is relatively compact if and only if it is uniformly bounded and equiregulated.

Fraňková's Theorem can be applied to show that the uniform boundedness condition can be weakened.

Corollary 1.2.7. (Fraňková 1991, Corollary 2.4) A subset M of the space $G([\alpha, \beta], \mathbb{R}^n)$ is relatively compact if and only if it is equiregulated and the set $\{f(t) \colon f \in M\}$ is bounded for each $t \in [\alpha, \beta]$.

Proof. If M is relatively compact, then it is equiregulated by Proposition 1.2.5, and it is evidently bounded. Assume that M is equiregulated and $\|f(t)\| \leq \beta_t$ for all $f \in M$ and $t \in [\alpha, \beta]$. Let $t \in (\alpha, \beta)$ be given. There exists $\delta > 0$ such that

$$\|f(\tau) - f(t^-)\| \leq 1 \quad \text{for all } f \in M, \ \tau \in (t - \delta, t),$$

and

$$\|f(\tau) - f(t^+)\| \leq 1 \quad \text{for all } \tau \in (t, t+\delta).$$

Let $\tau_1 \in (t-\delta, t)$ and $\tau_2 \in (t, t+\delta)$ be fixed. Then,

$$\|f(t) - f(t^-)\| \leq \|f(t)\| + \|f(\tau_1)\| + \|f(t^-) - f(\tau_1)\| \leq \beta_t + \beta_{\tau_1} + 1,$$

$$\|f(t^+) - f(t)\| \leq \|f(t^+)\| + \|f(\tau_2)\| + \|f(\tau_2) - f(t)\| \leq 1 + \beta_{\tau_2} + \beta_t.$$

Let $\gamma_t = 1 + \beta_t + \max\{\beta_{\tau_1}, \beta_{\tau_2}\}$. Similarly, define γ_α and γ_β. Hence, condition (1.2.1) is satisfied, and M is relatively compact by Proposition 1.2.5. □

1.3 Integral theory

In this section, we discuss about some general types of integrals. We make an introduction based on the history, from Riemann until we reach the Henstock-Kurzweil-Stieltjes integral. In our problem, we work with the Perron-Stieltjes integral, which is equivalent to the Henstock-Kurzweil-Stieltjes integral in bounded domains. The main references of this section are (Kurzweil 2000, Kurtz and Swartz 2004, Gordon 1994, Pesin 1970), which can be consulted for more details.

Let $f\colon [\alpha, \beta] \to \mathbb{R}$ be a function. Next, we present a discussion of the Fundamental Theorem of Calculus for the Riemann integral. The Fundamental Theorem of Calculus consists of two parts which relate the processes of differentiation and integration and show that in some sense these two operations are inverses of one another. We begin by considering the integration of derivatives, suppose that f is differentiable on $[\alpha, \beta]$ with derivative f'. The first part of the Fundamental Theorem of Calculus involves the familiar formula from calculus

$$\int_\alpha^\beta f'(t)\mathrm{d}t = f(b) - f(a). \tag{1.3.1}$$

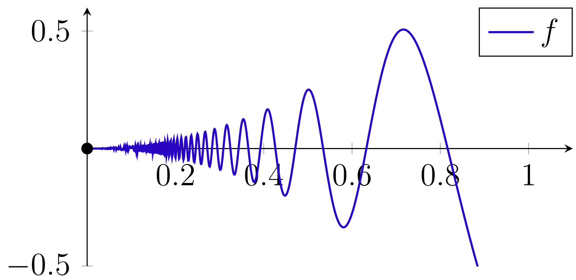
Theorem 1.3.1. (*Fundamental Theorem of Calculus for Riemann Integral- Part I*) Suppose $f\colon [\alpha, \beta] \to \mathbb{R}$ and f' are Riemann integrable on $[\alpha, \beta]$. Then, (1.3.1) holds.

The main assumption of Theorem 1.3.1 is that f' is Riemann integrable. The following example shows that (1.3.1) does not hold in general for the Riemann integral.

Example 1.3.2. (Kurtz and Swartz 2004, Exemple 2.31) Define $f : [0, 1] \to \mathbb{R}$ by

$$f(t) = \begin{cases} t^2 \cos\left(\dfrac{\pi}{t^2}\right) & \text{if} \quad 0 < t \leq 1; \\ 0 \ \text{ if } \ t = 0. \end{cases} \tag{1.3.2}$$

Figure 5 – Graph of f in $[0, 1]$



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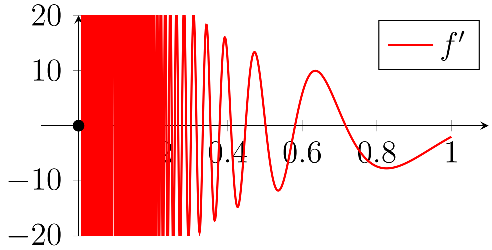
A differentiable function in $[0, 1]$, with derivative given by

$$f'(t) = \begin{cases} 2t\cos\left(\frac{\pi}{t^2}\right) + \frac{2\pi}{t}\sin\left(\frac{\pi}{t^2}\right) & \text{if} \quad 0 < t \leq 1; \\ 0 \quad \text{if} \quad t = 0. \end{cases} \tag{1.3.3}$$

Since f' is not bounded on $[0, 1]$, f' is not Riemann integrable on $[0, 1]$.

Figure 6 – Graph of f' in $[0, 1]$



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It is known in calculus that unbounded functions on compact intervals are not Riemann integrable, in Example 1.3.2, we gave an example of a derivative which is unbounded and is, therefore, not Riemann integrable, and we showed in Theorem that if f' is Riemann integrable, then (1.3.1) holds. That is, in order for (1.3.1) to hold, the assumption that the derivative f' is Riemann integrable is required.

It would be desirable to have an integration theory for which Part I of the Fundamental Theorem of Calculus holds in full generality. We will see later that the derivative f' in Example 1.3.2 is also not Lebesgue integrable so a general version of the Fundamental Theorem of Calculus for the Lebesgue integral also requires an integrability assumption on the derivative. Bellow, we will construct an integral, called the gauge or Henstock-Kurzweil integral, for which the Fundamental Theorem of Calculus holds in full generality; that is, the Henstock-Kurzweil integral integrates all derivatives and (1.3.1) holds.

The second part of the Fundamental Theorem of Calculus concerns the differentiation of indefinite integrals. Suppose that $f : [\alpha, \beta] \to \mathbb{R}$ is Riemann integrable on $[\alpha, \beta]$.

We define the indefinite integral of f at $x \in [\alpha, \beta]$ by

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

Theorem 1.3.3. (*Fundamental Theorem of Calculus for Riemann Integral- Part II*) Suppose $f\colon [\alpha, \beta] \to \mathbb{R}$ is Riemann integrable on $[\alpha, \beta]$. Let $F(x) = \int_\alpha^x f(t)\mathrm{d}t$. Then F is continuous on $[\alpha, \beta]$. If f is continuous on $\xi \in [\alpha, \beta]$, then F is differentiable on ξ and $F'(\xi) = f(\xi)$.

The theorem tells us that F must be differentiable at points where f is continuous. If f is not continuous at a point, F may or may not be differentiable.

In 1902, Henri Lebesgue introduced a concept of integral that allowed results involving limits, namely the Convergence Theorems (Monotone, Dominated, and Vitali) and Fatou's Lemma. With this, he constructed a more sophisticated concept of integral that generalized the Riemann integral, but still required a certain continuity of functions, since in the Fundamental Theorem of Calculus for Lebesgue integrals, the derivative of a function f will be Lebesgue integrable if f is absolutely continuous.

Theorem 1.3.4. Let $f\colon [\alpha, \beta] \to \mathbb{R}$ be differentiable on $[\alpha, \beta]$ and suppose that f' is bounded. Then, f' is Lebesgue integrable on $[\alpha, \beta]$ and satisfies $\int_\alpha^\beta f' = f(\beta) - f(\alpha)$.

The example below shows that the general form of Part I of the Fundamental Theorem of Calculus does not hold for the Lebesgue integral.

Example 1.3.5. (Kurtz and Swartz 2004, Example 4.1) In Example 1.3.2, we consider the function f, and we will show that f' is not Lebesgue integral. If $0 < \alpha < \beta < 1$, then f' is continuous on $[\alpha, \beta]$ and which implies that it is Riemann Integrable with the integral given by

$$\int_\alpha^\beta f' = \beta^2 \cos\left(\frac{\pi}{\beta^2}\right) - \alpha^2 \cos\left(\frac{\pi}{\alpha^2}\right).$$

Defining $\beta_k = \frac{1}{\sqrt{2k}}$ and $\alpha_k = \sqrt{\frac{2}{4k+1}}$, we see that $\displaystyle\int_{\alpha_k}^{\beta_k} f' = \frac{1}{2k}$. Since the intervals $[\alpha_k, \beta_k]$ are pairwise disjoint, we get

$$\int_0^1 |f'| \geq \sum_{k=1}^\infty \int_{\alpha_k}^{\beta_k} |f'| \geq \sum_{k=1}^\infty \frac{1}{2k} = \infty.$$

Therefore, f' is not absolutely integrable on $[0, 1]$ and, hence, is not Lebesgue integrable.

In the first decades of the twentieth century, the mathematicians Arnaud Denjoy and Oskar Perron, presented a theory of integration for which Part I of the Fundamental Theorem of Calculus holds , i.e., an integral for which all derivatives are integrable. In 1912, Denjoy (1884-1974) introduced such an integration theory, see (Gordon 1994). The integral presented by Denjoy is very technical, and we will not present its definition here. However the interested reader may consult (Pesin 1970) and (Gordon 1994). Lusin later gave a more elementary characterization of the Denjoy integral, but this is still quite technical and therefore, we will not focus in it here.

Between the years 1957 and 1961, Jaroslav Kurzweil and Ralph Henstock independently introduced a new concept of integral. This concept was able to encompass discontinuous functions of unbounded variation, that is, highly oscillatory functions. These functions became known as Henstock-Kurzweil integrable. And it has the property that functions that are Henstock-Kurzweil integrable are not absolutely integrable but when this class of functions are absolutely integrable, then this integral coincides to the Lebesgue integrable functions. On the other hand, we point out that any absolutely integrable function is Henstock-Kurzweil integrable.

Next, we will describe some important integrals including the Henstock-Kurzweil.

1.3.1 Perron Integral

In 1914, O. Perron (1880-1975) presented another theory of integration for which the part that assumes that the derivative f' is integrable in some sense in the Fundamental Theorem of Calculus is valid. The definition of the Perron integral is quite different from that of the Denjoy integral, although Alexandrov and Looman (Pesin 1970, Chapter 9) later showed that in fact these two integrals are equivalent. We will give a brief description of the Perron integral, since some of the basic ideas will be used later when we prove the equivalence of Henstock-Kurzweil absolute integrability.

The first step in defining the Perron integral on a line is to introduce the notion of **minor** and **major functions**. These functions are defined using the **upper** and **lower** derivatives, which establish the notation for various limits of difference quotients. These derivatives are often more useful than the ordinary derivative because they are defined at each point.

Definition 1.3.6. Let $F\colon [\alpha,\beta] \to \mathbb{R}$. The **upper right** and **lower right derivatives** of F at $t \in [\alpha,\beta]$ are defined by

$$D^+F(t) = \lim_{\delta\to 0^+} \sup\left\{\frac{F(s)-F(t)}{s-t} : t < s < t+\delta\right\},$$

$$D_+F(t) = \lim_{\delta\to 0^+} \inf\left\{\frac{F(s)-F(t)}{s-t} : t < s < t+\delta\right\}.$$

Similarly, the **upper left** and **lower left derivatives** of F at $t \in [\alpha,\beta]$ are defined by

$$D^-F(t) = \lim_{\delta\to 0^+} \sup\left\{\frac{F(s)-F(t)}{s-t} : t-\delta < s < t\right\},$$

$$D_-F(t) = \lim_{\delta\to 0^+} \inf\left\{\frac{F(s)-F(t)}{s-t} : t-\delta < s < t\right\}.$$

The function F is **differentiable at** α or β if the corresponding two derivatives (right and left) are finite and equal. The function F is **differentiable at** $t \in (\alpha, \beta)$ if all four derivatives are finite and equal.

Finally, the **upper** and **lower derivatives** of F at $t \in [\alpha, \beta]$ are defined by

$$\overline{D}F(t) = \lim_{\delta\to 0^+} \sup\left\{\frac{F(s)-F(t)}{s-t} : 0 < |s-t| < \delta\right\} = \max\{D^+F(t), D^-F(t)\},$$

$$\underline{D}F(t) = \lim_{\delta\to 0^+} \inf\left\{\frac{F(s)-F(t)}{s-t} : 0 < |s-t| < \delta\right\} = \min\{D_+F(t), D_-F(t)\}.$$

Therefore, F is differentiable at t if and only if $\overline{D}F(t) = \underline{D}F(t)$ and both upper and lower derivatives are finite.

Definition 1.3.7. Let $f\colon [\alpha, \beta] \to \mathbb{R}$. A function $M\colon [\alpha, \beta] \to \mathbb{R}$ is called a **major function** for f if M is continuous on $[\alpha, \beta]$, $\underline{D}M(t) > -\infty$, and $\underline{D}M(t) \geq f(t)$ for all $t \in [\alpha, \beta]$. A function $m\colon [\alpha, \beta] \to \mathbb{R}$ is called a **minor function** for f if m is continuous on $[\alpha, \beta]$, $\overline{D}m(t) < +\infty$ and $\overline{D}m(t) \leq f(t)$ for all $t \in [\alpha, \beta]$.

Notation 1.3.8. We will often write M_α^β for $M(\beta) - M(\alpha)$ to simplify the notation.

It follows that if f is differentiable on $[\alpha, \beta]$ and has a finite-valued derivative, then $f - f(\alpha)$ is both a major and a minor function for f'. If M is a **major function** and m is a **minor function** for f, then $M - m$ is increasing. Therefore, let $\sup\{m_\alpha^\beta : m \text{ is a minor function for } f\} = a$ and $\inf\{M_\alpha^\beta : M \text{ is a major function for } f\} = b$, then

$$-\infty < a \leq b < \infty.$$

Definition 1.3.9. A function $f\colon [\alpha, \beta] \to \mathbb{R}$ is called **Perron integrable** over $[\alpha, \beta]$ if, and only if, f has at least one major and one minor function on $[\alpha, \beta]$ and

$$\sup\{m_\alpha^\beta : m \text{ is a minor function for } f\} = \inf\{M_\alpha^\beta : M \text{ is a major function for } f\}.$$

This common value is the Perron integral of f over $[\alpha, \beta]$ and will be denoted by $\int_\alpha^\beta f(s)\mathrm{d}s$. We will use (P) before the integral symbol when we refer to this integral.

The following theorem is an immediate consequence of the definition, and bring an equivalence for a function be Perron integrable function.

Theorem 1.3.10. (Gordon 1994, Theorem 8.6) A function $f\colon [\alpha, \beta] \to \mathbb{R}$ is Perron integrable on $[\alpha, \beta]$ if and only if for each $\varepsilon > 0$ there exist a major function M and a minor function m of f on $[\alpha, \beta]$ such that

$$M_\alpha^\beta - m_\alpha^\beta < \varepsilon.$$

We next consider the relationship between Perron integrability on $[\alpha, \beta]$ and the integrability on subintervals. The next result is expected due to the known theory of integration.

Theorem 1.3.11. (Gordon 1994, Theorem 8.8) Let $f\colon [\alpha, \beta] \to \mathbb{R}$ and let $c \in (\alpha, \beta)$.

(a) If f is Perron integrable on $[\alpha, \beta]$, then f is Perron integrable on every subinterval of $[\alpha, \beta]$.

(b) If f is Perron integrable on each of the intervals $[\alpha, \gamma]$ and $[\gamma, \beta]$, then f is Perron integrable on $[\alpha, \beta]$ and
$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f.$$

Proof. Let M and m be a major and a minor function of f on $[\alpha, \beta]$, respectively. It is clear that M and m are a major and a minor function of f on every subinterval of $[\alpha, \beta]$ as well, respectively. This facts imply easily the item (a).

We turn now to a proof of (b). Let $\varepsilon > 0$. Choose a major function M and minor function m of f on $[\alpha, \beta]$ such that $M_\gamma^\alpha - m_\gamma^\alpha < \varepsilon$, and choose a major function ${}_1M$ and minor function ${}_1m$ of f on $[\gamma, \beta]$ such that ${}_1M_\gamma^\beta - {}_1m_\gamma^\beta < \varepsilon$, and choose a major function ${}_2M$ and minor function ${}_2m$ of f on $[\gamma, \beta]$ such that ${}_2M(\gamma) = {}_1M(\gamma)$, ${}_2m(\gamma) = {}_1m(\gamma)$ and ${}_2M_\gamma^\beta - {}_2m_\gamma^\beta < \varepsilon$. Let

$$M(t) = \begin{cases} {}_1M(t), & \text{if } \alpha \le t \le \gamma; \\ {}_2M(t), & \text{if } \gamma < t \le \beta; \end{cases} \quad \text{and} \quad m(t) = \begin{cases} {}_1m(t), & \text{if } \alpha \le t \le \gamma; \\ {}_2m(t), & \text{if } \gamma < t \le \beta. \end{cases}$$

Then M is a major function and m is a minor function of f on $[\alpha, \beta]$ and

$$M_\beta^\alpha - m_\beta^\alpha = ({}_2M_\gamma^\beta -{}_2 m_\gamma^\beta) + ({}_1M_\gamma^\alpha -{}_1 m_\gamma^\alpha) < 2\varepsilon.$$

Hence, the function f is Perron integrable on $[\alpha, \beta]$ by Theorem 1.3.10. With the same notation, we observe that

$$\int_\alpha^\beta f < m_\alpha^\beta + 2\varepsilon = {}_1m_\gamma^\alpha + {}_2m_\gamma^\beta + 2\varepsilon \le \int_\alpha^\gamma f + \int_\gamma^\beta f + 4\varepsilon;$$

$$\int_\alpha^\beta f > M_\alpha^\beta - 2\varepsilon = {}_1M_\alpha^\gamma + {}_2M_\gamma^\beta - 2\varepsilon \ge \int_\alpha^\gamma f + \int_\gamma^\beta f - 4\varepsilon.$$

It follows that
$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f.$$

□

Theorem 1.3.12. (Gordon 1994, Theorem 8.9) If $f\colon [\alpha, \beta] \to \mathbb{R}$ is Perron integrable on $[\alpha, \beta]$, then f is finite-valued almost everywhere on $[\alpha, \beta]$.

Proof. Let $C = \{t \in [\alpha, \beta] : |f(t)| = +\infty\}$. Let M be a major and m be a minor functions of f on $[\alpha, \beta]$ respectively, and define $W = M - m$. The function W is increasing on $[\alpha, \beta]$, so the set

$$T = \{t \in [\alpha, \beta] : \underline{D}W(t) = +\infty\}$$

has measure zero. We will prove that $C \subseteq T$.

Let $c \in C$ and suppose that $f(c) = +\infty$. By the properties of major and minor functions, $\underline{D}M(c) = +\infty$ and $\overline{D}m(c) < +\infty$, and hence

$$\underline{D}W(c) \geq \underline{D}M(c) - \overline{D}m(c) = +\infty$$

by the properties of major and minor functions. If $f(c) = -\infty$, then $\underline{D}M(c) > -\infty$ and $\overline{D}m(c) = -\infty$, and once again $\underline{D}W(c) = +\infty$. Hence $c \in D$. This completes the proof. □

Theorem 1.3.13. Let $f\colon [\alpha, \beta] \to \mathbb{R}$ be Perron integrable on $[\alpha, \beta]$. If $g = f$ almost everywhere on $[\alpha, \beta]$, then g is Perron integrable on $[\alpha, \beta]$, and

$$\int_\alpha^\beta g(s)\mathrm{d}s = \int_\alpha^\beta f(s)\mathrm{d}s.$$

Proof. See (Gordon 1994, Theorem 8.10). □

Theorem 1.3.14. (Gordon 1994, Theorem 8.11) Suppose $f, g\colon [\alpha, \beta] \to \Re$ are Perron integrable on $[\alpha, \beta]$. Then:

(a) kf is Perron integrable for each $k \in \mathbb{R}$, and $\int_\alpha^\beta kf = k\int_\alpha^\beta f$.

(b) $f + g$ is Perron integrable, and $\int_\alpha^\beta (f+g) = \int_\alpha^\beta f + \int_\alpha^\beta g$.

(c) If $f \leq g$ almost everywhere, then $\int_\alpha^\beta f \leq \int_\alpha^\beta g$.

(d) If $f = g$ almost everywhere, then $\int_\alpha^\beta f = \int_\alpha^\beta g$.

The fact that a derivative (when it exists at every point and is finite) is Perron-integrable follows directly from the fact that F is both a larger and a smaller function of its derivative F'. This holds only if F is differentiable at every point in $[\alpha, \beta]$. The corresponding theorem for the Denjoy integral required only that F' is differentiable at almost every point in $[\alpha, \beta]$. See (Kurtz and Swartz 2004, Theorem 8.26), where the authors prove that a stronger version holds for the Perron integral.

Theorem 1.3.15. Let $F\colon [\alpha, \beta] \to \mathbb{R}$ a function. If F is differentiable on $[\alpha, \beta]$, then F' is Perron integrable on $[\alpha, \beta]$ and $\int_\alpha^t F' = F(t) - F(\alpha)$ for each $t \in [\alpha, \beta]$.

The Perron integral just presented in this section is a generalization of the Lebesgue integral; it recovers a continuous function from its derivative. In the next section, we develop a generalization of the Riemann integral that solves the same problem. This integral, known as the Henstock-Kurzweil integral, is also a generalization of the Lebesgue integral.

1.3.2 Henstock-Kurzweil integral

The definition of the Henstock-Kurzweil integral is closely related to the Riemann integral. As we know, there are two approaches to the Riemann integral. The first is through the step function for this integral, which is called the Darboux definition. The other common method of defining the Riemann integral is through Riemann sums. To define the Henstock-Kurzweil integral, we will use the second approach.

Let $D = (\tau_i, [s_{i-1}, s_i])$ be a tagged partition of the fixed $[\alpha, \beta] \subset \mathbb{R}$, and $f\colon [\alpha, \beta] \to \mathbb{R}$ be a function.

Remark 1.3.16. We can denote $D = \{(\tau_i, I_i)\colon i = 1, \ldots, |D|\}$, where I_i is a closed subinterval $[s_{i-1}, s_i]$ of $[\alpha, \beta]$, $\tau_i \in I_i$. Observe that $\bigcup_{i=1}^{|D|} I_i = [\alpha, \beta]$ and the intervals have disjoint interiors, $\mathring{I}_i \cap \mathring{I}_j = \emptyset$ if $i \neq j$.

Definition 1.3.17. A **gauge** in $I = [\alpha, \beta]$ is any function $\delta\colon [\alpha, \beta] \to (0, \infty)$. Given a gauge δ on $[\alpha, \beta]$, we say that a tagged partition $D = (\tau_i, I_i)$ is $\delta-$**fine** if for all $i \in \{1, 2, \ldots, |D|\}$, we have

$$I_i \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)).$$

Remark 1.3.18. If δ is a constant, or more generally if δ is bounded away from 0, then the collection of tagged partitions subordinate to δ is not different to the collection of tagged partitions used in the definition of the Riemann integral.

Example 1.3.19. For a given gauge and a partition of the interval, not every tag will make the tagged partition δ–fine. In fact, let the function gauge $\delta\colon [0, 1] \to (0, +\infty)$ be given by

$$\delta(t) = \begin{cases} t^2, & \text{if } 0 < t \leq 1 \\ \frac{1}{2}, & \text{if } t = 0. \end{cases}$$

Let $D = \left\{\left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right]\right\}$ a division of $[0, 1]$.

The tagged partition $D_1 = \left\{\left(0, \left[0, \frac{1}{3}\right]\right), \left(\frac{1}{3}, \left[\frac{1}{2}, \frac{1}{3}\right]\right), \left(1, \left[\frac{1}{2}, 1\right]\right)\right\}$ is δ*–fine*. But, $D_2 = \left\{\left(\frac{1}{5}, \left[0, \frac{1}{3}\right]\right), \left(\frac{1}{2}, \left[\frac{1}{3}, \frac{1}{2}\right]\right), \left(1, \left[\frac{1}{2}, 1\right]\right)\right\}$ it is not δ–fine, since $\left[0, \frac{1}{3}\right] \not\subseteq \left(\frac{1}{5} - \delta\left(\frac{1}{5}\right), \frac{1}{5} + \delta\left(\frac{1}{5}\right)\right)$.

The next lemma, known as Cusin's Lemma, guarantees the existence of a $\delta-$fine tagged partition of $[\alpha, \beta]$.

Lemma 1.3.20. If $I = [\alpha, \beta]$ is a nondegenerate compact interval in $\mathbb{R}$ and δ is a gauge on I, then there exists a tagged partition of I that is δ–fine.

Proof. A proof this result can be found in (Lee 2011, Chapter 2, Lemma 1.1.5). □

Next, we will introduce the Henstock-Kurzweil integral and use this construction to prove that the Dirichlet function is Henstock-Kurzweil integrable.

If $D = \{(\tau_i, I_i) \colon i = 1, \cdots, |D|\}$ is a tagged partition of $[\alpha, \beta]$, we call

$$S(f, D) = \sum_{i=1}^{|D|} f(\tau_i)(s_i - s_{i-1})$$

the **Riemann sum** with respect to D.

Definition 1.3.21. A function $f \colon [\alpha, \beta] \to \mathbb{R}$ is **Henstock–Kurzweil integrable** on $[\alpha, \beta]$ if there is an element $A \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is a gauge $\delta \colon [\alpha, \beta] \to (0, \infty)$ such that

$$|S(f, D) - A| = \left| \sum_{i=1}^{|D|} [f(\tau_i)(s_i - s_{i-1})] - A \right| < \varepsilon$$

for all δ–fine tagged partition of $[\alpha, \beta]$. In this case, A is called the Henstock–Kurzweil integral of f over $[\alpha, \beta]$ and it will be denoted by $A = \int_\alpha^\beta f$. We will denote by $\mathcal{K}([\alpha, \beta], \mathbb{R})$ the space of functions from $[\alpha, \beta]$ to $\mathbb{R}$ that are Kurzweil-Henstock integrable.

However, there are Henstock-Kurzweil integrable functions that are not Riemann integrable. In fact, the Dirichlet function is one such examples, that were taken from the book (Kurtz and Swartz 2004, Example 4.15).

Example 1.3.22. Let $f : [0, 1] \to \mathbb{R}$ be the Dirichlet function.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } x \notin \mathbb{Q} \cap [0, 1]. \end{cases}$$

We will show that $\int_0^1 f = 0$. Let $\varepsilon > 0$. This function is equal to 0 almost everywhere. Let $\{r_i\}_{i=1}^{\infty}$ be an enumeration of the rational numbers in $\mathbb{Q} \cap [0, 1]$, and let $c > 0$, and define the gauge $\delta : [0, 1] \to (0, \infty)$ by

$$\delta(t) = \begin{cases} c & \text{if } t \notin \mathbb{Q} \cap [0, 1] \\ 2^{-i} c & \text{if } t = r_i \in \mathbb{Q} \cap [0, 1] \end{cases}$$

with $c = \frac{\varepsilon}{4}$. Let $D = \{(\tau_i, I_i) : i = 1, \ldots, |D|\}$ be a δ-fine tagged partition of $[0, 1]$ and note that

$$|S(f, D) - 0| = |S(f, D)| = \left| \sum_{i=1}^{|D|} f(\tau_i)(s_i - s_{i-1}) \right|$$

$$= \left| \sum_{\substack{(\tau_i, I_i) \in D \\ \tau_i \in \mathbb{Q} \cap [0,1]}} f(\tau_i)(s_i - s_{i-1}) + \sum_{\substack{(\tau_i, I_i) \in D \\ \tau_i \notin \mathbb{Q} \cap [0,1]}} f(\tau_i)(s_i - s_{i-1}) \right|$$

The sum for $\tau_i \notin \mathbb{Q} \cap [0,1]$ equals 0 since $f(t) = 0$ whenever $t \notin \mathbb{Q} \cap [0,1]$. To estimate the sum for $\tau_i \in \mathbb{Q} \cap [0,1]$, note that $f(\tau_i) = 1$ since $\tau_i \in \mathbb{Q} \cap [0,1]$ and recall that each tag τ_i can be a tag for at most two intervals. Since $\tau_i \in \mathbb{Q} \cap [0,1]$, there is a j so that $\tau_i = r_j$. Thus, if $(\tau_i, I_i) \in D$, then $I_i \subseteq (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$, such that

$$s_i - s_{i-1} \leq \tau_i + \delta(\tau_i) - (\tau_i - \delta(\tau_i)) = 2\delta(\tau_i) = r_j + \delta(r_j) - (r_j - \delta(r_j)) = 2^{-j+1}\frac{\varepsilon}{4}.$$

Thus,

$$\left| \sum_{\substack{i \\ \tau_i \notin \mathbb{Q} \cap [0,1]}} f(\tau_i)(s_i - s_{i-1}) + \sum_{\substack{i \\ \tau_i \notin \mathbb{Q} \cap [0,1]}} f(\tau_i)(s_i - s_{i-1}) \right| = \left| \sum_{\substack{i \\ \tau_i \in \mathbb{Q} \cap [0,1]}} f(\tau_i)(s_i - s_{i-1}) \right|$$

$$\leq \sum_{j=1}^{\infty} 2 \cdot 2^{1-j} \frac{\varepsilon}{4} = \varepsilon$$

We have shown that given any $\varepsilon > 0$, there is a gauge δ such that for any δ–fine tagged partition D, $|S(f, D) - 0| < \varepsilon$. In other words, the Dirichlet function is Henstock-Kurzweil integrable over $[0,1]$ with $\int_0^1 f(x)\mathrm{d}x = 0$.

Below, we present versions of the Fundamental Theorem of Calculus for Henstock–Kurzweil integral, as well as basic properties of these integrals. To demonstrate its version, we need Straddle's lemma. A proof of this result can be found in (Kurtz and Swartz 2004), Lemma 4.6.

Lemma 1.3.23 (Straddle Lemma)**.** Let $f\colon [\alpha, \beta] \to \mathbb{R}$ be differentiable at $y \in [\alpha, \beta]$. For each $\varepsilon > 0$, there is a $\delta > 0$, depending on y, such that

$$|f(v) - f(u) - f'(y)(v - u)| \leq \varepsilon(v - u)$$

whenever $u, v \in [\alpha, \beta]$ and $y - \delta < u \leq y \leq v < y + \delta$.

Theorem 1.3.24 (Fundamental Theorem of Calculus)**.** Suppose that $f\colon [\alpha, \beta] \to \mathbb{R}$ is differentiable on $[\alpha, \beta]$. Then, f' is Henstock-Kurzweil integrable on $[\alpha, \beta]$ and

$$\int_\alpha^\beta f'(s)\,\mathrm{d}s = f(\beta) - f(\alpha). \tag{1.3.4}$$

Proof. Fix $\varepsilon > 0$. For each $t \in [\alpha, \beta]$, choose $\delta > 0$ using the Straddle Lemma, and define a gauge $\delta(t) = \delta > 0$ on $[\alpha, \beta]$. Let $D = \{(\tau_i, I_i) : i = 1, \ldots, |D|\}$ be a δ-fine tagged partition of $[\alpha, \beta]$, where each $I_i = [s_{i-1}, s_i]$ and $\tau_i \in I_i$. Then,

$$f(\beta) - f(\alpha) = \sum_{i=1}^{m} [f(s_i) - f(s_{i-1})].$$

By Straddle Lemma, we have

$$|f(s_i) - f(s_{i-1}) - f'(\tau_i)(s_i - s_{i-1})| \leq \varepsilon(s_i - s_{i-1}),$$

so

$$\left|\sum_{i=1}^{m} f'(\tau_i)(s_i - s_{i-1}) - (f(\beta) - f(\alpha))\right| \leq \sum_{i=1}^{m} \varepsilon(s_i - s_{i-1}) = \varepsilon(\beta - \alpha).$$

Thus, f' is Henstock–Kurzweil integrable and satisfies the equation (1.3.4). □

Now, we are ready to prove that the derivative of the function in the Example 1.3.2 is Henstock–Kurzweil integrable.

Example 1.3.25. Let

$$f'(t) = \begin{cases} 2t\cos\left(\frac{\pi}{t^2}\right) + \frac{2\pi}{t}\sin\left(\frac{\pi}{t^2}\right) & \text{if } 0 < t \leq 1 \\ 0 & \text{if } t = 0, \end{cases}$$

where

$$f(t) = \begin{cases} t^2 \cos\left(\dfrac{\pi}{t^2}\right) & \text{if} \quad 0 < t \leq 1 \\ 0 \quad \text{if} \quad t = 0. \end{cases}$$

By Theorem 1.3.24, it follows that

$$\int_0^1 f'(s)\mathrm{d}s = f(1) - f(0) = -1.$$

Thus, f' is Henstock-Kurzweil integrable. However, from Examples 1.3.2 and 1.3.5, it is neither Riemann nor Lebesgue integrable.

Remark 1.3.26. The **improper Riemann integral** is an extension of the notion of a definite integral to cases that violate the usual assumptions for this type of integral, which involve non-boundedness either of the set over which the integral is taken or of the integrand, or both. It can also involve bounded but not closed sets, or bounded but not continuous functions. The improper integral represents a limit of a definite integral or a sum of such limits; therefore, improper integrals are said to be **convergent** or **divergent**.

Remark 1.3.27. If a regular definite integral, also called a Riemannian proper integral, is evaluated as improper, the same answer will result. Thus, proper integrals are contained within improper integrals.

We can calculate the improper Riemann integral from the previous example.

Example 1.3.28. Note that the function in Example 1.3.25 has an improper Riemann integral. In fact, when $t \to 0^+$, $2t\cos\left(\frac{\pi}{t^2}\right)$ is bounded in modulus by $2t \to 0$ while $\frac{2\pi}{t}\sin\left(\frac{\pi}{t^2}\right)$ oscillates between $-\frac{2\pi}{t}$ and $\frac{2\pi}{t}$ and this is unbounded (tends to $\pm\infty$ in magnitude). Calculating the improper Riemann integral we obtain

$$\int_0^1 f'(s)\mathrm{d}s = \lim_{a\to 0^+} \int_a^1 f'(s)\mathrm{d}s.$$

By the Fundamental Theorem of Calculus in $[a,1]$,

$$\begin{aligned}\lim_{a\to 0^+}\int_a^1 f'(s)\mathrm{d}s &= \lim_{a\to 0^+} f(1)-f(a)\\ &= \lim_{a\to 0^+}\left[-1-a^2\cos\left(\frac{\pi}{a^2}\right)\right]\\ &= -1.\end{aligned}$$

Which is equal to the Henstock-Kurzweil integral.

Remark 1.3.29. Not every function that is Henstock-Kurzweil integrable has a Riemann improper integral.

The basic properties of the Henstock-Kurzweil integrals work similarly to the Riemann integral, with only slight changes in the formal definition, such as of the gauge is a positive function function and not necessarily constant. We have the equivalence that if a function is Riemann integrable, then it is Henstock-Kurzweil integrable; however, from the examples presented above, the converse is not true, which is also extended to the Lebesgue integral: a Lebesgue integral function is H-K integral, but the converse is not true. Furthermore, if the Henstock-Kurzweil integral of a function exists, it is unique, and linear.

While the possibility to integrate all derivatives is an advantage of the Henstock-Kurzweil integral, we have an even better condition, which enlarges more our set of integrable functions. The derivative may not exist at a countable number of points and yet the integral may still exist.

Theorem 1.3.30 (Generalized Fundamental Theorem of Calculus)**.** Let $F, f\colon [\alpha,\beta]\to\mathbb{R}$. Suppose that F is continuous and $F'=f$ except for possibly a countable number of points in $[\alpha,\beta]$. Then, f is Henstock-Kurzweil integrable over $[\alpha,\beta]$ and

$$\int_\alpha^\beta f = F(\beta)-F(\alpha).$$

Proof. See (Kurtz and Swartz 2004), Theorem 4.24 □

Using Theorem 1.3.30, we can prove a general form of the well-known Integration by Parts Formula.

Theorem 1.3.31. (Lee 2011) Let $F, G, f, g\colon [\alpha,\beta]\to\mathbb{R}$. Suppose that F and G are continuous and $F'=f$ and $G'=g$, except for at most a countable number of points. Then, $Fg+fG$ is Henstock-Kurzweil integrable and

$$\int_\alpha^\beta (Fg+fG) = F(\beta)G(\beta)-F(\alpha)G(\alpha).$$

Moreover, Fg is Henstock-Kurzweil integrable if, and only if, fG is Henstock-Kurzweil integrable and, in this case,

$$\int_\alpha^\beta Fg + \int_\alpha^\beta fG = F(\beta)G(\beta) - F(\alpha)G(\alpha).$$

The next result is the well-known Saks-Henstock lemma, which is frequently used in the theory of the Henstock-Kurzweil integral. This lemma was first noted by Henstock in his initial explanation of this integral, but he refers to Saks for the idea behind it. Therefore, we will call this result as Saks-Henstock lemma. The lemma states that Riemann sums not only approximate the integral over the entire interval, but also over unions of subintervals. We will use it to show the equivalence between the Perron and Henstock-Kurweil integrals. The proof can be found at (Kurtz and Swartz 2004, Gordon 1994).

Lemma 1.3.32 (Saks-Henstock lemma)**.** Let $f\colon [\alpha,\beta] \to \mathbb{R}$ be Henstock-Kurzweil integrable on $[\alpha,\beta]$, let $F(t) = \int_\alpha^t f$ for each $t \in [\alpha,\beta]$, and let $\varepsilon > 0$. Suppose that δ is a gauge on $[\alpha,\beta]$ such that $|S(f,P) - S(F,P)| < \varepsilon$ whenever P is a tagged partition of $[\alpha,\beta]$ that is $\delta-$fine. If $P_0 = \{(t_i,[c_i,d_i]) : 1 \leq i \leq n\}$ is $\delta-$fine, then

$$|S(f,P_0) - S(F,P_0)| \leq \varepsilon$$

and

$$\sum_{i=1}^{n} |f(t_i)(d_i - c_i) - (F(d_i) - F(c_i))| \leq 2\varepsilon.$$

In what follows, we will demonstrate the equivalence between the Perron and Henstock-Kurzweil integrals, the proof of the result can be found in (Gordon 1994, Kurtz and Swartz 2004, Monteiro, Slavík and Tvdý 2019).

Theorem 1.3.33. Let $f\colon [\alpha,\beta] \to \mathbb{R}$ be a function. Then f is Perron integrable on $[\alpha,\beta]$ if and only if f is Henstock-Kurzweil integrable on $[\alpha,\beta]$, and the integrals are equal.

Proof. Let $\varepsilon > 0$. By definition, there exist a major function M and a minor function m for f on $[\alpha,\beta]$ such that

$$-\varepsilon < m_\alpha^\beta - (P)\int_\alpha^\beta f \leq 0 \leq M_\alpha^\beta - (P)\int_\alpha^\beta f < \varepsilon.$$

Since $\overline{D}m \leq f \leq \underline{D}M$ on $[\alpha,\beta]$, for each $t \in [\alpha,\beta]$, there exists $\delta(t) > 0$ such that

$$\frac{M(s) - M(t)}{s - t} \geq f(t) - \varepsilon \quad \text{and} \quad \frac{m(s) - m(t)}{s - t} \leq f(t) + \varepsilon,$$

whenever $0 < |s - t| < \delta(t)$ and $s \in [\alpha,\beta]$. Now suppose

$$P = \{(t_i,[c_i,d_i]) : 1 \leq i \leq q\}$$

is a $\delta-$fine tagged partition of $[\alpha, \beta]$, and compute

$$\begin{aligned}\sum_{i=1}^{q} f(t_i)(d_i - c_i) - (P)\int_\alpha^\beta f &= \sum_{i=1}^{q}\Big(f(t_i)(d_i - c_i) - M_{c_i}^{d_i}\Big) + M_\alpha^\beta - (P)\int_\alpha^\beta f \\ &< \sum_{i=1}^{q}\Big(f(t_i)(d_i - t_i) - M_{t_i}^{d_i}\Big) + \varepsilon \\ &= \sum_{i=1}^{q}\Big(f(t_i)(t_i - c_i) - M_{c_i}^{t_i}\Big) + \varepsilon \\ &\leq \sum_{i=1}^{q}(d_i - t_i) + \sum_{i=1}^{q}(t_i - c_i) + \varepsilon = \varepsilon(\beta - \alpha + 1).\end{aligned}$$

Similarly, using the minor function m,

$$\sum_{i=1}^{q} f(t_i)(d_i - c_i) - (P)\int_\alpha^\beta f > -\varepsilon(\beta - \alpha + 1).$$

It follows that f is Henstock–Kurzweil integrable on $[\alpha, \beta]$, and it coincides to the Perron integral

$$\int_\alpha^\beta f = (P)\int_\alpha^\beta f.$$

Reciprocally, suppose f is Henstock-Kurzweil integrable. Let $\varepsilon > 0$. By definition, there exists a gauge δ on $[\alpha, \beta]$ such that

$$|S(f, P) - \int_\alpha^\beta f| < \varepsilon$$

whenever P is a tagged partition δ-fine of $[\alpha, \beta]$.

For each $t \in (\alpha, \beta]$, define:

$$M(t) = \sup\{S(f, P) : P \text{ is a } \delta\text{–fine tagged partition of } [\alpha, t]\},$$

$$m(t) = \inf\{S(f, P) : P \text{ is a } \delta\text{–fine tagged partition of } [\alpha, t]\},$$

and let $M(\alpha) = 0 = m(\alpha)$.

By Saks-Henstock Lemma 1.3.32, the functions M and m are finite-valued on $[\alpha, \beta]$.

We claim that M is a major function of f on $[\alpha, \beta]$ and m is a minor function of f on $[\alpha, \beta]$. Since $|S(f, P_1) - S(f, P_2)| < 2\varepsilon$ for any two $\delta-$fine tagged partition P_1 and P_2 of $[\alpha, \beta]$, it follows that $M_\alpha^\beta - m_\alpha^\beta < 2\varepsilon$. By Cauchy criterion for the Perron integral [see (Kurtz and Swartz 2004, Subsec 4.3.1)], the function f is Perron-integrable on $[\alpha, \beta]$. This completes the proof. □

1.3.3 Stieltjes-type integral

The problems of this thesis involve integrals of the **Stieltjes type**. We consider integrals of the form $\int_\alpha^\beta f \mathrm{d}g$ in which a function $f\colon [\alpha, \beta] \longrightarrow \mathbb{R}$ (integrand) is integrated with respect to another function $g\colon [\alpha,\beta] \longrightarrow \mathbb{R}$ (integrator). The simplest integral of this type is the Riemann-Stieltjes integral, whose definition is based on integral sums having the form

$$\sum_{i=1}^{|D|} f(\tau_i)(g(s_i) - g(s_{i-1})),$$

where $\alpha = s_0 < s_1 < \cdots < s_{|D|} = \beta$ and $\tau_i \in [s_{i-1}, s_i]$ for each $i \in \{1, \cdots, |D|\}$. Such integrals appeared for the first time in a famous work (Stieltjes 1993) by T. J. Stieltjes.

Consider functions $f, g\colon [\alpha,\beta] \to \mathbb{R}$, a gauge $\delta\colon [\alpha,\beta] \to (0,\infty)$ and a $\delta-$fine tagged partition $D = \{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, |D|\}$, i.e

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$$

where $\alpha = s_0 < s_1 < \ldots < s_{|D|} = \beta$ is a partition of $[\alpha,\beta]$. We use the notation

$$S(f, \mathrm{d}g, D) = \sum_{i=1}^{|D|} f(\tau_i)(g(s_i) - g(s_{i-1}))$$

Definition 1.3.34. A function $f\colon [\alpha,\beta] \to \mathbb{R}$ is **Henstock–Kurzweil–Stieltjes integrable** on $[\alpha,\beta]$ with respect to a function $g\colon [\alpha,\beta] \to \mathbb{R}$, if there is an element $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is a gauge $\delta\colon [\alpha,\beta] \to (0,\infty)$ such that

$$\left| \sum_{i=1}^{|D|} f(\tau_i)\left(g(s_i) - g(s_{i-1})\right) - I \right| < \varepsilon$$

for all δ–fine tagged partition of $[\alpha,\beta]$. In this case, I is called the **Henstock–Kurzweil–Stieltjes** integral of f with respect to g over $[\alpha,\beta]$ and it will be denoted by $\int_\alpha^\beta f(s)\,\mathrm{d}g(s)$.

Lemma 1.3.35. If the integral $\int_\alpha^\beta f(s)\,\mathrm{d}g(s)$ exists, then its value is uniquely determined.

The classical properties of linearity, additivity with respect to adjacent intervals and integrability on subintervals are all valid for the Henstock-Kurzweil–Stieltjes integral. For more details, the reader can consult (Bonotto, Federson and Mesquita 2021, Kurtz and Swartz 2004, Monteiro, Slavík and Tvdý 2019).

Next, we provide some sufficient conditions for the existence of the Henstock-Kurzweil-Stieltjes integral. The existence result states that $\int_\alpha^\beta f\mathrm{d}g$ exists if f and g are regulated and one of them has bounded variation. Since regulated functions can be uniformly approximated by finite step functions, we will begin with the simple case in which one of the functions f or g is a finite step function.

Remark 1.3.36. Observe that if f is a constant function on $[\alpha,\beta]$, then

$$\int_\alpha^\beta f\mathrm{d}g = f(\alpha)(g(\beta) - g(\alpha)) \text{ and } \int_\alpha^\beta g\mathrm{d}f = 0$$

for every functions $f, g\colon [\alpha,\beta] \to \mathbb{R}$.

Lemma 1.3.37. Let $\chi_{[\alpha,\beta]}$ be the characteristic function

$$\chi_{[\alpha,\beta]}(t) = \begin{cases} 1, & \text{if } t \in [\alpha,\beta] \\ 0, & \text{otherwise.} \end{cases}$$

Then the following relations hold for any function $f : [\alpha,\beta] \to \mathbb{R}$:

1. $\displaystyle\int_\alpha^\beta f \, \mathrm{d}\chi_{(\tau,\beta]} = f(\tau), \quad \text{if} \quad \tau \in [\alpha,\beta);$

2. $\displaystyle\int_\alpha^\beta f \, \mathrm{d}\chi_{[\tau,\beta]} = f(\tau), \quad \text{if} \quad \tau \in (\alpha,\beta];$

3. $\displaystyle\int_\alpha^\beta f \, \mathrm{d}\chi_{[\alpha,\tau]} = -f(\tau), \quad \text{if} \quad \tau \in [\alpha,\beta);$

4. $\displaystyle\int_\alpha^\beta f \, \mathrm{d}\chi_{[\alpha,\tau)} = -f(\tau), \quad \text{if} \quad \tau \in (\alpha,\beta];$

5. $\displaystyle\int_\alpha^\beta f \, \mathrm{d}\chi_{[\tau]} = 0, \quad \text{if} \quad \tau \in (\alpha,\beta).$

Proof. See (Monteiro, Slavík and Tvdý 2019, Lemma 6.3.2) □

Lemma 1.3.38. The following relations hold for every regulated function $g\colon [\alpha,\beta] \to \mathbb{R}$:

1. $\displaystyle\int_\alpha^\beta \chi_{[\tau,\beta]} \, \mathrm{d}g = g(\beta) - g(\tau^+) \qquad \text{if } \tau \in [\alpha,\beta]$

2. $\displaystyle\int_\alpha^\beta \chi_{[\tau,\beta]} \, \mathrm{d}g = g(\beta) - g(\tau^-) \qquad \text{if } \tau \in (\alpha,\beta]$

3. $\displaystyle\int_\alpha^\beta \chi_{[\alpha,\tau]} \, \mathrm{d}g = g(\tau^+) - g(\alpha) \qquad \text{if } \tau \in [\alpha,\beta)$

4. $\displaystyle\int_\alpha^\beta \chi_{[\alpha,\tau]} \, \mathrm{d}g = g(\tau^-) - g(\alpha) \qquad \text{if } \tau \in (\alpha,\beta)$

5. $\displaystyle\int_\alpha^\beta \chi_{[\tau]} \, \mathrm{d}g = \Delta g(\tau) \qquad \text{if } \tau \in (\alpha,\beta)$

Proof. See (Monteiro, Slavík and Tvdý 2019, Lemma 6.3.3). □

Since each finite step function on $[\alpha,\beta]$ is a linear combination of functions of the form $\chi_{(\tau,\beta]}, \chi_{[\tau,\beta]}, \chi_{[\beta]}$ with $\tau \in [\alpha,\beta]$, by the Lemmas 1.3.37 and 1.3.38 we can show the following statement.

Theorem 1.3.39. Let $f, g\colon [\alpha,\beta] \to \mathbb{R}$ satisfy one of the following conditions:

(i) g is a finite step function.

(ii) f is a finite step function and g is regulated.

Then the integral $\displaystyle\int_\alpha^\beta f\,\mathrm{d}g$ exists.

Proof. See (Monteiro, Slavík and Tvdý 2019, Corollary 6.3.4) in the sense of Henstock-Kurzweil-Stieltjes. □

The next three theorems provide us with the basic estimates for Henstock-Kurzweil-Stieltjes integrals under the assumption that these integrals exist. The main goal below is to show that if f, g are regulated and one of them has bounded variation, then the integral exists.

Theorem 1.3.40. (Monteiro, Slavík and Tvdý 2019, Theorem 6.3.6) Let $f, g\colon [\alpha,\beta] \to \mathbb{R}$. Then

$$|S(f, \mathrm{d}g, D)| \leq \|f\|_\infty \cdot \mathrm{var}_\alpha^\beta g$$

holds for every tagged partition D of $[\alpha,\beta]$. If the integral $\int_\alpha^\beta f\,\mathrm{d}g$ exists, then:

$$\left|\int_\alpha^\beta f\,\mathrm{d}g\right| \leq \|f\|_\infty \cdot \mathrm{var}_\alpha^\beta\, g. \tag{1.3.5}$$

If the integral $\displaystyle\int_\alpha^\beta f(s)\,d(\mathrm{var}_\alpha^s\, g)$ also exists, then

$$\left|\int_\alpha^\beta f\,\mathrm{d}g\right| \leq \int_\alpha^\beta |f(s)|\,\mathrm{d}(\mathrm{var}_\alpha^s g) \leq \|f\|_\infty \cdot \mathrm{var}_\alpha^\beta\, g. \tag{1.3.6}$$

Proof. For every tagged partition $D = \{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, |D|\}$ of $[\alpha,\beta]$, we have:

$$\begin{aligned} |S(f, \mathrm{d}g, D)| &\leq \sum_{i=1}^{|D|} |f(\tau_i)|\,|g(s_i) - g(s_{i-1})| \\ &\leq \sum_{i=1}^{|D|} |f(\tau_i)| \left(\mathrm{var}_{s_{i-1}}^{s_i} g\right) \leq \|f\|_\infty \cdot \mathrm{var}_\alpha^\beta\, g. \end{aligned}$$

The estimates for the integral follow from the previous inequality. □

Theorem 1.3.41. (Monteiro, Slavík and Tvdý 2019, Theorem 6.3.8) Let $f, g\colon [\alpha,\beta] \to \mathbb{R}$. Then

$$|S(f, \mathrm{d}g, D)| \leq \left(|f(\alpha)| + |f(\beta)| + \mathrm{var}_\alpha^\beta\, f\right) \|g\|_\infty \leq 2\|f\|_{\mathrm{BV}} \cdot \|g\|_\infty$$

for every tagged partition D of $[\alpha,\beta]$. If the integral $\int_\alpha^\beta f\,\mathrm{d}g$ exists, then

$$\left|\int_\alpha^\beta f\,\mathrm{d}g\right| \leq \left(|f(\alpha)| + |f(\beta)| + \mathrm{var}_\alpha^\beta\, f\right) \|g\|_\infty \leq 2\|f\|_{\mathrm{BV}} \cdot \|g\|_\infty.$$

Proof. Consider an arbitrary tagged partition $D = \{(\tau_i, [s_{i-1}, s_i]), i = 1, \dots, m\}$ of $[\alpha,\beta]$ and let $m = |D|$, with $s_0 = \alpha$, $s_m = \beta$. Then:

$$\begin{aligned} S(f, \mathrm{d}g, D) &= f(\tau_1)(g(s_1) - g(\alpha)) + f(\tau_2)(g(s_2) - g(s_1)) + f(\tau_3)(g(s_3) - g(s_2)) \\ &\quad + \ldots + f(\tau_m)(g(\beta) - g(s_{m-1})) \\ &= f(\beta)g(\beta) - f(\alpha)g(\alpha) - (f(\tau_1) - f(\alpha))g(\alpha) \\ &\quad - (f(\tau_2) - f(\tau_1))g(s_1) - \ldots - (f(\beta) - f(\tau_m))g(\beta) \\ &= f(\beta)g(\beta) - f(\alpha)g(\alpha) - \sum_{i=1}^{m}(f(\tau_i) - f(\tau_{i-1}))g(s_{i-1}). \end{aligned}$$

This implies that

$$\begin{aligned} |S(f, \mathrm{d}g, D)| &\leq |f(\beta)g(\beta) - f(\alpha)g(\alpha)| + \sum_{i=1}^{m} |(f(\tau_i) - f(\tau_{i-1}))g(s_{i-1})| \\ &\leq |f(\beta)g(\beta)| + |f(\alpha)g(\alpha)| + \sum_{i=1}^{m} |f(\tau_i) - f(\tau_{i-1})||g(s_{i-1})| \\ &\leq |f(\beta)||g(\beta)| + |f(\alpha)||g(\alpha)| + \mathrm{var}_\alpha^\beta f \cdot \|g\|_\infty \\ &\leq (|f(\beta)| + |f(\alpha)|)\|g\|_\infty + \mathrm{var}_\alpha^\beta f \cdot \|g\|_\infty = (|f(\beta)| + |f(\alpha)| + \mathrm{var}_\alpha^\beta f) \cdot \|g\|_\infty \end{aligned}$$

since $\|f\|_{BV} = |f(\alpha)| + \mathrm{var}_\alpha^\beta(f)$. Then, the theorem follows immediately. □

Theorems 1.3.40 and 1.3.41 allow us to prove two convergence results for the Henstock-Kurzweil-Stieltjes integral.

Theorem 1.3.42. Let $f, g\colon [\alpha, \beta] \to \mathbb{R}$ and $f_n : [\alpha, \beta] \to \mathbb{R}$, $n \in \mathbb{N}$, be such that the integrals $\int_\alpha^\beta f_n \,\mathrm{d}g$ exist for all $n \in \mathbb{N}$. Suppose at least one of the following conditions holds:

(i) $g \in BV([\alpha, \beta])$ and $f_n \rightrightarrows f$,

(ii) g is bounded and $\lim_{n\to\infty} \|f_n - f\|_{BV} = 0$.

Then the integral $\int_\alpha^\beta f \,\mathrm{d}g$ also exists, and

$$\lim_{n\to\infty} \int_\alpha^\beta f_n \,\mathrm{d}g = \int_\alpha^\beta f \,\mathrm{d}g.$$

Proof. See (Monteiro, Slavík and Tvdý 2019, Theorem 6.3.8). □

Theorem 1.3.43. Let $f, g\colon [\alpha, \beta] \to \mathbb{R}$ and $g_n : [\alpha, \beta] \to \mathbb{R}$, $n \in \mathbb{N}$, be such that the integrals $\int_\alpha^\beta f \,\mathrm{d}g_n$ exist for all $n \in \mathbb{N}$. Suppose at least one of the following conditions is satisfied:

(i) $f \in BV([\alpha, \beta])$ and $g_n \rightrightarrows g$,

(ii) f is bounded and $\lim_{n\to\infty} \mathrm{var}_\alpha^\beta(g_n - g) = 0$.

Then the integral $\int_{\alpha}^{\beta} f \, \mathrm{d}g$ also exists, and

$$\lim_{n\to\infty} \int_{\alpha}^{\beta} f \, \mathrm{d}g_n = \int_{\alpha}^{\beta} f \, \mathrm{d}g.$$

Proof. See (Monteiro, Slavík and Tvdý 2019, Theorem 6.3.9). □

Next, we have the main result of this section, the theorem that guarantees the sufficient condition of existence.

Theorem 1.3.44. If $f, g\colon [\alpha, \beta] \to \mathbb{R}^n$ are regulated and one of them has bounded variation, then the integral $\int_{\alpha}^{\beta} f \, \mathrm{d}g$ exists.

Proof. Assume first that $f \in BV([\alpha, \beta])$. There exists a sequence $\{f_n\}$ of finite step functions on $[\alpha, \beta]$ such that

$$f_n \rightarrow f.$$

By Theorem 1.3.39, the integrals $\int_{\alpha}^{\beta} f_n \, \mathrm{d}g$ exist for all $n \in \mathbb{N}$. By Theorem 1.3.42, this implies that the integral $\int_{\alpha}^{\beta} f \, \mathrm{d}g$ also exists.

Next, assume that $g \in BV([\alpha, \beta])$. There exists a sequence $\{g_n\}$ of finite step functions on $[\alpha, \beta]$ such that

$$g_n \rightarrow g.$$

By Theorem 1.3.39, the integrals $\int_{\alpha}^{\beta} f \, \mathrm{d}g_n$ exist for all $n \in \mathbb{N}$. By Theorem 1.3.43, this implies that the integral $\int_{\alpha}^{\beta} f \, \mathrm{d}g$ also exists. □

1.3.4 Relationship between Perron-Stieltjes and Henstock–Kurzweil-Stieltjes integrals

The definition *Perron–Stieltjes* integral is based on the notion of major and minor functions, the same way as Perron integral.

Definition 1.3.45. Let $f, g\colon [\alpha, \beta] \longrightarrow \mathbb{R}$. We say that $M\colon [\alpha, \beta] \longrightarrow \mathbb{R}$ is a **major** function of f with respect to g if there exists a gauge $\delta\colon [\alpha, \beta] \longrightarrow (0, \infty)$ such that the inequality

$$f(\tau)(g(y) - g(x)) \leq M(y) - M(x)$$

holds whenever $[x, y] \subset (\tau - \delta(\tau), \tau + \delta(\tau)) \cap [\alpha, \beta]$ and $\tau \in [x, y]$. Similarly, $m\colon [\alpha, \beta] \longrightarrow \mathbb{R}$ is a **minor** function of f with respect to g if there exists a gauge $\delta\colon [\alpha, \beta] \longrightarrow (0, \infty)$ such that the inequality

$$f(\tau)(g(y) - g(x)) \geq m(y) - m(x)$$

hold whenever $[x, y] \subset (\tau - \delta(\tau), \tau + \delta(\tau)) \subset [\alpha, \beta]$ and $\tau \in [x, y]$.

The set of all major functions of f with respect to g is denoted by $\mathfrak{M}(f, \mathrm{d}g)$, whereas $\mathfrak{m}(f, \mathrm{d}g)$ stands for the set of all minor functions of f with respect to g.

Definition 1.3.46. Given a pair of functions $f, g\colon [\alpha,\beta] \longrightarrow \mathbb{R}$, we define the **upper Perron-Stieltjes integral** of f with respect to g by

$$\overline{\int_\alpha^\beta} f \mathrm{d}g = \inf\{M(\beta) - M(\alpha)\colon M \in \mathfrak{M}(f, \mathrm{d}g)\},$$

and the **lower Perron-Stieltjes integral** of f with respect to g by

$$\underline{\int_\alpha^\beta} f \mathrm{d}g = \inf\{m(\beta) - m(\alpha)\colon m \in \mathfrak{m}(f, \mathrm{d}g)\}.$$

We use the convention that

$$\overline{\int_\alpha^\beta} f \mathrm{d}g = \infty \quad \text{if} \quad \mathfrak{M}(f, \mathrm{d}g) = \emptyset \quad \text{and} \quad \underline{\int_\alpha^\beta} f \mathrm{d}g = -\infty \quad \text{if} \quad \mathfrak{m}(f, \mathrm{d}g) = \emptyset$$

The next statement shows that the value of the lower integral never exceeds the value of the upper integral.

Lemma 1.3.47. If $f, g : [\alpha,\beta] \to \mathbb{R}$, then the following statements hold:

(i) If $M \in \mathfrak{M}(f, dg)$ and $m \in \mathfrak{m}(f, dg)$, there is a gauge δ on $[\alpha,\beta]$ such that

$$m(\beta) - m(\alpha) \leq S(P) \leq M(\beta) - M(\alpha)$$

for each δ-fine partition P of $[\alpha,\beta]$.

(ii) $\underline{\int_\alpha^\beta} f \,\mathrm{d}g \leq \overline{\int_\alpha^\beta} f \,\mathrm{d}g.$

Definition 1.3.48. If $\overline{\int_\alpha^\beta} f \mathrm{d}g = \underline{\int_\alpha^\beta} f \mathrm{d}g$ in $\mathbb{R}$, then the common value of the upper and lower integrals is called the **Perron-Stieltjes integral** of f with respect to g, and is denoted by

$$\int_\alpha^\beta f(x)\mathrm{d}g(x) \quad \text{or} \quad \int_\alpha^\beta f \mathrm{d}g.$$

In the special case when $g(x) = x$ for all $x \in [\alpha,\beta]$, the integral coincides to the Perron integral and denote it by $\int_\alpha^\beta f(x)\mathrm{d}x$.

The following result shows the equivalence relation between the Perron-Stieltjes (PS) integral and the Henstock-Kurzweil-Stieltjes integral.

Theorem 1.3.49. (Monteiro, Slavík and Tvdý 2019, Theorem 6.2.9). The Perron-Stieltjes integral $(PS)\int_\alpha^\beta f \mathrm{d}g$ exists if and only if the Henstock-Kurzweil-Stieltjes integral $\int_\alpha^\beta f \mathrm{d}g$ exists; in that case, both integrals have the same value.

Proof. Assume that the integral $(PS)\int_\alpha^\beta f \,\mathrm{d}g$ exists and equals $I \in \mathbb{R}$. Given an arbitrary $\varepsilon > 0$, there exist a major function $M \in \mathfrak{M}(f, \mathrm{d}g)$ and a minor function $m \in \mathfrak{m}(f, \mathrm{d}g)$ such that:

$$m(\beta) - m(\alpha) > I - \varepsilon, \qquad M(\beta) - M(\alpha) < I + \varepsilon.$$

If δ is the gauge specified in Lemma 1.3.47 (i), then the inequalities

$$I - \varepsilon < m(\beta) - m(\alpha) \leq S(f, \mathrm{d}g, P) \leq M(\beta) - M(\alpha) < I + \varepsilon$$

hold, and it implies that

$$-\varepsilon < S(f, \mathrm{d}g, P) - I < \varepsilon,$$

which implies

$$|S(f, \mathrm{d}g, P) - I| < \varepsilon$$

hold for every δ-fine partition P of $[\alpha, \beta]$. This means that the Henstock-Kurzweil–Stieltjes integral $\int_\alpha^\beta f\,\mathrm{d}g$ exists and equals I.

Reciprocally assume that $\int_\alpha^\beta f\,\mathrm{d}g$ exists and equals $I \in \mathbb{R}$. Given an arbitrary $\varepsilon > 0$, there is a gauge $\delta_\varepsilon : [\alpha, \beta] \to (0, \infty)$ such that:

$$I - \varepsilon < S(f, \mathrm{d}g, P) < I + \varepsilon$$

for all δ_ε-fine tagged partition P of $[\alpha, \beta]$.

Define $M(\alpha) = m(\alpha) = 0$, and

$$M(t) = \sup\left\{S(f, \mathrm{d}g, Q) : Q \text{ is a } \delta_\varepsilon\text{-fine tagged partition of } [\alpha, t]\right\}, \text{ for } t \in (\alpha, \beta],$$

$$m(t) = \inf\left\{S(f, \mathrm{d}g, Q) : Q \text{ is a } \delta_\varepsilon\text{-fine tagged partition of } [\alpha, t]\right\}, \text{ for } t \in (\alpha, \beta].$$

We claim that M is a major function of f with respect to g. To see this, consider an arbitrary $\tau \in [\alpha, \beta]$. Using the definition of M, we get

$$M(s) \geq M(\tau) + f(\tau)(g(s) - g(\tau)), \quad \text{for } s \in [\tau, \tau + \delta_\varepsilon(\tau)) \cap [\alpha, \beta],$$

since each δ_ε-fine partition of $[\alpha, \tau]$ can be extended to a δ_ε-fine partition P of $[\alpha, t]$ by combining the interval $[\tau, s]$ and the tag τ, whose contribution to the sum $S(f, \mathrm{d}g, P)$ is $f(\tau)(g(s) - g(\tau))$. Analogously,

$$M(\tau) \geq M(t) + f(\tau)(g(\tau) - g(t)), \quad \text{for } t \in (\tau - \delta_\epsilon(\tau), \tau] \cap [\alpha, \beta],$$

since each δ_ε-fine partition of $[\alpha, t]$ can be extended to a δ_ϵ-fine partition P of $[\alpha, \tau]$ by combining the interval $[t, \tau]$ and the tag τ, whose contribution to the sum $S(f, \mathrm{d}g, P)$ is $f(\tau)(g(\tau) - g(t))$. By adding the previous two inequalities, we see that

$$M(s) - M(t) \geq f(\tau)(g(s) - g(t))$$

whenever $[t, s] \subset (\tau - \delta_\varepsilon(\tau), \tau + \delta_\varepsilon(\tau)) \cap [\alpha, \beta]$ and $\tau \in [t, s]$.

Similarly, one can show that m is a minor function of f with respect to g. By definition major and minor functions, we have

$$I - \varepsilon \leq m(\beta) - m(\alpha) \leq I + \varepsilon, \quad I - \varepsilon \leq M(\beta) - M(\alpha) \leq I + \varepsilon.$$

Consequently, having in mind the definition of the upper and lower integral and Lemma 1.3.47-(ii), we see that

$$I - \varepsilon \leq (\mathrm{PS})\underline{\int_{\alpha}^{\beta}} f \,\mathrm{d}g \leq (\mathrm{PS})\overline{\int_{\alpha}^{\beta}} f \,\mathrm{d}g \leq I + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we necessarily have:

$$(\mathrm{PS}) \int_{\alpha}^{\beta} f \,\mathrm{d}g = I.$$

Hence, the Perron–Stieltjes integral $(\mathrm{PS}) \int_{\alpha}^{\beta} f \,\mathrm{d}g$ exists and equals I. □

To conclude this section, we will present the definition of the Henstock-Kurzweil-Stieltjes integral for vector-valued functions in $\mathbb{R}^n$. From now on, we will consider vector-valued functions defined on a compact interval $I \subset \mathbb{R}$ with values $f(I) \subset \mathbb{R}^n$, the so-called vector-valued functions, where each component is a real function $f_i\colon [\alpha, \beta] \to \mathbb{R}$. That is, given $\{e_1, e_2, \ldots, e_n\}$ a canonical basis of $\mathbb{R}^n$, we can write f in the form

$$f(t) = f_1(t)e_1 + f_2(t)e_2 + \ldots + f_n(t)e_n.$$

The concept of limit, derivative and integral of a function in $\mathbb{R}^n$ is generally simpler than the ones of vector-valued functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ for example, for more details see (Fonda 2018). Recall that $f\colon [\alpha, \beta] \to \mathbb{R}^n$ has derivative and integral if and only if each of its component function f_i has derivative and integral for all $i = 1, 2, \ldots, n$. Thus, we consider that if f is integrable in some sense, then

$$\int_{\alpha}^{\beta} f = \int_{\alpha}^{\beta} f_1(t)e_1 + \int_{\alpha}^{\beta} f_2(t)e_2 + \ldots + \int_{\alpha}^{\beta} f_n(t)e_n.$$

This implies that it is enough to investigate the theory for each component function $f_i\colon [\alpha, \beta] \to \mathbb{R}$, in order to obtain the one for vector-valued functions in $\mathbb{R}^n$ for instance.

Definition 1.3.50. A function $f\colon [\alpha, \beta] \to \mathbb{R}^n$ is **Henstock–Kurzweil–Stieltjes integrable** on $[\alpha, \beta]$ with respect to a function $g\colon [\alpha, \beta] \to \mathbb{R}$, if there is an element $A \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there is a gauge $\delta\colon [\alpha, \beta] \to (0, \infty)$ such that

$$\left\| \sum_{i=1}^{|\mathcal{D}|} f(\tau_i)\left(g(s_i) - g(s_{i-1})\right) - A \right\| < \varepsilon$$

for all δ–fine tagged partition $\mathcal{D}$ of $[\alpha, \beta]$. In this case, A is called the *Henstock–Kurzweil–Stieltjes* integral of f with respect to g over $[\alpha, \beta]$ and it will be denoted by $\int_{\alpha}^{\beta} f(s) \,\mathrm{d}g(s)$.

Theorem 1.3.51. Let $[\alpha, \beta] \subset \mathbb{R}$. A function $f\colon [\alpha, \beta] \to \mathbb{R}^n$ is Henstock-Kurzweil-Stieltjes integrable if, and only if, for all $i = 1, \cdots, n$, $f_i\colon [\alpha, \beta] \to \mathbb{R}$ is also Henstock-Kurzweil-Stieltjes integrable; in the case,

$$\int_{[\alpha,\beta]} f(s)\mathrm{d}g(s) = \int_{[\alpha,\beta]} f_1(s)\mathrm{d}g(s)e_1 + \int_{[\alpha,\beta]} f_2(s)\mathrm{d}g(s)e_2 + \ldots + \int_{[\alpha,\beta]} f_n(s)\mathrm{d}g(s)e_n$$

The classical properties of linearity, additivity with respect to adjacent intervals, and integrability on subintervals all remain valid for the Henstock–Kurzweil–Stieltjes integral taking values in $\mathbb{R}^n$, since it holds for each f_i with $i = 1, 2, \ldots, n$. Furthermore, the following result is valid.

Theorem 1.3.52. Let $f_1, f_2\colon [\alpha, \beta] \to \mathbb{R}^n$ be Henstock-Kurzweil–Stieltjes integrable functions with respect to a nondecreasing function $g\colon [\alpha, \beta] \to \mathbb{R}$ such that $f_1(t) \le f_2(t)$ for $t \in [\alpha, \beta]$. Then,

$$\int_\alpha^\beta f_1(s)\,\mathrm{d}g(s) \le \int_\alpha^\beta f_2(s)\,\mathrm{d}g(s).$$

The following result, found in (Schwabik 1992), ensures that the class of regulated functions is integrable according to Henstock-Kurzweil-Stieltjes with respect to a nondecreasing function. It also provides an upper bound for the absolute value of the definite integral on an interval.

Theorem 1.3.53. Let $f\colon [\alpha, \beta] \to \mathbb{R}^n$ be a regulated function and $g\colon [\alpha, \beta] \to \mathbb{R}$ be a nondecreasing function. Then, the following statements hold:

(i) The integral $\int_\alpha^\beta f(s)\,\mathrm{d}g(s)$ exists;

(ii) $\left\| \int_\alpha^\beta f(s)\,\mathrm{d}g(s) \right\| \le \int_\alpha^\beta \|f(s)\|\,\mathrm{d}g(s) \le \|f\|_\infty (g(\beta) - g(\alpha))$.

In the following results, weuse the fact that the Perron-Stieltjes and Henstock-Kurzweil-Stieltjes integrals are equivalent. We will perform the proofs using the definition of the latter. Equivalence allows us to use the most appropriate definition and facilitate the proofs.

The following lemma, known as the Saks-Henstock Lemma, is an indispensable tool for obtaining basic properties of the Kurzweil-Stieltjes indefinite integral, given by the function $t \to \int_\alpha^t f(s)\mathrm{d}g(s),\ t \in [\alpha, \beta]$. The results can be found in (Schwabik 1996) and (Monteiro, Slavík and Tvdý 2019).

Lemma 1.3.54 (Saks-Henstock)**.** Let $f\colon [\alpha, \beta] \to \mathbb{R}^n$ and $g\colon [\alpha, \beta] \to \mathbb{R}$,be such that the integral $\int_\alpha^\beta f\,\mathrm{d}g$ exists. Let $\varepsilon > 0$ be given and let δ be a gauge on $[\alpha, \beta]$ such that

$$\left\| S(f, \mathrm{d}g, P) - \int_a^b f\, dg \right\| < \varepsilon \quad \text{for all } \delta\text{-fine tagged partition } P \text{ of } [\alpha, \beta].$$

If $\{([s_j, t_j], \theta_j) : j = 1, 2, \ldots, n\}$ is an arbitrary system satisfying

$$\begin{cases} \alpha \le s_1 \le \theta_1 \le t_1 \le s_2 \le \cdots \le s_n \le \theta_n \le t_n \le \beta, \\ [s_j, t_j] \subset (\theta_j - \delta(\theta_j), \theta_j + \delta(\theta_j)) \quad \text{for } j = 1, \ldots, n, \end{cases}$$

then

$$\left\| \sum_{j=1}^n \left(f(\theta_j)\,(g(t_j) - g(s_j)) - \int_{s_j}^{t_j} f\,\mathrm{d}g \right) \right\| \le \varepsilon. \tag{1.3.7}$$

The Saks-Henstock lemma allows us to obtain some basic properties of the Henstock-Kurzweil-Stieltjes indefinite integral. The following results show that the indefinite integral does not need to be continuous, but we ensure it is regulated if the integrator is regulated.

Theorem 1.3.55. Let $\int_\alpha^\beta f\,\mathrm{d}g$ exist and let $\gamma \in [\alpha,\beta]$. Then

$$\lim_{t\to\gamma^+\in[\alpha,\beta]} \left(\int_\alpha^t f\,\mathrm{d}g + f(\gamma)(g(\gamma)-g(t))\right) = \int_\alpha^\gamma f\,\mathrm{d}g,$$

$$\lim_{t\to\gamma^-\in[\alpha,\beta]} \left(\int_t^\beta f\,\mathrm{d}g + f(\gamma)(g(t)-g(\gamma))\right) = \int_\gamma^\beta f\,\mathrm{d}g.$$

Proof. We prove only the first statement and leave the second to the reader.

Let $\varepsilon > 0$ be given and let δ_ε be a gauge on $[\alpha,\beta]$ such that

$$\left\| S(f,\mathrm{d}g,P) - \int_\alpha^\beta f\,\mathrm{d}g \right\| < \varepsilon \quad \text{for all } \delta_\varepsilon\text{-fine tagged partition } P \text{ of } [\alpha,\beta].$$

For each $t \in (\gamma, \gamma+\delta_\varepsilon(\gamma)) \cap [\alpha,\beta]$, the system $\{([\gamma,t],\gamma)\}$ satisfies the assumptions of Lemma 1.3.54, and we get

$$\left\| f(\gamma)(g(t)-g(\gamma)) - \int_\gamma^t f\,\mathrm{d}g \right\| \le \varepsilon. \tag{1.3.8}$$

Similarly, if $t \in (\gamma-\delta_\varepsilon(\gamma), \gamma) \cap [\alpha,\beta]$, then, applying Lemma 1.3.54 to the system $\{([t,\gamma],\gamma)\}$, we get

$$\left\| f(\gamma)(g(\gamma)-g(t)) - \int_t^\gamma f\,\mathrm{d}g \right\| \le \varepsilon.$$

Thus, inequality (1.3.8) holds for each

$$t \in (\gamma-\delta_\varepsilon(\gamma), \gamma+\delta_\varepsilon(\gamma)) \cap [\alpha,\beta].$$

Hence,

$$\left\| \int_\alpha^\gamma f\,\mathrm{d}g - \int_\alpha^t f\,\mathrm{d}g - f(\gamma)(g(\gamma)-g(t)) \right\| = \left\| \int_t^\gamma f\,\mathrm{d}g - f(\gamma)(g(t)-g(\gamma)) \right\| \le \varepsilon,$$

and the proof is complete □

Corollary 1.3.56. The following statements hold:

(i) If $\gamma \in (\alpha,\beta]$, $\int_\alpha^\gamma f\,\mathrm{d}g$ exists, and $\lim_{t\to\gamma^-} g(t)$ exists as well, then

$$\int_\alpha^\gamma f\,\mathrm{d}g = \lim_{t\to\gamma^-} \left(\int_\alpha^t f\,\mathrm{d}g\right) + f(\gamma)\Delta^- g(\gamma).$$

(ii) If $\gamma \in [\alpha,\beta)$, $\int_\gamma^\beta f\,\mathrm{d}g$ exists, and $\lim_{t\to\gamma^+} g(t)$ exists as well, then

$$\int_\gamma^\beta f\,\mathrm{d}g = \lim_{t\to\gamma^+} \left(\int_t^\beta f\,\mathrm{d}g\right) + f(\gamma)\Delta^+ g(\gamma).$$

Corollary 1.3.57. Suppose that $\int_\alpha^\beta f\,\mathrm{d}g$ exists, g is regulated on $[\alpha,\beta]$, and let

$$h(t) = \int_\alpha^t f\,\mathrm{d}g \quad \text{for } t\in[\alpha,\beta].$$

Then the following statements hold:

(i) h is regulated and satisfies

$$h(t^+) = h(t) + f(t)\Delta^+ g(t) \quad \text{for } t\in[\alpha,\beta),$$

$$h(t^-) = h(t) - f(t)\Delta^- g(t) \quad \text{for } t\in(\alpha,\beta].$$

(ii) If f is bounded and g has bounded variation, then

$$\mathrm{var}_\alpha^\beta h \le \|f\|_\infty \,\mathrm{var}_\alpha^\beta g < \infty.$$

We conclude this section with results that are crucial for the development of the following chapters. The following lemma is useful in proving that an impulsive Volterra-Stieltjes integral equation can always be transformed into a non-impulsive Volterra-Stieltjes integral equation.

Lemma 1.3.58. (Federson, Mesquita and Slavík 2012, Lemma 2.4) Let $m\in\mathbb{N}$, and let $\alpha \le t_1 < t_2 < \cdots < t_m \le \beta$. Consider a pair of functions $f\colon[\alpha,\beta]\to\mathbb{R}^n$ and $g\colon[\alpha,\beta]\to\mathbb{R}$, where g is regulated, left-continuous on $[\alpha,\beta]$, and continuous at $t_1,\dots,t_m$. Let $\tilde{f}\colon[\alpha,\beta]\to\mathbb{R}^n$ and $\tilde{g} : [\alpha,\beta]\to\mathbb{R}$ be such that:

- $\tilde{f}(t) = f(t)$ for every $t\in[\alpha,\beta]\setminus\{t_1,\dots,t_m\}$,
- $\tilde{g} - g$ is constant on each of the intervals

$$[\alpha,t_1],\ (t_1,t_2],\ \dots,\ (t_{m-1},t_m],\ (t_m,\beta].$$

Then the integral $\int_\alpha^\beta \tilde{f}\,\mathrm{d}\tilde{g}$ exists if and only if the integral $\int_\alpha^\beta f\,\mathrm{d}g$ exists in the sense of Henstock-Kurzweil-Stieltjes. In that case, we have:

$$\int_\alpha^\beta \tilde{f}(s)\,\mathrm{d}\tilde{g}(s) = \int_\alpha^\beta f(s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\}\\ t_k<\beta}} \tilde{f}(t_k)\Delta^+\tilde{g}(t_k).$$

The next, presenting a substitution–type theorem for the , as well as analogous versions of Gronwall Inequality and Dominated Convergence Theorem for this integral. We also present a way of interchanging the order of integrals.

Theorem 1.3.59. Let $f, h\colon [\alpha,\beta] \to \mathbb{R}^n$ and $g\colon [\alpha,\beta] \to \mathbb{R}$ functions, such that the function $h\colon [\alpha,\beta] \to \mathbb{R}^n$ is bounded and that the integral $\int_\alpha^\beta \tilde{f}(s)\,\mathrm{d}\tilde{g}(s)$ exists. If one of the integrals

$$\int_\alpha^\beta h(t)\mathrm{d}\left(\int_\alpha^t f(\xi)\mathrm{d}g(\xi)\right), \int_\alpha^\beta h(t)f(t)\mathrm{d}g(t),$$

exists, then the other one exists as well, and

$$\int_\alpha^\beta h(t)\mathrm{d}\left(f(\xi)\mathrm{d}g(\xi)\right) = \int_\alpha^\beta h(t)f(t)\mathrm{d}g(t)$$

holds.

Proof. This theorem is a consequence of the definition of the Henstock-Kurzweil-Stieltjes indefinite integral and the Saks-Henstock Lemma, see (Monteiro, Slavík and Tvdý 2019), Theorem 6.6.1. □

Lemma 1.3.60 (Gronwall Inequality). Let $\psi\colon [\alpha,\beta] \to [0,+\infty)$, $h\colon [\alpha,\beta] \to [t_0,+\infty)$ be given where ψ is bounded and h is nondescreasing and left–continuous in $[\alpha,\beta]$. Assume that for some $\xi \in [\alpha,\beta]$ the inequality

$$\psi(\xi) = k + L\int_\alpha^\xi \psi(\tau)\mathrm{d}h(\tau) \qquad \xi \in [\alpha,\beta]$$

holds with constants $L \geq 0$ and $k > 0$, then for every $\xi \in [\alpha,\beta]$ the inequality

$$\psi(\xi) \leq ke^{L(h(\xi)-h(\alpha)}$$

is satisfied.

Proof. See (Schwabik 1992), Corollary 1.43. □

Remark 1.3.61. In a completely analogous way also the following statement can be provided using 1.3.60, for inequality

$$\psi(\xi) = k + L\int_\xi^\beta \psi(\tau)\mathrm{d}h(\tau) \qquad \xi \in [\alpha,\beta].$$

Theorem 1.3.62. (Dominated Convergence Theorem) Let $g\colon [\alpha,\beta] \to \mathbb{R}$ be a non-decreasing function. Assume that the functions $\varphi_n\colon [\alpha,\beta] \to \mathbb{R}$ are such that the integral $\int_\alpha^\beta \varphi_n(s)\,\mathrm{d}g(s)$ exists for all $n \in \mathbb{N}$. Suppose that

$$\lim_{n\to\infty} \varphi_n(s) = \varphi(s) \quad \text{for } s \in [\alpha,\beta]$$

and the inequalities

$$\kappa(s) \leqslant \varphi_n(s) \leqslant \omega(s) \quad \text{for } n \in \mathbb{N},\ s \in [\alpha,\beta]$$

hold, where $\omega, \kappa\colon [\alpha,\beta] \to \mathbb{R}$ are such that the integrals $\int_\alpha^\beta \kappa(s)\,\mathrm{d}g(s)$ and $\int_\alpha^\beta \omega(s)\,\mathrm{d}g(s)$ exist. Then the integral $\int_\alpha^\beta \varphi(s)\,\mathrm{d}g(s)$ exists and

$$\lim_{n\to\infty} \int_\alpha^\beta \varphi_n(s)\,\mathrm{d}g(s) = \int_\alpha^\beta \varphi(s)\,\mathrm{d}g(s).$$

CHAPTER

2

TIME SCALES THEORY

In this chapter, we introduce calculus on time scales, as presented in by Stefan Hilger in (Hilger 1988). A time scale $\mathbb{T}$ is an arbitrary closed and nonempty subset of the reals numbers. For functions $f\colon \mathbb{T} \to \mathbb{R}^n$, we introduce the derivative and the integral for this case. The theory of time scales calculus is a tool that unifies continuous and discrete calculus as well as the cases in between. The choice $\mathbb{T} = \mathbb{R}$ leads to classical continuous calculus, while $\mathbb{T} = \mathbb{Z}$ corresponds to discrete calculus. Another frequently studied time scale is $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^n\colon\ n \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$; this leads to quantum calculus. The main goal of this section is to show that delta integrals are special cases of the Henstock-Kurzweil-Stieltjes integral, see (Federson, Mesquita and Slavík 2012). The main references here are (Bohner and Peterson 2001), (Bohner and Peterson 2003) and (Streipert 2023).

2.1 Fundamentals of time scales

We define a time scale as any closed nonempty subset of $\mathbb{R}$ and denote by the symbol $\mathbb{T}$. Given $\alpha, \beta \in \mathbb{T}$, denote by $[\alpha, \beta]_{\mathbb{T}}$ a set $\{t \in \mathbb{T} : \alpha \leq t \leq \beta\}$ which is called a closed interval in $\mathbb{T}$. The notation $[\alpha, \infty)_{\mathbb{T}}$ means that given α in $\mathbb{T}$, $[\alpha, \infty)_{\mathbb{T}} = \{t \in \mathbb{T} \,|\, \alpha \leq t < \infty\}$ i.e. $[\alpha, \infty)_{\mathbb{T}} = [\alpha, \infty) \cap \mathbb{T}$. Similarly, we define the open interval $[\alpha, \beta)_{\mathbb{T}}$ and $(\alpha, \beta]_{\mathbb{T}}$.

Example 2.1.1. $\mathbb{R}, \mathbb{Z}, h\mathbb{Z}$ and $q^{\mathbb{N}_0} = \{1, q, q^2, \ldots\}$ $(q > 1)$ are examples of time scales.

Definition 2.1.2. For all $t \in \mathbb{T}$, define the *forward jump operator* $\sigma\colon \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the *backward jump operator* by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

In this definition we put $\sup \emptyset = \inf \mathbb{T}$ and $\inf \emptyset = \sup \mathbb{T}$. We also define the **graininess function** $\mu\colon \mathbb{T} \to [0, \infty)$ by $\mu(t) = \sigma(t) - t$.

Table 1 – Time scale operators σ, ρ, and μ for different examples of time scales

Time Scale $\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$\mathbb{Z}$	$t+1$	$t-1$	1
$q^{\mathbb{N}_0}$ $(q>1)$	qt	t/q	$t(q-1)$

According to the definitions above, we can classify a point $t \in \mathbb{T}$ as: *right-scattered* if $\sigma(t) > t$, *left-scattered* if $\rho(t) < t$, *right–dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, *left–dense* if $t > \inf \mathbb{T}$ and $\rho(t) = t$.

2.2 Functions defined on time scales

The following are some properties of $f: \mathbb{T} \to \mathbb{R}^n$ that later will be important to ensure the $\Delta-$integrability.

Definition 2.2.1. A function $f: \mathbb{T} \to \mathbb{R}^n$ is called **regulated** provided its right-sided limit exists (as a finite value) for all right-dense points and its left-sided limit exists (as a finite value) for all left-dense points.

Definition 2.2.2. A function $f: \mathbb{T} \to \mathbb{R}^n$ is rd$-$continuous provided it is continuous at all right dense points of t and its left sided limit exists (finite) at left dense points of $\mathbb{T}$. The set of all right dense continuous functions on $\mathbb{T}$ is denoted by

$$\mathrm{C}_{\mathrm{rd}} = \mathrm{C}_{\mathrm{rd}}(\mathbb{T}) = \mathrm{C}_{\mathrm{rd}}(\mathbb{T}; \mathbb{R}^n).$$

Furthermore, we define the set $\mathbb{T}^\kappa$ that is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a maximum point that is left-scattered m, then $\mathbb{T}^\kappa = \mathbb{T} \backslash \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. In summary

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty \\ \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}), & \text{if} \quad \sup \mathbb{T} < \infty. \end{cases} \tag{2.2.1}$$

We define the subset $\mathbb{T}^*$ as follows: Given a time scale $\mathbb{T}$ and a real number $t \leqslant \sup \mathbb{T}$ we define

$$t^* := \inf\{s \in \mathbb{T} \colon s \geqslant t\}.$$

This operator was introduced for the first time by (Slavík 2012). We define the set $\mathbb{T}^*$ as an extension of $\mathbb{T}$ in the following way

$$\mathbb{T}^* = \begin{cases} (-\infty, +\infty), & \text{if} \quad \sup \mathbb{T} = +\infty \\ (-\infty, \sup \mathbb{T}], & \text{if} \quad \sup \mathbb{T} < \infty. \end{cases}$$

Definition 2.2.3. Given a function $f\colon \mathbb{T} \to \mathbb{R}^n$ we can extend it to the set $\mathbb{T}^*$ by defining the function $f^*\colon \mathbb{T}^* \to \mathbb{R}^n$ by

$$f^*(t) = f(t^*).$$

Furthermore, if $S \subset \mathbb{R}^n$ and $f\colon S \times \mathbb{T} \to \mathbb{R}^n$ we define $f^*(x,t) = f(x,t^*)$ for all $x \in S$ and $t \in \mathbb{T}^*$. Assuming the function $\beta\colon \mathbb{T} \times \mathbb{T} \to \mathbb{R}^n$ we define $\beta^{**}\colon \mathbb{T}^* \times \mathbb{T}^* \to \mathbb{R}^n$ given by

$$\beta^{**}(t,s) := \beta(t^*, s^*), \qquad (t,s) \in \mathbb{T}^* \times \mathbb{T}^*.$$

Lemma 2.2.4. (Federson, Grau and Mesquita 2019, Lemma 5.1) Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$, and let $t_0 \in \mathbb{T}$. Let $g\colon [t_0, +\infty) \to \mathbb{R}$ be given by

$$g(t) = t^* \quad \text{for all } t \in [t_0, +\infty).$$

Then g satisfies the following conditions:

i) g is a nondecreasing function;

ii) g is left–continuous on $(t_0, +\infty)$.

Proof. Item (i) is a direct consequence of the definition of the function g.

To demonstrate item (ii), let $\tau_0 \in (t_0, +\infty)$. We analyze three distinct cases depending on whether $\tau_0 \in \mathbb{T}$ and its local density properties.

- **Case 1:** If $\tau_0 \in \mathbb{T}$ and is left-dense, then

$$\lim_{t \to \tau_0^-} g(t) = \lim_{t \to \tau_0^-} t^* = \lim_{t \to \tau_0^-} t = \tau_0 = \tau_0^* = g(\tau_0),$$

  where this last one is the identity, which is left continuous on 0 τ_0.

- **Case 2:** If $\tau_0 \in \mathbb{T}$ and is left-scattered, then

$$\lim_{t \to \tau_0^-} g(t) = \lim_{t \to \tau_0^-} t^* = \tau_0^* = g(\tau_0).$$

- **Case 3:** If $\tau_0 \notin \mathbb{T}$, then

$$\lim_{t \to \tau_0^-} g(t) = \lim_{t \to \tau_0^-} t^* = (\tau_0)^* = g(\tau_0),$$

  which completes the proof.

□

Lemma 2.2.5. (Slavík 2012, Lemma 4) If $f\colon \mathbb{T} \to \mathbb{R}^n$ is a regulated function, then $f^*\colon \mathbb{T}^* \to \mathbb{R}^n$ is also regulated. If f is left-continuous on $\mathbb{T}$, then f^* is left–continuous on $\mathbb{T}^*$. If f is right-continuous on $\mathbb{T}$, then f^* is right–continuous at right-dense points of $\mathbb{T}$.

Proof. Let us calculate $\lim_{t\to t_0^-} f^*(t)$, where $t_0 \in \mathbb{T}^*$. If $t_0 \in \mathbb{T}$ and it is left-dense, then

$$\lim_{t\to t_0^-} f^*(t) = \lim_{t\to t_0^-} f(t) = f(t_0).$$

If $t_0 \in \mathbb{T}$ and it is left-scattered, then

$$\lim_{t\to t_0^-} f^*(t) = \lim_{t\to t_0^-} f(t) = f(t_0^-).$$

If $t_0 \notin \mathbb{T}$, then

$$\lim_{t\to t_0^-} f^*(t) = f^*(t_0^-).$$

Now consider $\lim_{t\to t_0^+} f^*(t)$, where $t_0 \in \mathbb{T}$ and $t_0 < \sup \mathbb{T}^*$. If $t_0 \in \mathbb{T}$ and it is right-dense, then

$$\lim_{t\to t_0^+} f^*(t) = \lim_{t\to t_0^+} f(t) = f(t_0).$$

If $t_0 \in \mathbb{T}$ and it is right-scattered, then

$$\lim_{t\to t_0^+} f^*(t) = \lim_{t\to t_0^+} f(t) = f(\sigma(t_0)).$$

If $t_0 \notin \mathbb{T}$, then

$$\lim_{t\to t_0^+} f^*(t) = f^*(t_0^+).$$

□

2.3 Delta derivative

Definition 2.3.1. Assume $f\colon \mathbb{T} \to \mathbb{R}^n$ is a function and let $t \in \mathbb{T}^\kappa$. Define $f^\Delta(t)$ to be the vector with the property that given any $\varepsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t, i.e. $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$\|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\| \leqslant \varepsilon|\sigma(t) - s|, \qquad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the **delta a (or Hilger) derivative** of f at $t \in \mathbb{T}^\kappa$.

Moreover, we say that f is **delta or Hilger differentiable** on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta\colon \mathbb{T}^\kappa \to \mathbb{R}^n$ is then called the **delta derivative of** f on $\mathbb{T}^\kappa$.

Theorem 2.3.2 (Delta Derivative, (Bohner and Peterson 2003, Theorem 1.16))**.** Let f : $\mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. Then:

(i) If f is $\Delta-$differentiable at t, then f is continuous at t.

(ii) If t is right-dense, then f is $\Delta-$differentiable at t if, and only if, the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

exists as a finite number. In this case,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iii) If f is continuous at the right-scattered point t, then f is $\Delta-$differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iv) If f is $\Delta-$differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Example 2.3.3. Applying Theorem 2.3.2 for the case of:

1. For $\mathbb{T} = \mathbb{R}$, the delta derivative coincides with the classical derivative:

$$f^{\Delta}(t) = f'(t), \quad t \in \mathbb{R}.$$

2. For $\mathbb{T} = \mathbb{Z}$, the delta derivative becomes the forward Euler operator:

$$f^{\Delta}(t) = f(t+1) - f(t), \quad t \in \mathbb{Z}.$$

3. For $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$, the delta derivative is given by

$$\frac{f(qt) - f(t)}{t(q-1)}.$$

Theorem 2.3.4 (Properties of the $\Delta-$Derivative on time scales)**.** Let $a \in \mathbb{R}$, $t \in \mathbb{T}^*$, and let $f, g : \mathbb{T} \to \mathbb{R}$ be delta differentiable functions. Then:

(i) The sum $f + g \colon \mathbb{T} \to \mathbb{R}$ is $\Delta-$differentiable at t with $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$.

(ii) For any constant a, $af \colon \mathbb{T} \to \mathbb{R}$ is $\Delta-$differentiable at t with $(af)^{\Delta}(t) = af^{\Delta}(t)$.

(iii) The product $fg \colon \mathbb{T} \to \mathbb{R}$ is $\Delta-$differentiable at t with $(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$.

(iv) If $g(t) \neq 0$ and $g^{\sigma}(t) \neq 0$, then

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}.$$

2.4 Delta Integral

In this section, we describe the classes of functions that are $\Delta-$integrable and let us introduce some basic concepts below.

Theorem 2.4.1. (Bohner and Peterson 2003) (Existence of Pre-Antiderivatives). Let f be a regulated function. Then there exists a function F which is pre-differentiable with region of differentiation D such that

$$F^{\Delta}(t) = f(t) \quad \text{for all } t \in D.$$

Definition 2.4.2. Let $f : \mathbb{T} \to \mathbb{R}^n$ be a regulated function. Any function F as in Theorem 2.4.1 is called an **antiderivative** of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. The Cauchy integral is defined for all antiderivatives of f. We define the Cauchy integral by

$$\int_{\alpha}^{\beta} f(t)\Delta t = F(\beta) - F(\alpha).$$

A function $F : \mathbb{T} \to \mathbb{R}^n$ is called an antiderivative provided

$$F^{\Delta}(t) = f(t)$$

holds for all $t \in \mathbb{T}^{\kappa}$.

Theorem 2.4.3. (Existence of Antiderivatives)(Bohner and Peterson 2003). Every rd–continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then the function F defined by

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau$$

for $t \in \mathbb{T}$ is an antiderivative of f.

The following theorem gives several elementary properties of the delta integral.

Theorem 2.4.4. If $f, g \in \mathrm{C_{rd}}$, $\alpha, \beta, \gamma \in \mathbb{T}$, and $a \in \mathbb{R}$, then

(i) $\int_{\alpha}^{\beta} [f(s) + g(s)]\Delta s = \int_{\alpha}^{\beta} f(s)\Delta s + \int_{\alpha}^{\beta} g(s)\Delta s;$

(ii) $\int_{\alpha}^{\alpha} f(s)\Delta s = 0;$

(iii) $\int_{\alpha}^{\beta} af(s)\Delta s = a\int_{\alpha}^{\beta} f(s)\Delta s;$

(iv) $\int_{\alpha}^{\beta} f(s)\Delta s = \int_{\alpha}^{\gamma} f(s)\Delta s + \int_{\gamma}^{\beta} f(s)\Delta s$ for $\alpha < \gamma < \beta;$

(v) $\int_{\alpha}^{\beta} f(s)\Delta s = -\int_{\beta}^{\alpha} f(s)\Delta s$;

(vi) $\int_{\alpha}^{\beta} f(\sigma(s))g^{\Delta}(s)\Delta s = (fg)(\beta) - (fg)(\alpha) - \int_{\alpha}^{\beta} f^{\Delta}(s)g(s)\Delta s$;

(vii) If $f(s) \geq 0$ for all $\alpha \leq s < \beta$, then $\int_{\alpha}^{\beta} f(s)\Delta s \geq 0$;

(viii) If $\|f(s)\| \leq g(s)$ on $[\alpha, \beta)$, then

$$\left\| \int_{\alpha}^{\beta} f(s)\Delta s \right\| \leq \int_{\alpha}^{\beta} \|f(s)\|\Delta s.$$

Proof. See Theorem 1.28 from (Bohner and Peterson 2003). □

We will introduce the $\Delta-$integral in the sense of Henstock-Kurzweil, here the main reference is (Bohner and Peterson 2003). Let (δ_L, δ_S) be a pair of nonnegative functions defined on $[\alpha, \beta]_{\mathbb{T}}$. We say that $\delta = (\delta_L, \delta_S)$ is a $\Delta-$**gauge** for $[\alpha, \beta]_{\mathbb{T}}$ provided $\delta_L(t) > 0$ in $[\alpha, \beta)_{\mathbb{T}}$, $\delta_S(t) > 0$ in $(\alpha, \beta]_{\mathbb{T}}$ and $\delta_S(t) \geqslant \mu(t)$ for all $t \in [\alpha, \beta)_{\mathbb{T}}$.

A **tagged partition** of $[\alpha, \beta]_{\mathbb{T}}$ consists of division points $s_0, \ldots, s_m \in [\alpha, \beta]_{\mathbb{T}}$ such that $\alpha = s_0 < s_1 < \ldots < s_m = \beta$, and tags $\tau_1, \ldots, \tau_m \in [\alpha, \beta]_{\mathbb{T}}$ such that $\tau_i \in [s_{i-1}, s_i]_{\mathbb{T}}$ for every $i \in \{1, \ldots, m\}$. Such a partition is called $\delta-$**fine** if

$$\tau_i - \delta_L(\tau_i) \leqslant s_{i-1} \leqslant s_i < \tau_i + \delta_S(\tau_i), \quad i \in \{1, \ldots, m\}.$$

Definition 2.4.5. We say that $f\colon [\alpha, \beta]_{\mathbb{T}} \to \mathbb{R}^n$ is **Henstock–Kurzweil $\Delta-$integrable** on $[\alpha, \beta]_{\mathbb{T}}$, if there exists a vector $I \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there is a $\Delta-$gauge δ on $[\alpha, \beta]_{\mathbb{T}}$ such that

$$\left\| \sum_{i=1}^{m} f(\tau_i)(s_i - s_{i-1}) - I \right\| < \varepsilon$$

for every $\delta-$fine tagged partition of $[\alpha, \beta]_{\mathbb{T}}$. In this case, $I = \int_{\alpha}^{\beta} f(t)\Delta t$ is Henstock-–Kurzweil $\Delta-$integrable of f over $[\alpha, \beta]_{\mathbb{T}}$. See (Slavík 2012) and (Peterson and Thompson) for more details.

Remark 2.4.6. We will use the acronym HK before the Henstock-Kurzweil integral symbol when working with different integral definitions, thus $I = (HK)\int_{\alpha}^{\beta} f(t)\Delta t$.

Lemma 2.4.7. If δ is a Δ-gauge for $[\alpha, \beta]_{\mathbb{T}}$, then there is a $\delta-$fine partition for $[\alpha, \beta]_{\mathbb{T}}$.

Proof. See (Peterson and Thompson, Lemma 1.9). □

We now give Riemann's definition of $\Delta-$integrability on time scale.

Definition 2.4.8. Let f be a bounded function on $[\alpha, \beta]_{\mathbb{T}}$ and let D be a partition of the $[\alpha, \beta]_{\mathbb{T}}$ given by $\alpha = s_0 < s_1 < \cdots < s_{|D|} = \beta$. In each interval $[s_{i-1}, s_i]_{\mathbb{T}}$ with $1 \leq i \leq |D|$, choose an arbitrary point τ_i and form the sum

$$S(f, D) = \sum_{i=1}^{|D|} f(\tau_i)(s_i - s_{i-1}). \tag{2.4.1}$$

We call $S(f, D)$ a **Riemann Δ-sum** of f corresponding to D. We say that f is **Riemann Δ-integrable** from α to β on $[\alpha, \beta]_{\mathbb{T}}$ if there exists a number I with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|S(f, D) - I| < \varepsilon$$

for every Riemann Δ-sum S of f corresponding to any partition D and independent of the choice of $\tau_i \in [s_{i-1}, s_i]$ with $1 \leq i \leq |D|$. It is easy to see that such an I is unique. The number I is called the **Riemann Δ-integral** of f from α to β.

Theorem 2.4.9. Let $\alpha, \beta \in \mathbb{T}$. Then, if $\mathbb{T} = \mathbb{R}$, then a bounded function f on $[\alpha, \beta]_{\mathbb{T}}$ is Riemann Δ-integrable from α to β if and only if f is Riemann integrable on $[\alpha, \beta]$ in the classical sense, and in this case

$$\int_{\alpha}^{\beta} f(s)\, \Delta s = \int_{\alpha}^{\beta} f(s)\, \mathrm{d}s,$$

where the integral on the right is the ordinary Riemann integral.

Proof. See (Bohner and Peterson 2003), Theorem 5.14 (i). □

Remark 2.4.10. If f is Riemann delta integrable on $[\alpha, \beta]_{\mathbb{T}}$ according to the definition of Riemann sums given for a bounded function $f(t)$ on $[\alpha, \beta]_{\mathbb{T}}$ in Definition 2.4.8, then it is Henstock-–Kurzweil $\Delta-$integrable on $[\alpha, \beta]_{\mathbb{T}}$ and

$$(HK) \int_{\alpha}^{\beta} f(s)\, \Delta s = \int_{\alpha}^{\beta} f(s)\, \Delta s.$$

The proof of this remark follows from the fact that if we define $\delta(t) = (\delta_L(t), \delta_S(t))$, by $\delta_L(t) = \delta_L > 0$ a constant for $t \in [\alpha, \beta]_{\mathbb{T}}$ and $\delta_S(t) = \delta_L$ for all right-dense points in $[\alpha, \beta]_{\mathbb{T}}$ and $\delta_S(t) = \mu(t)$ for all right-scattered points in $[\alpha, \beta]_{\mathbb{T}}$, then δ is a Δ-gauge of $[\alpha, \beta]_{\mathbb{T}}$.

Corollary 2.4.11. (Peterson and Thompson, Corollary 2.7) If $f\colon \mathbb{T} \to \mathbb{R}^n$ is regulated and $\alpha, \beta \in \mathbb{T}$, then f is Henstock-–Kurzweil $\Delta-$integrable on $[\alpha, \beta]_{\mathbb{T}}$ and

$$(HK) \int_{\alpha}^{\beta} f(s)\, \Delta s = \int_{\alpha}^{\beta} f(s)\, \Delta s.$$

Moreover, if $G(t) := \int_{\alpha}^{t} f(s)\, \Delta s$, then $G^{\Delta}(t) = f(t)$ except for a countable set.

The following theorem from (Slavík 2012) describes the relation between the Riemann Δ-integral and the Kurzweil-Henstock-Stieltjes integral.

Theorem 2.4.12. Let $f\colon \mathbb{T} \to \mathbb{R}^n$ be an rd-continuous function. Choose an arbitrary $\alpha \in \mathbb{T}$ and define

$$F_1(t) = \int_\alpha^t f(s)\,\Delta s, \quad t \in \mathbb{T},$$

$$F_2(t) = \int_\alpha^t f^*(s)\,\mathrm{d}g(s), \quad t \in \mathbb{T}^*,$$

where $g(s) = s^*$ for every $s \in \mathbb{T}$. Then $F_2 = F_1^*$. In particular, if $f\colon [\alpha,\beta]_\mathbb{T} \to \mathbb{R}^n$ is rd-continuous, we obtain

$$\int_\alpha^\beta f(s)\,\Delta s = \int_\alpha^\beta f^*(s)\mathrm{d}g(s).$$

Proof. See (Slavík 2012, Theorem 5). □

The following results are important for the development of results applied to time scales, since with them we can see that it is possible to transport the Henstock-Kurzweil-Stieltjes integral of a function f to its time scale version, with the converse also being valid.

The next result brings a very important property of the Kurzweil–Henstock–Stieltjes integral for the case $g(t) = t^*$, show how it is possible to write a Henstock–Kurzweil–Stieltjes integrable function f in a Henstock–Kurzweil $\Delta-$integrable version and also the reciprocal.

Theorem 2.4.13. (Federson, Mesquita and Slavík 2013, Theorem 4.2). Let $\mathbb{T}$ be a time scale such that $\alpha, \beta \in \mathbb{T}$ and $f\colon [\alpha,\beta]_\mathbb{T} \to \mathbb{R}^n$ be a function. Define $g(t) = t^*$, for every $t \in [\alpha,\beta]$. Then the Henstock–Kurzweil $\Delta-$integral $\int_\alpha^\beta f(s)\Delta s$ exists if, and only if, the Kurzweil-Henstock-Stieltjes integral $\int_\alpha^\beta f^*(s)\mathrm{d}g(s)$ exists. In this case, both integrals have the same value.

Proof. Assuming that the Kurzweil-Henstock Δ-integral $\int_\alpha^\beta f(s)\Delta s$ exists. For an arbitrary tagged partition D of $[\alpha,\beta]$ consisting of division points $\alpha = s_0 < s_1 < \cdots < s_{|D|} = \beta$ and tags $\tau_1, \ldots, \tau_{|D|}$, let

$$S(f,D) = \sum_{i=1}^{|D|} f^*(\tau_i)(g(s_i) - g(s_{i-1})) = \sum_{i=1}^{|D|} f(\tau_i^*)(s_i^* - s_{i-1}^*). \tag{2.4.2}$$

Given an arbitrary $\varepsilon > 0$, there is a Δ-gauge $\delta = (\delta_L, \delta_S)$ on $[\alpha,\beta]_\mathbb{T}$ such that

$$\left\| \sum_{i=1}^{|D|} f(\tau_i)(s_i - s_{i-1}) - \int_\alpha^\beta f(s)\,\Delta s \right\| < \varepsilon$$

for every δ-fine tagged partition of $[\alpha,\beta]_\mathbb{T}$. We construct a gauge $\tilde{\delta}\colon [\alpha,\beta] \to [0,\infty)$ in the following way:

$$\tilde{\delta}(t) = \begin{cases} \min\left(\delta_L(t), \sup\{d : t + d \in [\alpha,\beta]_\mathbb{T}, d \leqslant \delta_S(t)\}\right) & \text{if } t \in (\alpha,\beta) \cap \mathbb{T}, \\ \sup\{d; \alpha + d \in [\alpha,\beta]_\mathbb{T}, d \leqslant \delta_S(\alpha)\} & \text{if } t = \alpha, \\ \delta_L(\beta) & \text{if } t = \beta, \\ \dfrac{1}{2}\inf\{|t - s| : s \in T\} & \text{if } t \in [\alpha,\beta] \setminus \mathbb{T}. \end{cases}$$

Let D be an arbitrary $\tilde{\delta}$–fine tagged partition of $[\alpha,\beta]$ with division points $\alpha = s_0 < s_1 < \ldots < s_{|D|} = \beta$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i \in \{1,\ldots,|D|\}$. For every $i \in \{1,\ldots,|D|\}$, there are two possibilities: either $\tau_i \in \mathbb{T}$, or $[s_{i-1}, s_i] \cap \mathbb{T} = \emptyset$. The division points $s_0,\ldots,s_{|D|}$ and tags $\tau_1,\ldots,\tau_{|D|}$ need not belong to $\mathbb{T}$, but we can find a partition D' whose division points and tags belong to $\mathbb{T}$, $S(f,D) = S(f,D')$, and D' is $\delta-$fine. We proceed by induction: Clearly, $s_0 = \alpha \in \mathbb{T}$. Now, consider an interval $[s_{i-1}, s_i]$ with $s_{i-1} \in \mathbb{T}$. Since $[s_{i-1}, s_i] \cap \mathbb{T} \neq \emptyset$, we must have $\tau_i \in \mathbb{T}$. If $s_i \notin \mathbb{T}$, we replace the division point s_i by s_i^*, delete all division points s_j belonging to (s_i, s_i^*), and also all tags τ_j belonging to (s_i, s_i^*). This operation keeps the value of the integral sum (2.4.2) unchanged: The contributions of the intervals $[s_{i-1}, s_i]$ and $[s_{i-1}, s^*]$ to the value of the sum are the same, and the contributions of intervals $[s_{j-1}, s_j]$ contained in (s_i, s_i^*) are zero because $s_{j-1}^* = s_j^* = s_i^*$. It remains to check that the modified partition is $\delta-$fine. Let $M = \sup\{[\alpha, \tau_i + \delta_S(\tau_i)] \cap \mathbb{T}\}$. Obviously, $M \in [\alpha,\beta]_{\mathbb{T}}$. Since our original partition was $\tilde{\delta}-$fine, it follows that

$$s_i \leqslant \tau_i + \tilde{\delta}(\tau_i) \leqslant \tau_i + \sup\{d; \tau_i + d \in [\alpha,\beta]_{\mathbb{T}}, d \leqslant \delta_S(\tau_i)\} = M.$$

But $s_i \notin \mathbb{T}$ and $M \in \mathbb{T}$ implies $s_i^* \leqslant M$, because s_i^* is the smallest time scale point larger than s_i. Consequently, $s_i^* \leqslant M \leqslant \tau_i + \delta_S(\tau_i)$. Now, D' is a δ-fine tagged partition of $[\alpha,\beta]_{\mathbb{T}}$, and therefore

$$\left\| S(f,D) - \int_\alpha^\beta f(s)\Delta s \right\| = \left\| S(f,D') - \int_\alpha^\beta f(s)\Delta s \right\| < \varepsilon,$$

which proves that $\int_\alpha^\beta f^*(s)\,\mathrm{d}g(s)$ exists and equals $\int_\alpha^\beta f(s)\Delta s$.

Now assume that $\int_\alpha^\beta f^*(s)\,\mathrm{d}g(s)$ exists. Then, given an arbitrary $\varepsilon > 0$, there is a gauge $\delta\colon [\alpha,\beta] \to \mathbb{R}^+$ such that

$$\left\| \sum_{i=1}^m f^*(\tau_i)(s_i^* - s_{i-1}^*) - \int_\alpha^\beta f^*(s)\,\mathrm{d}g(s) \right\| < \varepsilon$$

for every δ-fine tagged partition of $[\alpha,\beta]$. We construct a Δ-gauge $\delta = (\delta_L, \delta_S)$ on $[\alpha,\beta]_{\mathbb{T}}$ by letting

$$\delta_L(t) = \tilde{\delta}(t), \quad \delta_S(t) = \max(\tilde{\delta}(t), \mu(t))$$

for every $t \in [\alpha,\beta]_{\mathbb{T}}$.

Consider an arbitrary δ-fine tagged partition D of $[\alpha,\beta]_{\mathbb{T}}$ with division points $\alpha = s_0 < s_1 < \cdots < s_m = \beta$ and tags $\tau_i \in [s_{i-1}, s_i]$, for $i \in \{1,\ldots,m\}$; by definition, all these points belong to $\mathbb{T}$.

Our δ-fine partition need not be $\tilde{\delta}$-fine: for certain values of $i \in \{1,\ldots,m\}$, it can happen that

$$\delta_S(\tau_i) + \tau_i \geq s_i > \tilde{\delta}(\tau_i) + \tau_i.$$

In this case, we have $\delta(\tau_i) = \mu(\tau_i)$, the point τ_i is right-scattered, and $s_i = \sigma(\tau_i)$. We claim that it is possible to find a modified tagged partition D' of $[\alpha,\beta])$ that is $\tilde{\delta}$-fine and satisfies $S(f, D') = S(f, D)$. To this end, replace the division point s_i by $\tau_i + \tilde{\delta}(\tau_i)$, while keeping τ_i as the tag for the interval $[s_{i-1}, \tau_i + \tilde{\delta}(\tau_i)]$, and cover the interval $[\tau_i + \tilde{\delta}(\tau_i), s_i]$ by an arbitrary δ-fine partition. The equality $S(f, D') = S(f, D)$ follows from the fact that $t_i^* = s_i$ for every $t \in [\tau_i, s_i]$.

The proof is concluded by observing that

$$\begin{aligned}\left\| \sum_{i=1}^{m} f(\tau_i)(s_i - s_{i-1}) - \int_\alpha^\beta f^*(s)\,\mathrm{d}g(s) \right\| &= \left\| S(f, D) - \int_\alpha^\beta f^*(s)\,\mathrm{d}g(s) \right\| \\ &= \left\| S(f, D') - \int_\alpha^\beta f^*(s)\,\mathrm{d}g(s) \right\| < \varepsilon,\end{aligned}$$

which implies that

$$\int_\alpha^\beta f(s)\Delta s \text{ exists and equals } \int_\alpha^\beta f(s)\,\mathrm{d}g(s).$$

□

Remark 2.4.14. (Federson, Mesquita and Slavík 2012, Remark 4.3) The definition of the Riemann-Stieltjes $\Delta-$integral of a function $f\colon [\alpha, \beta] \to \mathbb{R}^n$ with respect to a function $g\colon [\alpha, \beta] \to \mathbb{R}$ can be obtained in a straightforward way by taking the definition of the Riemann $\Delta-$integral and replacing the usual integral sums by

$$\sum_{i=1}^{m} f(\tau_i)(g(s_i) - g(s_{i-1})).$$

Alternatively, we can start with the definition of the Henstock-Kurzweil $\Delta-$integral and modify the integral sums in the same way. Using exactly the same reasoning as in the proof of Theorem 2.4.13, one can show that the resulting Stieltjes-type $\Delta-$integral satisfies

$$\int_\alpha^\beta f(s)\Delta g(s) = \int_\alpha^\beta f^*(s)\,\mathrm{d}g^*(s).$$

Lemma 2.4.15. Let $\mathbb{T}$ be a time scale such that $\alpha, \beta \in \mathbb{T}$, $\alpha < \beta$ and $g(t) = t^*$ for every $t \in [\alpha, \beta]$. If $f\colon [\alpha, \beta] \to \mathbb{R}^n$ is such that the integral $\int_\alpha^\beta f(s)\mathrm{d}g(s)$ exists, then

$$\int_\gamma^\eta f(s)\mathrm{d}g(s) = \int_{\gamma^*}^{\eta^*} f(s)\mathrm{d}g(s),$$

for every $\alpha \leqslant \gamma < \eta \leqslant \beta$.

Proof. Using the definition of the Kurzweil-Henstock-Stieltjes integral, note that g is constant on $[\gamma, \gamma^*]$ and on $[\eta, \eta^*]$ by definition, thus g is of zero variation, we see that $\int_\gamma^{\gamma^*} f(s)\mathrm{d}g(s) = 0$ and $\int_\eta^{\eta^*} f(s)\mathrm{d}g(s) = 0$. Therefore

$$\int_\gamma^\eta f(s)\mathrm{d}g(s) = \int_\gamma^{\gamma^*} f(s)\mathrm{d}g(s) + \int_{\gamma^*}^\eta f(s)\mathrm{d}g(s) + \int_\eta^{\eta^*} f(s)\mathrm{d}g(s) = \int_{\gamma^*}^{\eta^*} f(s)\mathrm{d}g(s)$$

□

Theorem 2.4.16. (Federson, Mesquita and Slavík 2012, Theorem 4.5) Let $\mathbb{T}$ be a time scale and $f\colon \mathbb{T} \to \mathbb{R}^n$ be a rd-continuous function such that the Henstock-Kurzweil $\Delta-$integral $\int_\alpha^\beta f(s)\Delta s$ exists for every $\alpha, \beta \in \mathbb{T}$, $\alpha < \beta$. Choose an arbitrary $\alpha \in [t_0, \infty)_\mathbb{T}$ and define

$$F_1(t) = \int_\alpha^t f(s)\Delta s, \quad t \in \mathbb{T}$$

and

$$F_2(t) = \int_\alpha^t f^*(s)\mathrm{d}g(s), \quad t \in \mathbb{T}^*,$$

where $g(s) = s^*$ for every $s \in \mathbb{T}^*$. Then $F_2 = F_1^*$; in particular, $F_2(t) = F_1(t)$ for every $t \in \mathbb{T}$.

Theorem 2.4.17. (Federson, Mesquita and Slavík 2012, Theorem 4.2) Let $\mathbb{T}$ be a time scale, $g(s) = s^*$ for every $s \in \mathbb{T}^*$ and $[\alpha, \beta] \subset \mathbb{T}^*$. Consider a pair of functions $f_1, f_2\colon [\alpha, \beta] \to \mathbb{R}^n$ such that $f_1(t) = f_2(t)$ for every $t \in [\alpha, \beta] \cap \mathbb{T}$. If $\int_\alpha^\beta f_1(s)\mathrm{d}g(s)$ exists, then $\int_\alpha^\beta f_2(s)\mathrm{d}g(s)$ exists as well, and both integrals have the same value.

Proof. Let an arbitrary $\varepsilon > 0$, there exists is a gauge $\delta_1\colon [\alpha, \beta] \to [t_0, \infty)$ such that

$$\left\| \sum_{i=1}^{|D|} f_1(\tau_i)(g(s_i) - g(s_{i-1})) - \int_\alpha^\beta f_1(s)\mathrm{d}g(s) \right\|$$

for every δ_1-fine partition with division points $\alpha = s_0 < s_1 < \ldots < s_{|D|} = \beta$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i = 1, \ldots, |D|$. Now, let

$$\delta_2(t) = \begin{cases} \delta_1(t) & \text{if } t \in [\alpha, \beta] \cap \mathbb{T}, \\ \min(\delta_1(t), \dfrac{1}{2}\inf\{|t - s| : s \in \mathbb{T}\} & \text{if } t \in [\alpha, \beta] \setminus \mathbb{T}. \end{cases}$$

Note that each δ_2-fine partition is also δ_1-fine. Consider an arbitrary δ_2-fine partition with division points $\alpha = s_0 < s_1 < \ldots < s_{|D|} = \beta$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i \in \{1, \ldots, |D|\}$. For every $i \in \{1, \ldots, |D|\}$, there are two possibilities, either $[s_{i-1}, s_i] \cap \mathbb{T} = \emptyset$ or $\tau_i \in \mathbb{T}$. In the first case, $g(s_i) = g(s_{i-1})$, and therefore

$$f_1(\tau_i)(g(s_i) - g(s_{i-1})) = f_2(\tau_i)(g(s_i) - g(s_{i-1})) = 0.$$

In the second case, $f_1(\tau_i) = f_2(\tau_i)$ and

$$f_1(\tau_i)(g(s_i) - g(s_{i-1})) = f_2(\tau_i)(g(s_i) - g(s_{i-1})).$$

Thus we have

$$\left\| \sum_{i=1}^{|D|} f_2(\tau_i)(g(s_i) - g(s_{i-1})) - I \right\| = \left\| \sum_{i=1}^{|D|} f_1(\tau_i)(g(s_i) - g(s_{i-1})) - I \right\| < \varepsilon,$$

were $I = \int_\alpha^\beta f_1(s)\mathrm{d}g(s)$. Since ε can be arbitrarily small, we conclude that $\int_\alpha^\beta f_2(s)\mathrm{d}g(s) = I$. □

Remark 2.4.18. The results obtained in the Theorems 2.4.13, 2.4.16, 2.4.17 and Lemma 2.4.15 are also valid for half-open intervals, see (Federson, Grau and Mesquita 2019).

CHAPTER 3

VOLTERRA-STIELJTES FUNCTIONAL EQUATION WITH FINITE DELAY

The aim of this chapter is to establish the basic theory for a class of Volterra–Stieltjes functional integral equations with finite delay. This type of equation naturally arises in models that incorporate memory effects of limited duration, where the current state of the system depends not only on present conditions but also on a finite interval of past states. The mathematical formulation considered is given by

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s)\, dg(s), \quad t \geq \tau_0, \\ x_{\tau_0} = \phi, \end{cases}$$

where the integral is understood in the Henstock-Kurzweil–Stieltjes sense, $a(t,\cdot)$ represents the kernel of the system, f is a nonlinear functional term, and g is a function.

This chapter is divided into three sections. In the first one, we introduce the formal definition of a solution and the appropriate phase space for the problem. In the second, we establish fundamental results concerning existence and uniqueness of solutions, together with properties of regularity and continuous dependence on the initial data. Finally, the third section is devoted to the correspondence theorems, which show how the Volterra–Stieltjes equation with finite delay generalizes important classes of equations, including fractional equations, impulsive equations, and dynamic equations on time scales. These results provide the theoretical foundation for the subsequent chapters, where criteria for boundedness, stability, and instability will be developed, along with applications to different mathematical and physical contexts.

3.1 Volterra-Stieltjes functional integral equation with finite delay

Below we present the fundamental notation and definitions that we will refer to throughout this section. Given $r > 0$ fixed, we denote $G_0 := G([-r,0],\mathbb{R}^n)$ the Banach space of all regulated functions $x\colon [-r,0] \to \mathbb{R}^n$ (i.e. $x(t^-) = \lim_{s\to t^-} x(s)$, exists for all $t \in (-r,0]$, and $x(t^+) = \lim_{s\to t^+} x(s)$, exist for all $t \in [-r,0)$). Suppose $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,+\infty) \to \mathbb{R}^n$ is a given function, we call the integral equation

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s), & t \geqslant \tau_0 \\ x_{\tau_0} = \phi & t \in [\tau_0 - r, \tau_0], \end{cases} \tag{3.1.1}$$

as **Volterra-Stieltjes functional integral equation with finite delay**. Here $\tau_0 \geqslant t_0$, $g\colon [t_0,+\infty) \to \mathbb{R}$ is a function, $a\colon [t_0,+\infty)^2 \to \mathbb{R}$ is the kernel of the integral equation with $[t_0,+\infty)^2 = [t_0,+\infty) \times [t_0,+\infty)$ and $\phi \in G([-r,0],\mathbb{R}^n)$ is the function.

For $s \geqslant t_0$ and $x\colon [t_0-r,\infty) \to \mathbb{R}^n$, let $x \in G([t_0-r,+\infty),\mathbb{R}^n)$ and $s \in [t_0,+\infty)$, we define the **history function** $x_s\colon [-r,0] \to \mathbb{R}^n$ by $x_s(\theta) := x(s+\theta)$, for all $-r \leqslant \theta \leqslant 0$. It is known that $x_t \in G_0$, see (Grau, Lafetá and Mesquita 2024, Lemma 3.1.1). Also, we will denote by $G_\infty := G([t_0-r,\infty),\mathbb{R}^n)$ the space of all regulated functions $x\colon [t_0-r,\infty) \to \mathbb{R}^n$.

The formulation is more general than the one initially described by Volterra, since we consider here the integral in the Henstock-Kurzweil–Stieltjes sense and the kernel a is considered to be all those such that the function $a\colon [t_0,\infty)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable. Depending on the definition of a, g and r, the equation studied here can cover a wide variety of problems, as we describe below:

- If $a(t,s) = 1$, $g(s) = s$, $r = 0$ and $\phi = x_0$, we obtain the standard formulation of ODEs:
$$x(t) = x_0 + \int_{\tau_0}^{t} f(x(s),s)\,\mathrm{d}s \quad t \geqslant \tau_0$$
with initial condition $x(\tau_0) = x_0$.

- If we choose $a(t,s) = 1$ and $g(s) = s$, we obtain the classical functional differential equations:
$$x(t) = \phi(0) + \int_{\tau_0}^{t} f(x_s,s)\,\mathrm{d}s \quad t \geqslant \tau_0$$
with initial condition $x_{\tau_0} = \phi$.

- If $a(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$, $g(s) = s$ and $r = 0$, then we are discussing the standard Caputo fractional differential equations:
$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{\tau_0}^{t} (t-s)^{\alpha-1} f(x(s),s)\,\mathrm{d}s \quad t \geqslant \tau_0$$

with initial condition $x(\tau_0) = x_0$ and $1 < \alpha < 2$.

Recently in (Grau, Lafetá and Mesquita 2024), they presented sufficient conditions to guarantee the existence, uniqueness and prolongation of solutions for this type of equations. They also proved the correspondence between these equations and the Volterra functional delta integral equations on time scales, as well as with the Volterra-Stieltjes impulsive functional integral equations, we will make extensive use of such results.

The following result ensures that if $x \in G([t_0 - r, +\infty), \mathbb{R}^n)$, then the history function $x_t \in G([-r, 0], \mathbb{R}^n)$ for all $t \geqslant t_0$. This property will be very important to our purposes and it can be found in (Grau, Lafetá and Mesquita 2024, Lemma 3.1.1).

Lemma 3.1.1. Let $x \in G_\infty$ and $t \geqslant t_0$ be given. Then $x_t \in G([-r, 0], \mathbb{R}^n)$.

Proof. We will show that $\lim_{s\to\tau^-} x_t(s)$ exists for all $\tau \in (-r, 0]$ and $\lim_{s\to\tau^+} x_t(s)$ exists for all $\tau \in [-r, 0)$. First, let $\tau \in (-r, 0]$ be fixed, we will show that $\lim_{s\to\tau^-} x_t(s)$ exists. By hypothesis $t \geqslant t_0$. Moreover, since $-r < \tau \leqslant 0$, it follows that

$$t_0 - r \leqslant t - r < t + \tau \leqslant t$$

thus $t_0 - r < t + \tau$ which implies that $t + \tau \in (t_0 - r, +\infty)$. Since x is regulated on $[t_0 - r, +\infty)$, the limit $L = \lim_{\xi\to(t+\tau)^-} x(\xi)$ exists. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x(\xi) - L\| < \varepsilon \qquad \xi \in (t + \tau - \delta, t + \tau).$$

By the change of variables $\xi = s + t$, we get

$$\|x(s + t) - L\| < \varepsilon \qquad s \in (\tau - \delta, \tau).$$

Consequently,

$$\|x_t(s) - L\| < \varepsilon \qquad s \in (\tau - \delta, \tau),$$

obtaining $\lim_{s\to\tau^-} x_t(s)$ exists and is equal to L.

Now, let us prove the existence of $\lim_{s\to\tau^+} x_t(s)$. Similarly, since f is a regulated function on $[t_0 - r, +\infty)$, the limit $L_1 = \lim_{\eta\to(t+\tau)^+} x(\eta)$ exists. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x(\eta) - L_1\| < \varepsilon \qquad \eta \in (t + \tau, t + \tau + \delta).$$

This implies that

$$\|x(t + s) - L_1\| < \varepsilon \qquad s \in (\tau, \tau + \delta).$$

It follows that

$$\|x_t(s) - L_1\| < \varepsilon \qquad s \in (\tau, \tau + \delta),$$

therefore $\lim_{s\to\tau^+} x_t(s)$ exists and is equal to L_1. □

Lemma 3.1.2. (Federson, Mesquita and Slavík 2012, Lemma 3.5) If $x \in G([t_0-r,+\infty),\mathbb{R}^n)$, then for each compact interval $[t_0, t_0+\sigma] \subset [t_0-r,+\infty)$ for $\sigma > 0$, the function $s \mapsto \|x_s\|_{\infty,[t_0,t_0+\sigma]}$ is regulated on $[t_0, t_0+\sigma]$.

Proof. We will show that $\lim_{s\to s_0^-} \|x_s\|_{\infty,[t_0,t_0+\sigma]}$ exists for every $s_0 \in (t_0, t_0+\sigma]$. The function x is regulated on $[t_0-r,+\infty)$, and, therefore it satisfies the Cauchy condition at s_0-r and s_0, that is, given an arbitrary $\varepsilon > 0$, there exists a $\delta \in (0, s_0-t_0)$ such that

$$\|x(\mu)-x(\nu)\| < \varepsilon, \quad \mu,\nu \in (s_0-r-\delta, s_0-r), \tag{3.1.2}$$

and

$$\|x(\mu)-x(\nu)\| < \varepsilon, \quad \mu,\nu \in (s_0-\delta, s_0). \tag{3.1.3}$$

Consider a pair of numbers s_1, s_2 such that $s_0-\delta < s_1 < s_2 < s_0$. For every $s \in [s_1-r, s_2-r]$, it follows from (3.1.2) that

$$\|x(s)\| \leq \|x(s_2-r)\| + \varepsilon \leq \|x_{s_2}\|_{\infty,[t_0,t_0+\sigma]} \tag{3.1.4}$$

It is also clear that $\|x(s)\| \leq \|x_{s_1}\|_\infty$ for every $s \in [s_2-r, s_1]$. Consequently, $\|x_{s_1}\|_\infty \leq \|x_{s_2}\|_\infty + \varepsilon$. Using (3.1.3) in a similar way, we obtain $\|x_{s_2}\|_\infty \leq \|x_{s_1}\|_\infty + \varepsilon$. It follows that

$$\left| \|x_{s_1}\|_{\infty,[t_0,t_0+\sigma]} - \|x_{s_2}\|_{\infty,[t_0,t_0+\sigma]} \right| \leq \varepsilon, \quad s_1, s_2 \in (s_0-\delta, s_0), \tag{3.1.5}$$

that is, the Cauchy condition for the existence of $\lim_{s\to s_0^-} \|x_s\|_{\infty,[t_0,t_0+\sigma]}$ is satisfied. The existence of $\lim_{s\to s_0^+} \|x_s\|_{\infty,[t_0,t_0+\sigma]}$ for $s_0 \in [t_0, t_0+\sigma)$ can be proved similarly. □

We begin by recalling below the definition of a solution of equation (3.1.1).

Definition 3.1.3. (Grau, Lafetá and Mesquita 2024, Definition 3.1) A function $x\colon [\tau_0 - r, \gamma] \to \mathbb{R}^n$ is called a **solution of the equation** (3.1.1) on $[\tau_0-r,\gamma]$ if the following conditions are satisfied:

- For every $\tau_0 \leqslant t \leqslant \gamma$, the equality
$$x(t) = \phi(0) + \int_{\tau_0}^{t} a(t,s) f(x_s,s)\,\mathrm{d}g(s)$$
holds, where the integral in the right hand side is in the sense of Henstock-Kurzweil-Stieltjes.

- $x(\tau_0+\theta) = \phi(\theta)$ for all $\theta \in [-r,0]$, i.e., $x_{\tau_0} = \phi$.

The initial condition $x_{\tau_0} = \phi$ is equivalent to $x(s) = \phi(s-\tau_0)$ for all $s \in [\tau_0-r, \tau_0]$. Indeed, if $x_{\tau_0} = \phi$ then $x(\tau_0+s) = \phi(s)$ for all $s \in [-r,0]$. Thus, in our problem, the initial condition gives us the behaviour of the solution not only in a single point, but in all the interval $[\tau_0-r,\tau_0]$.

Definition 3.1.4. A function $x\colon [\tau_0 - r, \omega) \to \mathbb{R}^n$, $\tau_0 \leqslant \omega < \infty$, is called a **solution of the equation** (3.1.1) **on** $[\tau_0 - r, \omega)$ if, for each $\tau_0 < \alpha < \omega$, the restriction of x to $[\tau_0 - r, \alpha]$ is a solution of the equation (3.1.1).

By Definitions 3.1.3 and 3.1.4 and the equivalence between the Perron-Stieltjes and Henstock-Kurzweil-Stieltjes integrals proved in Chapter 1 (see Theorem 1.3.49), in order to ensure the well-posedness of the problem (3.1.1), we need to assume some conditions, observing the most recent results in (Bonotto, Federson and Mesquita 2021), (Grau, Lafetá and Mesquita 2024) and (Federson, Mesquita and Slavík 2012).

(P1) The function $g\colon [t_0, +\infty) \to \mathbb{R}$ is nondecreasing and left–continuous on $(t_0, +\infty)$.

(P2) The function $a\colon [t_0, +\infty)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable.

(P3) The Henstock-Kurzweil–Stieltjes integral

$$\int_{\tau_1}^{\tau_2} a(t, s) f(x_s, s) \mathrm{d}g(s)$$

exists, for each compact interval $[\tau_1, \tau_2] \subset [t_0, +\infty)$, $x \in G([t_0 - r, +\infty), \mathbb{R}^n)$ and $t \in [t_0, +\infty)$.

(P4) (**Carathéodory condition**) There exists a locally Henstock-Kurzweil–Stieltjes integrable function $M\colon [t_0, +\infty) \to \mathbb{R}^+$ with respect to g such that for each compact interval $[\tau_1, \tau_2] \subset [t_0, +\infty)$, we have

$$\left\| \int_{\tau_1}^{\tau_2} \left(\beta_2 a(\tau_2, s) + \beta_1 a(\tau_1, s)\right) f(x_s, s) \mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} \left|\beta_2 a(\tau_2, s) + \beta_1 a(\tau_1, s)\right| M(s) \mathrm{d}g(s),$$

for all $x \in G([t_0 - r, +\infty), \mathbb{R}^n), \beta_1, \beta_2 \in \mathbb{R}$.

(P5) (**Local Lipschitz condition**) There exists a locally regulated function $L\colon [t_0, +\infty) \to \mathbb{R}^+$ such that for each compact interval $[\tau_1, \tau_2] \subset [t_0, +\infty)$, we have

$$\left\| \int_{\tau_1}^{\tau_2} a(\tau_2, s)[f(x_s, s) - f(z_s, s)] \mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, L(s) \|x_s - z_s\|_\infty \mathrm{d}g(s),$$

for all $x, z \in G([t_0 - r, +\infty), \mathbb{R}^n)$.

With these conditions in hand, in the work of (Grau, Lafetá and Mesquita 2024) they showed the existence and uniqueness theorem for solutions of (3.1.1). For continuous dependence of the data, stability using the Lyapunov method and periodic boundary value problem, applying each result to the impulsive Volterra equation and on time scales, see (Lafetá 2022). Our work is to develop and to give continuity to this work, adding to the theory results of instability, stability with infinite delay and making a robust theory of boundedness. For this, we will need to present some results of existence and uniqueness, whose proofs can be easily consulted in (Grau, Lafetá and Mesquita 2024).

One of the most pertinent conditions lies in the fact that the solution of the equation may contain several points of discontinuity, as follows in the next lemma.

Lemma 3.1.5. (Grau, Lafetá and Mesquita 2024, Lemma 3.1.7) Assume $f\colon G([-r,0],\mathbb{R}^n)\times [t_0,d) \to \mathbb{R}^n$ satisfies conditions (P3) and (P4), $a\colon [t_0,d)^2 \to \mathbb{R}$ satisfies condition (P2) and $g\colon [t_0,d) \to \mathbb{R}$ satisfies condition (P1). If $x\colon [t_0-r,t_0+\sigma] \to \mathbb{R}^n$, $t_0+\sigma < d$ is a solution of the equation (3.1.1), then x is a regulated function on $[t_0-r,t_0+\sigma]$.

The following theorem provides sufficient conditions to guarantee the existence and uniqueness of a local solution of (3.1.1). The proof of this result is similar to that found in (Alvarez *et al.* 2021), with the necessary adaptations, since in (Alvarez *et al.* 2021) the equation is without delay. The crucial point is to apply Schauder's fixed-point theorem, that is, to consider the set of points

$$H_\phi := \{\varphi \in G([\tau_0-r,\tau_0+\sigma],\mathbb{R}^n)\colon \varphi_{\tau_0} = \phi\}$$

and we show that H_ϕ is a nonempty convex and closed subset of $G([t_0-r,t_0+\sigma],\mathbb{R}^n)$ where $\tau \in [t_0,d)$ and that the operator $T\colon H_\phi \to H_\phi$ defined by

$$(Tx)(t) = \begin{cases} \phi(t-\tau_0), & t \in [\tau_0-r,\tau_0] \\ \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s)f(x_s,s)\mathrm{d}g(s) & t \in [\tau_0,\tau_0+\sigma] \end{cases}$$

has a fixed point in H_ϕ.

Theorem 3.1.6. Let $d \in \mathbb{R}$ such that $d < \infty$. Assume $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,d) \to \mathbb{R}^n$ satisfies the conditions (P3), (P4) and (P5), $a\colon [t_0,d)^2 \to \mathbb{R}$ satisfies the condition (P2) and $g\colon [t_0,d) \to \mathbb{R}$ satisfies condition (P1). Then for all $\tau_0 \in [t_0,d)$ and all $\phi \in G([-r,0],\mathbb{R}^n)$, there exists a $\sigma > 0$ and a unique solution $x\colon [\tau_0-r,\tau_0+\sigma] \to \mathbb{R}^n$ of the initial value problem:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s)f(x_s,s)\mathrm{d}g(s) & t \in [\tau_0,\tau_0+\sigma] \\ x_{\tau_0} = \phi \qquad t \in [\tau_0-r,\tau_0]. \end{cases}$$

Proof. See (Grau, Lafetá and Mesquita 2024, Theorem 3.1.9). □

With these results and the definition of right-hand extension solutions and maximal solutions for (3.1.1) the authors in (Grau, Lafetá and Mesquita 2024) showed the existence and uniqueness of a maximal solution in a nondegenerate interval $I = [\tau_0-r,\omega)$ where $\omega \leqslant +\infty$, applying the Theorem 3.1.6.

Definition 3.1.7 (Prolongation to the right)**.** Let $\tau_0 \geq t_0$, $\phi \in G([-r,0],\mathbb{R}^n)$ and $x\colon J \to \mathbb{R}^n$, $J \subset [t_0-r,+\infty)$, be a solution of (3.1.1) on the interval J with $\tau_0 - r = \min J$. The solution $y\colon \tilde{J} \to \mathbb{R}^n$, $\tilde{J} \subset [t_0-r,+\infty)$ with $\tau_0 - r = \min \tilde{J}$, of (3.1.1) is called a **prolongation to the right** of x, if $J \subset \tilde{J}$ and $x(t) = y(t)$ for all $t \in J$. If $J \subsetneq \tilde{J}$, then y is called a **proper prolongation** of x to the right.

Definition 3.1.8 (Maximal solution)**.** Let $\tau_0 \geq t_0$, $\phi \in G([-r,0],\mathbb{R}^n)$. A solution $y\colon I \to \mathbb{R}^n$, $I \subset [t_0-r,+\infty)$ and I is such that $\tau_0 - r = \min I$, of equation (3.1.1), is called **maximal** if there is no proper prolongation of y to the right. In this case, I is called the **maximal interval of existence** of y.

Lemma 3.1.9. Assume $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,+\infty) \to \mathbb{R}^n$ satisfies the conditions (P3), (P4) and (P5). Let $a\colon [t_0,+\infty) \to \mathbb{R}$ satisfies the condition (P2) and $g\colon [t_0,+\infty) \to \mathbb{R}$ satisfies the condition (P1). Let $\tau_0 \geq t_0$, $\phi \in G([-r,0],\mathbb{R}^n)$ and consider equation (3.1.1). Let $x\colon I_x \to \mathbb{R}^n$ and $y\colon I_y \to \mathbb{R}^n$ be solutions of (3.1.1), where I_x and I_y are intervals such that $\tau_0 - r = \min I_x = \min I_y$, then $x(t) = y(t)$ for all $t \in I_x \cap I_y$.

Proof. Suppose $x\colon I_x \to \mathbb{R}^n$ and $y\colon I_y \to \mathbb{R}^n$ are solutions of the (3.1.1), where J_x and J_y are intervals with $\tau_0 - r = \min I_x = \min I_y$. Then, since $x_{\tau_0}(s) = x(\tau_0+s) = \phi(s)$ and $y_{\tau_0}(s) = y(\tau_0+s) = \phi(s)$, we have $x(t) = y(t)$ for all $t \in [\tau_0-r,\tau_0]$. We will show that $x(t) = y(t)$ for all $t \in I_x \cap I_y$. Denote $I = I_x \cap I_y$ and note that I is an interval of the form $[\tau_0-r,d]$ or $[\tau_0-r,d)$ with $d \leq +\infty$.

Case 1: $I = [\tau_0-r,d]$. Define

$$\Lambda = \{t \in [\tau_0-r,d] : x(s) = y(s) \text{ for all } s \in [\tau_0-r,t]\}.$$

Observe that if $d = \tau_0$, there is nothing to prove. Since $\tau_0 \in \Lambda$, we have $\Lambda \neq \emptyset$. Assume $d > \tau_0$, let $\lambda = \sup \Lambda$. Clearly $\lambda \leq d$ and $[\tau_0-r,\lambda) \subset \Lambda$, that is,

$$x(s) = y(s), \quad \forall s \in [\tau_0-r,\lambda).$$

The functions x and y are solutions of the (3.1.1) on I, it follows by Lemma 3.1.5 that x and y are regulated on I. From equation (3.1.1) and by condition (P5)

$$\begin{aligned}
\|x(\lambda)-y(\lambda)\| &= \left\|\int_{\tau_0}^{\lambda} a(\lambda,s)f(x_s,s)\,\mathrm{d}g(s) - \int_{\tau_0}^{\lambda} a(\lambda,s)f(y_s,s)\,\mathrm{d}g(s)\right\| \\
&= \left\|\int_{\tau_0}^{\lambda} a(\lambda,s)[f(x_s,s)-f(y_s,s)]\,\mathrm{d}g(s)\right\| \\
&\leqslant \int_{\tau_0}^{\lambda} |a(\lambda,s)|L(s)\|x_s-y_s\|_\infty \mathrm{d}g(s),
\end{aligned}$$

But for $s \in [\tau_0,\lambda)$, we have $s \in [\tau_0-r,\lambda)$ since $\lambda > \tau_0$, and from the definition of Λ, we know that $x(u) = y(u)$ for all $u \in [\tau_0-r,\lambda)$. Therefore, $x_s = y_s$ for $s \in [\tau_0,\lambda)$, so

$\|x_s - y_s\|_\infty = 0$. Thus, the integral vanishes and we obtain

$$\|x(\lambda) - y(\lambda)\| = 0 \quad \Rightarrow \quad x(\lambda) = y(\lambda).$$

Therefore, $\lambda \in \Lambda$.

Now, suppose by contradiction that $\lambda < d$. Since x and y are solutions, they satisfy (3.1.1) on I. For $t > \lambda$ sufficiently close to λ, we have

$$x(t) - y(t) = \int_{\tau_0}^{t} a(t,s)[f(x_s,s) - f(y_s,s)]\, ds.$$

Due to the properties of the Henstock–Kurzweil–Stieltjes integral, we can decompose

$$x(t) - y(t) = \int_{\tau_0}^{\lambda} a(t,s)[f(x_s,s) - f(y_s,s)]\, ds + \int_{\lambda}^{t} a(t,s)[f(x_s,s) - f(y_s,s)]\, ds.$$

The first integral is zero because $x_s = y_s$ for $s \in [\tau_0, \lambda]$. Using condition (P5) for the second integral,

$$\|x(t) - y(t)\| \leq \int_{\lambda}^{t} |a(t,s)| L(s) \|x_s - y_s\|_\infty\, ds.$$

Since x and y are regulated and L is locally regulated, there exists a constant $K > 0$ such that for t close enough to λ,

$$\int_{\lambda}^{t} |a(t,s)| L(s)\, ds \leq K(t - \lambda).$$

Also, by continuity of x and y at λ, we have $\|x_s - y_s\|_\infty \to 0$ as $s \to \lambda^+$. Hence, for $\varepsilon > 0$ small, we can choose t such that

$$\|x(t) - y(t)\| \leq \varepsilon K(t - \lambda).$$

Taking t sufficiently close to λ forces $x(t) = y(t)$, contradicting the definition of λ as $\sup \Lambda$. Therefore, $\lambda = d$.

Case 2: $I = [\tau_0 - r, d)$. The argument is analogous. Define

$$\Lambda = \{t \in [\tau_0 - r, d) : x(s) = y(s) \text{ for all } s \in [\tau_0 - r, t]\},$$

and let $\lambda = \sup \Lambda$. The same reasoning as in Case 1 shows that $\lambda \in \Lambda$ and $\lambda = d$.

In both cases, we conclude that $x(t) = y(t)$ for all $t \in I_x \cap I_y$. □

Theorem 3.1.10. Suppose $f : G([-r,0],\mathbb{R}^n) \times [t_0,+\infty) \to \mathbb{R}^n$ satisfies conditions (P3), (P4) and (P5); $a : [t_0,+\infty)^2 \to \mathbb{R}$ satisfies condition (P2) and $g : [t_0,+\infty) \to \mathbb{R}$ satisfies the condition (P1). Then, for every $\tau_0 \geq t_0$ and $\phi \in G([-r,0],\mathbb{R}^n)$, there exists a unique maximal solution $x : I \to \mathbb{R}^n$ of equation (3.1.1), where I is a nondegenerate interval with $\tau_0 - r = \min I$. Also, $I = [\tau_0 - r, \omega)$, with $\omega \leq +\infty$.

Proof. See (Lafetá 2022, Theorem 3.2.4). □

3.2 Correspondence among equations

In this section, we make an inherent part of our work, the correspondences between the Volterra-Stieltjes functional integral equation given in (3.1.1) with the impulsive, fractional functional equations and dynamic equations on time scales. To this end, we will provide a basic introduction to the theory of each one, with results on the existence and uniqueness of solutions and finally the correspondence theorem.

3.2.1 Functional Volterra delta integral equations on time scales

Below, we present some investigations on the functional Volterra $\Delta-$integral equations on time scales, such as the relationship between the Volterra-Stieltjes functional integral equations and the Volterra functional delta integral equations on time scales and present results on the existence and uniqueness of solutions and the existence of maximal solutions for these equations.

Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 - r, t_0 \in \mathbb{T}$. In this section, we consider the functional Volterra $\Delta-$integral equation on time scales given by

$$\begin{cases} x(t) &= x(t_0) + \int_{t_0}^{t} a(t,s) f(x_s^*, s)\Delta s, \quad t \in [t_0, +\infty)_{\mathbb{T}} \\ x(t) &= \phi(t) \qquad t \in [t_0 - r, t_0)_{\mathbb{T}}, \end{cases} \tag{3.2.1}$$

where $\phi \in G([t_0 - r, t_0]_{\mathbb{T}}, \mathbb{R}^n)$, $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$.

Let us assume the following conditions for a, f and $\Delta-$integral:

(E1) The function $a\colon [t_0, \infty)^2_{\mathbb{T}} \to \mathbb{R}$ is nondecreasing and rd$-$continuous with respect to the first variable, regulated with respect to the second variable.

(E2) The Henstock–Kurzweil $\Delta-$integral

$$\int_{s_1}^{s_2} a(\tau, s) f(x_s, s)\Delta s$$

exists for all $x \in G([t_0 - r, +\infty)_{\mathbb{T}}, \mathbb{R}^n)$, and $s_1, s_2 \in [t_0, +\infty)_{\mathbb{T}}$, $s_1 \leqslant s_2$.

(E3) There exists a locally Henstock–Kurzweil $\Delta-$integrable function $M_1\colon [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ such that

$$\left\| \int_{s_1}^{s_2} (c_1 a(s_2, s) + c_2 a(s_1, s)) f(x_s, s)\Delta s \right\| \leqslant \int_{s_1}^{s_2} M_1(s) |c_1 a(s_2, s) + c_2 a(s_1, s)| \Delta s,$$

for all $x \in G([t_0 - r, \infty)_{\mathbb{T}}, \mathbb{R}^n)$ and $s_1, s_2 \in [t_0, \infty)_{\mathbb{T}}$, $s_1 \leqslant s_2$.

(E4) There exists a locally regulated function $L_1\colon [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ such that

$$\left\|\int_{s_1}^{s_2} a(s_2,s)[f(x_s,s)-f(z_s,s)]\Delta s\right\| \leqslant \int_{s_1}^{s_2} L_1(s)|a(s_2,s)|\,\|x_s-z_s\|_\infty\,\Delta s,$$

for all $x,z\in G([t_0-r,\infty)_{\mathbb{T}},\mathbb{R}^n)$ and $s_1,s_2\in[t_0,\infty)_{\mathbb{T}}$, $s_1\leqslant s_2$.

Furthermore, given $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,+\infty)_{\mathbb{T}}\to\mathbb{R}^n$, $a\colon [t_0,+\infty)^2_{\mathbb{T}}\to\mathbb{R}$ and $g\colon [t_0,+\infty)\to\mathbb{R}$ arbitrary functions, we define the functions

$$\begin{cases} g(s) := s^* & \text{for} \quad s\in[t_0,+\infty) \\ f^*(\psi,s) := f(\psi,s^*) & \text{for} \quad s\in[t_0,+\infty), \psi\in G([-r,0],\mathbb{R}^n), \\ a^{**}(t,s) := a(t^*,s^*) & \text{for} \quad (t,s)\in[t_0,+\infty)^2. \end{cases} \tag{3.2.2}$$

The extensions defined above allow rewriting conditions (E1) to (E4) in the framework of Henstock-Kurzweil-Stieltjes integral, as follows in the lemma below.

Lemma 3.2.1. (Grau, Lafetá and Mesquita 2024) Let $\mathbb{T}$ be a time scale such that $\sup\mathbb{T} = +\infty$ and $t_0\in\mathbb{T}$. Let us assume $a\colon [t_0,+\infty)^2_{\mathbb{T}}\to\mathbb{R}$ and $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,+\infty)_{\mathbb{T}}\to\mathbb{R}^n$. Define g, f^* and a^{**} as in (3.2.2). Then the following statements hold.

(E_1^*) If $a\colon [t_0,+\infty)^2_{\mathbb{T}}\to\mathbb{R}$ satisfies condition (E1), then the function $a^{**}\colon [t_0,+\infty)^2\to\mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable and locally bounded on $[t_0,+\infty)^2$.

(E_2^*) If $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,+\infty)_{\mathbb{T}}\to\mathbb{R}^n$ satisfies the condition (E2), then the Henstock-Kurzweil-Stieltjes integral

$$\int_{v_1}^{v_2} a^{**}(t,s)f^*(x_s,s)\mathrm{d}g(s)$$

exists for each compact interval $[\tau_0,\tau_0+\eta]\subset[t_0,+\infty)$, $x\in G([\tau_0-r,\tau+\eta],\mathbb{R}^n)$ and $\tau_0\leqslant v_1\leqslant v_2\leqslant\tau_0+\eta$.

(E_3^*) If $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,\infty)_{\mathbb{T}}\to\mathbb{R}^n$ satisfies the condition (E3), then $f^*\colon G([-r,0],\mathbb{R}^n)\times[t_0,+\infty)\to\mathbb{R}^n$ satisfies the condition

$$\left\|\int_{u_1}^{u_2}(c_1a^{**}(u_2,s)+c_2a^{**}(u_1,s))f^*(x_s,s)\mathrm{d}g(s)\right\|$$
$$\leqslant\int_{u_1}^{u_2} M_1^*(s)|c_1a^{**}(u_2,s)+c_2a^{**}(u_1,s)|\mathrm{d}g(s)$$

for each compact interval $[\tau_0,\tau_0+\eta]\subset[t_0,+\infty)$, $x\in G([\tau_0-r,\tau_0+\eta]),\mathbb{R}^n)$, $\tau_0\leqslant u_1\leqslant u_2<\tau_0+\eta$ and $c_1,c_2\in\mathbb{R}$.

(E_4^*) If $f\colon G([-r,0],\mathbb{R}^n)\times[t_0,+\infty)_{\mathbb{T}}\to\mathbb{R}^n$ satisfies the condition (E4), then $f^*\colon G([-r,0],\mathbb{R}^n)\times[t_0,\infty)\to\mathbb{R}^n$ satisfies the condition

$$\left\|\int_{v_1}^{v_2} a^{**}(v_2,s)[f^*(x_s,s)-f^*(z_s,s)]\mathrm{d}g(s)\right\|\leqslant\int_{v_1}^{v_2} L_1^*(s)|a^{**}(u_2,s)|\|x_s-z_s\|_\infty\mathrm{d}g(s)$$

for each compact interval $[\tau_0, \tau_0+\eta] \subset [t_0, +\infty)$, $x, z \in G([\tau_0 - r, \tau_0+\eta), \mathbb{R}^n)$, and $\tau_0 \leqslant v_1 \leqslant v_2 < \tau_0 + \eta$.

The next result plays an important role in this section, as it establishes a relationship between the solutions of the Δ-integral Volterra functional equation on time scales and the solutions of the Volterra–Stieltjes integral functional equation. The proof is omitted and may be found in (Lafetá 2022, Theorem 2.2.1).

Throughout this section, we are considering $a(\cdot, s)f(x_s^*, s)$ is Henstock–Kurzweil Δ–integrable in $[t_0, +\infty)_{\mathbb{T}}$ and $a^{**}(\cdot, s)f^*(x_s, s)$ is Henstock–Kurzweil–Stieltjes integrable for all $s \in [t_0, +\infty)$.

Theorem 3.2.2. Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $t_0 \in \mathbb{T}$. Let $a\colon [t_0, +\infty)^2_{\mathbb{T}} \to \mathbb{R}$ and $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$. Define $g\colon [t_0, +\infty) \to \mathbb{R}$ by $g(s) = s^*$ for every $s \in [t_0, +\infty)$. If $x\colon [t_0 - r, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of the functional Volterra $\Delta-$integral equation on time scales

$$\begin{cases} x(t) &= x(t_0) + \displaystyle\int_{t_0}^{t} a(t,s)f(x_s^*, s)\Delta s, \quad t \in [t_0, +\infty)_{\mathbb{T}} \\ x(t) &= \phi(t) \qquad t \in [t_0 - r, t_0)_{\mathbb{T}}. \end{cases} \tag{3.2.3}$$

Then $x^*\colon [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of the functional Volterra–Stieltjes integral equation

$$\begin{cases} y(t) &= y(t_0) + \displaystyle\int_{t_0}^{t} a^{**}(t,s)f^*(y_s, s)\mathrm{d}g(s) \\ y_{t_0} &= \phi^*_{t_0}. \end{cases} \tag{3.2.4}$$

Conversely, if $y\colon [t_0 - r, +\infty) \to \mathbb{R}^n$ satisfies the equation (3.2.4), then it must have the form $y = x^*$, where $x\colon [t_0 - r, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of the equation (3.2.3).

Remark 3.2.3. The symbol x_t^* describes the term $(x^*)_t$, note that x_t^* is always defined on the entire interval $[-r, 0]$ while x_t is defined only on a subset of $[-r, 0]$; moreover, this subset depends on t. For more details, see (Federson, Mesquita and Slavík 2013).

The next result guarantees the existence and uniqueness of a solution to the functional Volterra Δ-integral equation on time scales. The proof is omitted and may be found in (Grau, Lafetá and Mesquita 2024).

Theorem 3.2.4. Let $[t_0 - r, t_0]_{\mathbb{T}}$ be a time scale interval and let $d \in \mathbb{T}$ be a left-dense point such that $d > t_0$. Assume $a\colon [t_0, +\infty)^2_{\mathbb{T}} \to \mathbb{R}$ satisfies condition (C1), and $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, +\infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies conditions (E2)–(E4). Then, for all $\varphi \in G([t_0 - r, t_0]_{\mathbb{T}}, \mathbb{R}^n)$ there exists $\eta > 0$ such that $\eta \geq \mu(t_0)$ and $t_0 + \eta \in \mathbb{T}$, and a function $x\colon [t_0 - r, t_0 + \eta]_{\mathbb{T}} \to \mathbb{R}^n$ which is a unique solution of the functional Volterra Δ-integral equation on time scales given by

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^{t} \alpha(t,s) f(x_s, s)\Delta s, & t \in [t_0, t_0 + \eta]_{\mathbb{T}}, \\ x(t) = \varphi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{cases} \tag{3.2.5}$$

The following theorem establishes the existence of maximal solutions for the Volterra $\Delta-$integral equation. Its proof can be found in (Lafetá 2022, Theorem 4.3.2) and is therefore omitted.

Theorem 3.2.5. Let $[t_0 - r, t_0]_{\mathbb{T}}$ be a time scale interval. Assume $a\colon [t_0, \infty)^2_{\mathbb{T}} \to \mathbb{R}$ satisfies condition (E1), where $d = \infty$, and $f\colon G([-r, 0], \mathbb{R}^n) \times [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ satisfies conditions (E2)–(E4). Then, for all $\phi \in G([-r, 0], \mathbb{R}^n)$, there exists a function $x\colon [t_0 - r, \omega)_{\mathbb{T}} \to \mathbb{R}^n$, $\omega \leq \infty$, which is a unique maximal solution of the functional Volterra Δ-integral equation on time scales given by

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^{t} a(t,s) f(x_s)\, \Delta s, & t \in [t_0, t_0 + \omega)_{\mathbb{T}}, \\ x(t) = \phi(t), & t \in [t_0 - r, t_0]_{\mathbb{T}}. \end{cases} \tag{3.2.6}$$

Moreover, if $\omega < \infty$, then $\omega \in \mathbb{T}$ and ω is left-dense.

3.2.2 Impulsive functional Volterra–Stieltjes integral equations

In this section, we are interested in the study of impulsive Volterra–Stieltjes integral equations. The theory of impulsive equations is an important branch of differential equations. The first paper in this theory is related to A. D. Mishkis and V. D. Mil'man in 1960 and 1963 (Mil'man and Myshkis 1960). These equations are described by three components: a continuous-time integral equation, which governs the state of the system between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which defines a set of jump events in which the impulse equation is active.

There are several types of impulses. In **classical mechanics**, an impulse is defined as the change in the **linear momentum** of an object. That is, if an object has an initial momentum x_1 and a subsequent momentum x_2, then the impulse it receives is

$$I = x_2 - x_1.$$

The notion of impulse considered here is based on the same idea. In the theory of differential equations, impulses appear in two different forms. The first corresponds to the case of **pre-assigned impulse moments**, that is, when the impulses occur at fixed, predetermined times.

$$\Delta x(\tau_k) = I_k(x(\tau_k)), \quad k = 1, \ldots, m$$

where τ_k, $k = 1, \ldots, m$ with $\tau_0 < \tau_1 < \ldots < \tau_k < \ldots < \tau_m \leqslant \tau_0 + \sigma$ are pre-assigned impulse moments, $x \mapsto I_k(x)$ maps $\mathbb{R}^n$ into itself and

$$\Delta x(\tau_k) := x(\tau_k^+) - x(\tau_k^-) = x(\tau_k^+) - x(\tau_k), \quad k = 1, 2, \ldots, m,$$

in the case x is left-continuous function.

Our attention will be focused on the case of pre–assigned moments of impulses. Thus, let us assume that $\{\tau_k\}_{k=1}^m$ are moments of impulses and each $\tau_k \in [\tau_0, \tau_0 + \sigma]$. Suppose also that the condition $\Delta x(\tau_k) = I_k(x(\tau_k))$ where $I_k \colon \mathbb{R}^n \to \mathbb{R}^n$ is the impulse operator, is satisfied for each $k = 1, \ldots, m$. Therefore, consider the following equation

$$\begin{cases} x(\mu) - x(\nu) &= \displaystyle\int_{\tau_0}^{\mu} a(\mu, s) f(x_s, s) \mathrm{d}g(s) - \int_{\tau_0}^{\nu} a(\nu, s) f(x_s, s) \mathrm{d}g(s) \quad \text{for } \mu, \nu \in T_k, \\ \Delta x(\tau_k) &= I_k(x(\tau_k)) \quad k = 1, \cdots, m, \\ x_{\tau_0} &= \phi, \end{cases}$$

where $T_0 = [\tau_0, \tau_1]$, $T_k = (\tau_k, \tau_{k+1}]$ for $k = 1, \ldots, m-1$ and $T_m = (\tau_m, \tau_0 + \sigma]$.

Remark 3.2.6 (Invariance of the integral with respect to of g)**.** Let $g \colon [\tau_0, \tau_0 + \sigma] \to \mathbb{R}$ be a function, $T_k = (\tau_k, \tau_{k+1}] \subset [\tau_0, \tau_0 + \sigma]$. Then the value of both integrals

$$\int_{\tau_0}^{\mu} a(\mu, s) f(x_s, s) \mathrm{d}g(s) \quad \text{and} \quad \int_{\tau_0}^{\nu} a(\nu, s) f(x_s, s) \mathrm{d}g(s)$$

where $\mu, \nu \in T_k$, do not change if we replace g by a function $\tilde{g}$ such that $g - \tilde{g}$ is a constant function on T_k. This follows from the properties of Henstock–Kurzweil–Stieltjes integral.

Assume also that a is continuous with respect to the first variable at $\{\tau_k\}_{k=1}^m$ and also, a satisfies condition (P2) presented in Section 3.1. Moreover, we assume that g is left-continuous on the moments of impulse τ_k for $k = 1, 2, \ldots, m$. Further, suppose that f and g satisfy conditions (P1), (P3) and (P4) presented in Section 3.1. Under these assumptions, our problem can be rewritten as

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t, s) f(x_s, s) \mathrm{d}g(s) + \sum_{\tau_0 \leqslant \tau_k \leqslant t} I_k(x(\tau_k)) \quad t \in [\tau_0, \tau_0 + \sigma] \\ x_{\tau_0} = \phi. \end{cases} \tag{3.2.7}$$

Remark 3.2.7. Note that by the assumptions above, the function

$$t \mapsto \int_{\tau_0}^{t} a(t, s) f(x_s, s) \mathrm{d}g(s)$$

is continuous at $\tau_1, \cdots, \tau_m$ and, therefore, $\Delta^+ x(\tau_k) = I_k(x(\tau_k))$ for every $k \in \{1, \cdots, m\}$. In fact, observe that given $\tau_0 \leqslant t < \tau \leqslant \tau_0 + \sigma$, we get

$$\begin{aligned}
&\left\|\int_{\tau_0}^{t} a(t,s)f(x_s,s)\mathrm{d}g(s) - \int_{\tau_0}^{\tau} a(\tau,s)f(x_s,s)\mathrm{d}g(s)\right\| \\
= \ & \left\|\int_{\tau_0}^{\tau} a(t,s)f(x_s,s)\mathrm{d}g(s) + \int_{\tau}^{t} a(t,s)f(x_s,s)\mathrm{d}g(s) - \int_{\tau_0}^{\tau} a(\tau,s)f(x_s,s)\mathrm{d}g(s)\right\| \\
\leqslant \ & \left\|\int_{\tau}^{t} a(t,s)f(x_s,s)\mathrm{d}g(s)\right\| + \left\|\int_{\tau_0}^{\tau} (a(t,s)-a(\tau,s))f(x_s,s)\mathrm{d}g(s)\right\| \\
\leqslant \ & \int_{\tau}^{t} |a(t,s)|M(s)\mathrm{d}g(s) + \int_{\tau_0}^{\tau} |a(t,s)-a(\tau,s)|M(s)\mathrm{d}g(s) \\
\leqslant \ & \int_{\tau}^{t} cM(s)\mathrm{d}g(s) + \int_{\tau_0}^{\tau} |a(t,s)-a(\tau,s)|M(s)\mathrm{d}g(s) \\
\leqslant \ & \int_{\tau}^{t} cM(s)\mathrm{d}g(s) + \int_{\tau_0}^{\tau_0+\sigma} |a(t,s)-a(\tau,s)|M(s)\mathrm{d}g(s),
\end{aligned}$$

where $c = \sup_{(t,s)\in[\tau_0,\tau_0+\sigma]^2} |a(t,s)|$. Define $\upsilon\colon [\tau_0, \tau_0+\sigma] \to \mathbb{R}$ such that

$$\upsilon(t) := \int_{\tau_0}^{t} cM(s)\mathrm{d}g(s) + \int_{\tau_0}^{\tau_0+\sigma} a(t,s)M(s)\mathrm{d}g(s), \tag{3.2.8}$$

for every $t \in [\tau_0, \tau_0+\sigma] \to \mathbb{R}$. Since M is a Henstock-Kurzweil-Stieltjes integrable function, $\int_{\tau_0}^{t} cM(s)\mathrm{d}g(s)$ exists for all $t \in [\tau_0, \tau_0+\sigma]$, the same happens for $\int_{\tau_0}^{\tau_0+\sigma} a(t,s)M(s)\mathrm{d}g(s)$. Then υ is well-defined. It is easily verified that υ is nondecreasing. Using (3.2.8), we have

$$\begin{aligned}
\upsilon(t) - \upsilon(\tau) \ &= \ \int_{\tau_0}^{t} cM(s)\mathrm{d}g(s) + \int_{\tau_0}^{\tau_0+\sigma} a(t,s)M(s)\mathrm{d}g(s) \\
&- \ \int_{\tau_0}^{\tau} cM(s)\mathrm{d}g(s) - \int_{\tau_0}^{\tau_0+\sigma} a(\tau,s)M(s)\mathrm{d}g(s) \\
&= \ \int_{\tau}^{t} cM(s)\mathrm{d}g(s) + \int_{\tau_0}^{\tau_0+\sigma} (a(t,s)-a(\tau,s))M(s)\mathrm{d}g(s),
\end{aligned}$$

that is,

$$\left\|\int_{\tau_0}^{t} a(t,s)f(x_s,s)\mathrm{d}g(s) - \int_{\tau_0}^{\tau} a(\tau,s)f(x_s,s)\mathrm{d}g(s)\right\| \leqslant |\upsilon(t)-\upsilon(\tau)| \tag{3.2.9}$$

for all $t, \tau \in [\tau_0, \tau_0+\sigma]$. Next, we will show the continuity of υ in $\tau_1, \tau_2, \cdots, \tau_m$. Notice that every point in $[\tau_0, \tau_0+\sigma]$ at which the function υ is continuous, is a continuity point of the function $t \mapsto \displaystyle\int_{\tau_0}^{t} a(t,s)f(x_s,s)\mathrm{d}g(s)$ by inequality (3.2.9). Let $\upsilon_1(t) = \int_{\tau_0}^{t} cM(s)\mathrm{d}g(s)$, $t \in [\tau_0, \tau_0+\sigma]$, clearly υ_1 is continuous at $t_1, t_2, \cdots, t_m$.

Now, set $\upsilon_2(t) = \int_{\tau_0}^{\tau_0+\sigma} a(t,s)M(s)\mathrm{d}g(s)$, $t \in [\tau_0, \tau_0+\sigma]$. Let us show that υ_2 is continuous at $\tau_1, \ldots, \tau_m$. For this purpose, let $i \in \{1, \cdots, m\}$ and $(t_n)_{n\in\mathbb{N}} \subset [\tau_0, \tau_0+\sigma]$ such that $t_n \to \tau_i$ when $n \to \infty$. Define the sequence of functions

$$\varphi_n(s) := a(t_n,s)M(s), \quad s \in [\tau_0, \tau_0+\sigma],$$

and a function $\varphi\colon [\tau_0, \tau_0+\sigma] \to \mathbb{R}$ by $\varphi(s) := a(\tau_i,s)M(s), s \in [\tau_0, \tau_0+\sigma]$. We have

$$\lim_{n\to\infty} a(t_n,s) = a(\tau_i,s)$$

because a is continuous at τ_i and $(t_n)_{n\in\mathbb{N}} \subset [\tau_0, \tau_0+\sigma]$ converges to τ_i, and therefore

$$\lim_{n\to\infty} \varphi_n(s) = \lim_{n\to\infty} a(t_n, s)M(s) = a(\tau_i, s)M(s) = \varphi(s).$$

Furthermore, according to condition (P3) Section 3.1, $\int_{\tau_0}^{\tau_0+\sigma} a(t_n, s)M(s)\mathrm{d}g(s)$ exists for all $n \in \mathbb{N}$, we get $\int_{\tau_0}^{\tau_0+\sigma} \varphi_n(s)M(s)\mathrm{d}g(s)$ exists for all $n \in \mathbb{N}$. On the other hand, for all $t \in [\tau_0, \tau_0+\sigma]$, $n \in \mathbb{N}$, we have

$$|\varphi_n(t)| = |a(t_n, t)M(t)| \leq c|M(t)| = cM(t).$$

This implies that

$$-cM(t) \leqslant \varphi_n(t) \leqslant cM(t).$$

Also, observe that the integrals $-\int_{\tau_0}^{\tau_0+\sigma} cM(s)\mathrm{d}g(s)$ and $\int_{\tau_0}^{\tau_0+\sigma} cM(s)\mathrm{d}g(s)$ exist, since M is a Henstock-Kurzweil-Stieltjes integrable functions. Since all the hypotheses of Theorem 1.3.62 are satisfied, we obtain

$$\lim_{n\to\infty} \int_{\tau_0}^{\tau_0+\sigma} \varphi_n(s)\mathrm{d}g(s) = \int_{\tau_0}^{\tau_0+\sigma} \varphi(s)\mathrm{d}g(s).$$

Hence, the function υ_2 is continuous at τ_i , for each $i = 1, \cdots, m$. From these facts and by the equality $\upsilon(t) = \upsilon_1(t) + \upsilon_2(t)$, it follows that υ is continuous at $\tau_1, \cdots, \tau_m$. We concluded that $t \mapsto \displaystyle\int_{\tau_0}^{t} a(t,s)f(x_s, s)\mathrm{d}g(s)$ is continuous.

One of our interests lies in the fact that we want to transfer the information from the impulsive Volterra-Stieltjes functional equation in (3.2.7) to the Volterra-Stieltjes functional equation (3.1.1), as done on time scales. This helps us to find results for FDEs in a more economical way. In the next result we do this, we describe how we can transfer the conditions of the Volterra–Stieltjes impulsive functional integral equation to the conditions of the Volterra–Stieltjes functional integral equation.

Lemma 3.2.8. (Grau, Lafetá and Mesquita 2024, Lemma 2.1.1) Let $m \in \mathbb{N}$, $\tau_0 \leqslant t_1 < \cdots < \tau_m \leqslant \tau_0 + \sigma$, $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$ for $k \in \{1, \cdots, m\}$, $a\colon [\tau_0, \tau_0+\sigma]^2 \to \mathbb{R}$ nondecreasing with respect to the first variable, regulated with respect to the second variable and locally bounded on $[\tau_0, \tau_0+\sigma]^2$. Assume that $g\colon [\tau_0, \tau_0+\sigma] \to \mathbb{R}$ is a left-continuous and nondecreasing function. Let $f\colon G([-r,0],\mathbb{R}^n) \times [\tau_0, \tau_0+\sigma] \to \mathbb{R}^n$ be an arbitrary function. Define $\tilde{f}\colon G([-r,0],\mathbb{R}^n) \times [\tau_0, \tau_0+\sigma] \to \mathbb{R}^n$, $\tilde{g}\colon [\tau_0, \tau_0+\sigma] \to \mathbb{R}$ and $\tilde{a}\colon [\tau_0, \tau_0+\sigma]^2 \to \mathbb{R}$ by

$$\tilde{f}(x,s) = \begin{cases} f(x,s), & \text{for } x \in \mathbb{R}^n \text{ and } s \in [\tau_0, \tau_0+\sigma] \setminus \{\tau_k\}_{k=1}^{\infty}, \\ I_k(x(0)), & \text{for } x \in \mathbb{R}^n \text{ and } \tau = \tau_k,\ k \in \{1, \ldots, m\}, \end{cases} \tag{3.2.10}$$

$$\tilde{g}(s) = \begin{cases} g(s), & s \in [\tau_0, \tau_1], \\ g(\tau_k) + k, & s \in (\tau_k, \tau_{k+1}],\ k \in \{1, \dots, m-1\}, \\ g(\tau_m) + m, & s \in (\tau_m, \tau_0 + \sigma], \end{cases} \tag{3.2.11}$$

$$\tilde{a}(t,s) = \begin{cases} a(t,s), & t \in [\tau_0, \tau_0+\sigma] \text{ and } s \in [\tau_0, \tau_0+\sigma] \backslash \{\tau_k\}_{k=1}^m, \\ 1, & t \in [\tau_0, \tau_0+\sigma] \text{ and } s \in \{\tau_k\}_{k=1}^m. \end{cases} \tag{3.2.12}$$

Also, suppose that $I_1, \cdots, I_m \colon \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following condition:

(I) There exists constants $M_2, L_2 > 0$ such that

$$\|I_k(x)\| \leq M_2$$

for every $k \in \{1, \cdots, m\}$ and $x \in \mathbb{R}^n$, and

$$\|I_k(x) - I_k(y)\| \leq L_2 \|x - y\|$$

for every $k \in \{1, \cdots, m\}$ and $x, y \in \mathbb{R}^n$.

Then the functions $\tilde{a}$, $\tilde{f}$ and $\tilde{g}$ also satisfy conditions (P1)–(P5) with $\tilde{a}$, $\tilde{f}$ and $\tilde{g}$ respectively in the place of a, f and g.

Proof. **Assertion 1:** The function $\tilde{g} \colon [\tau_0, \tau_0+\sigma] \to \mathbb{R}$ defined by (3.2.11) is nondecreasing and left-continuous.

Note that by definition of $\tilde{g}$ in terms of g, in addition, we have to $\tilde{g}(u) - \tilde{g}(\vartheta) \geqslant g(u) - g(\vartheta)$ for all $u, \vartheta \in [\tau_0, \tau_0+\sigma]$, since g is nondecreasing and left-continuous, it follows therefore condition (P1) holds for $\tilde{g}$.

Assertion 2: The function $\tilde{a} \colon [\tau_0, \tau_0+\sigma]^2 \to \mathbb{R}$ defined by (3.2.12) is nondecreasing with respect to the first variable, regulated with respect to the second variable.

In fact, the condition (P2) is an immediate consequence from the definition of $\tilde{a}$.

Assertion 3: If the Henstock–Kurzweil–Stieltjes integral

$$\int_{u_1}^{u_2} \tilde{a}(t,s) \tilde{f}(x_s, s)\, \mathrm{d}\tilde{g}(s)$$

exists for each compact interval $[u_1, u_2] \subset [t_0, +\infty)$, all $x \in G([t_0 - r, +\infty), \mathbb{R}^n)$, $t \in [t_0, +\infty)$, and $\tau_0 \leq u_1 < u_2 \leq \tau_0 + \sigma$, then the Henstock-Kurzweil–Stieltjes integral

$$\int_{u_1}^{u_2} a(t,s) f(x_s, s)\, \mathrm{d}g(s)$$

exists for each compact interval $[u_1, u_2] \subset [t_0, +\infty)$, all $x \in G([t_0 - r, +\infty), \mathbb{R}^n)$, $t \in [t_0, +\infty)$ and $\tau_0 \leq u_1 < u_2 \leq \tau_0 + \sigma$.

Assertion 4: If there exists a locally Henstock-Kurzweil–Stieltjes integrable function $M_1 : [t_0, +\infty) \to \mathbb{R}^+$ with respect to g such that for each compact interval $[\tau_0, \tau_0+\sigma] \subset [t_0, +\infty)$, we have

$$\left\| \int_{u_1}^{u_2} b_{u_1,u_2}(s) f(x, s)\, \mathrm{d}g(s) \right\| \leq \int_{u_1}^{u_2} M_1(s) |b_{u_1,u_2}(s)|\, \mathrm{d}g(s)$$

for all $x \in G([\tau_0, \tau_0 + \sigma], \mathbb{R}^n)$, $b_{u_1,u_2}(s) = c_1 a(u_1, s) + c_2 a(u_2, s)$ in $G_2([\tau_0, \tau_0 + \sigma]^2, \mathbb{R})$ with $c_1, c_2 \in \mathbb{R}$ and $\tau_0 \leq u_1 < u_2 \leq \tau_0 + \sigma$. Then there exists a locally Henstock—Kurzweil–Stieltjes integrable function $M\colon [t_0, +\infty) \to \mathbb{R}^+$ with respect to $\tilde{g}$ such that for each compact interval $[\tau_0, \tau_0 + \sigma] \subset [t_0, +\infty)$ we have

$$\left\| \int_{u_1}^{u_2} b_{u_1,u_2}(s) \tilde{f}(x, s)\, \mathrm{d}\tilde{g}(s) \right\| \leq \int_{u_1}^{u_2} M(s) |b_{u_1,u_2}(s)|\, \mathrm{d}\tilde{g}(s),$$

or all $x \in G([\tau_0, \tau_0 + \sigma], \mathbb{R}^n)$, $b_{u_1,u_2}(s) = c_1 a(u_1, s) + c_2 a(u_2, s)$ in $G([\tau_0, \tau_0 + \sigma]^2, \mathbb{R})$ with $c_1, c_2 \in \mathbb{R}$ and $\tau_0 \leq u_1 < u_2 \leq \tau_0 + \sigma$.

Indeed, note that $\tilde{g} - g$ is constant on each of the intervals $[\tau_0, \tau_1], (\tau_1, \tau_2], \cdots, (t_m, \tau_0 + \sigma]$, $\tilde{f}(x, s) = f(x, s)$ for all $s \in [\tau_0, +\tau_0 + \sigma] \setminus \{\tau_k\}_{k=1}^{\infty}$ for hypotheses. applying the Lemma 1.3.58 and the definition of $\tilde{f}$ on (3.2.10) we obtain

$$\begin{aligned}
\int_{u_1}^{u_2} b_{u_2,u_1} \tilde{f}(x_s, s) \mathrm{d}\tilde{g}(s) &= \int_{u_1}^{u_2} b_{u_2,u_1}(s) f(x_s, s) \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \cdots, m\} \\ u_2 \leqslant \tau_k < u_2}} b_{u_2,u_1}(\tau_k) \tilde{f}(x_{\tau_k}, \tau_k) \Delta^+ \tilde{g}(\tau_k) \\
&= \int_{u_1}^{u_2} b_{u_2,u_1}(s) f(x_s, s) \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \cdots, m\} \\ u_2 \leqslant \tau_k < u_2}} b_{u_2,u_1}(\tau_k) I_k(x_{\tau_k}(0)) \Delta^+ \tilde{g}(\tau_k) \\
&= \int_{u_1}^{u_2} b_{u_2,u_1}(s) f(x_s, s) \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \cdots, m\} \\ u_2 \leqslant \tau_k < u_2}} b_{u_2,u_1}(\tau_k) I_k(x(\tau_k)) \Delta^+ \tilde{g}(\tau_k),
\end{aligned}$$

and, therefore,

$$\begin{aligned}
\left\| \int_{u_1}^{u_2} b_{u_2,u_1} \tilde{f}(x_s, s) \mathrm{d}\tilde{g}(s) \right\| &\leq \left\| \int_{u_1}^{u_2} b_{u_2,u_1}(s) f(x_s, s) \mathrm{d}g(s) \right\| \\
&\quad + \left\| \sum_{\substack{k \in \{1, \cdots, m\} \\ u_2 \leqslant \tau_k < u_2}} b_{u_2,u_1}(\tau_k) I_k(x(\tau_k)) \Delta^+ \tilde{g}(\tau_k) \right\| \\
&\leq \int_{u_1}^{u_2} M_1(s) |b_{u_2,u_1}(s)| \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \cdots, m\} \\ u_2 \leqslant \tau_k < u_2}} M_2 |b_{u_2,u_1}(\tau_k)| \Delta^+ \tilde{g}(\tau_k).
\end{aligned}$$

On the other hand, notice that $\tilde{g}(u) - \tilde{g}(\vartheta) \geq g(u) - g(\vartheta)$ whenever $\tau_0 \leq u < \vartheta \leq \tau_0 + \sigma$. It implies together with the definition of the Henstock–Kurzweil–Stieltjes integral

$$\begin{aligned}
\int_{u_1}^{u_2} M_1(s) |b_{u_2,u_1}(s)| \mathrm{d}g(s) &= \sum_{i=1}^{|D|} M_1(\tau_i) |b_{u_2,u_1}(\tau_i)| (g(s_i) - g(s_{i-1})) \\
&\leq \sum_{i=1}^{|D|} M_1(\tau_i) |b_{u_2,u_1}(\tau_i)| (\tilde{g}(s_i) - \tilde{g}(s_{i-1})) \\
&= \int_{u_1}^{u_2} M_1(s) |b_{u_2,u_1}(s)| \mathrm{d}\tilde{g}(s),
\end{aligned}$$

for every δ–fine tagged partition D of $[u_1, u_2]$. Define $\tilde{M}(s) = 1 + M_2 + M_1(s)$ for all $s \in [t_0, t_0 + \sigma]$, notice that $\tilde{M}(s) \geq M_1(s)$, it follows that

$$\int_{u_1}^{u_2} M_1(s)|b_{u_2,u_1}(s)|\mathrm{d}\tilde{g}(s) \leq \int_{u_1}^{u_2} \tilde{M}(s)|b_{u_2,u_1}(s)|\mathrm{d}\tilde{g}(s).$$

Therefore

$$\left\|\int_{u_1}^{u_2} b_{u_2,u_1}\tilde{f}(x_s, s)\mathrm{d}\tilde{g}(s)\right\| \leq \int_{u_1}^{u_2} \tilde{M}(s)|b_{u_2,u_1}(s)|\mathrm{d}\tilde{g}(s) + \sum_{\substack{k\in\{1,\cdots,m\}\\ u_2\leqslant t_k<u_2}} M_2|b_{u_2,u_1}(t_k)|\Delta^+\tilde{g}(t_k). \tag{3.2.13}$$

On the other hand, the function

$$h(t) := \int_{t_0}^{t} \tilde{M}(s)|b_{u_2,u_1}(s)|\mathrm{d}\tilde{g}(s), \qquad \text{for } t \in [\tau_0, \tau_0 + \sigma]$$

is nondecreasing and by definition $\Delta^+h(\tau_k) = \tilde{M}(\tau_k)|b_{u_2,u_1}(\tau_k)|\Delta^+\tilde{g}(\tau_k)$ for $k \in \{1, \cdots, m\}$ by Corallary 1.3.57. Hence

$$\sum_{\substack{k\in\{1,\cdots,m\}\\ u_2\leqslant\tau_k<u_2}} M_2|b_{u_2,u_1}(\tau_k)|\Delta^+\tilde{g}(\tau_k) \leq \sum_{\substack{k\in\{1,\cdots,m\}\\ u_2\leqslant\tau_k<u_2}} \tilde{M}(s)|b_{u_2,u_1}(\tau_k)|\Delta^+\tilde{g}(\tau_k) \leq h(u_2) - h(u_1), \tag{3.2.14}$$

as a result

$$\begin{aligned}\sum_{\substack{k\in\{1,\cdots,m\}\\ u_2\leqslant\tau_k<u_2}} M_2|b_{u_2,u_1}(\tau_k)|\Delta^+\tilde{g}(\tau_k) &\leq \int_{\tau_0}^{u_2} \tilde{M}(s)|b_{u_2,u_1}(s)|\mathrm{d}\tilde{g}(s) - \int_{\tau_0}^{u_1} \tilde{M}(s)|b_{u_2,u_1}(s)|\mathrm{d}\tilde{g}(s)\\ &= \int_{u_1}^{u_2} \tilde{M}(s)|b_{u_2,u_1}(s)|\mathrm{d}\tilde{g}(s).\end{aligned} \tag{3.2.15}$$

we have from (3.2.13), (3.2.14) and (3.2.15)

$$\left\|\int_{u_1}^{u_2} b_{u_2,u_1}\tilde{f}(x_s, s)\mathrm{d}\tilde{g}(s)\right\| \leq 2\int_{u_1}^{u_2} \tilde{M}(s)|b_{u_2,u_1}(s)|\mathrm{d}\tilde{g}(s).$$

Now, defining $M(t) = 2\tilde{M}(t)$ for all $t \in [\tau_0, \tau_0 + \sigma]$, we get the Assertion 4.

Assertion 5: If there exists a regulated function $L_1 : [t_0, +\infty) \to \mathbb{R}^+$ such that

$$\left\|\int_{u_1}^{u_2} a(u_2, s)[f(x_s, s) - f(z_s, s)]\,\mathrm{d}g(s)\right\| \leq \int_{u_1}^{u_2} L_1(s)\|x_s - z_s\|_\infty\,\mathrm{d}g(s)$$

for all $x, z \in G([t_0 - r, +\infty), \mathbb{R}^n)$ and $t_0 \leq u_1 < u_2 \leq t_0 + \sigma$ and there exists a constant $L_2 > 0$ such that

$$L_1(t) \leq L_2, \quad \text{for all } t \in [t_0, +\infty),$$

then there exists a locally regulated function $L\colon [t_0, +\infty) \to \mathbb{R}^+$ such that for each compact interval $[u_1, u_2] \subset [t_0, +\infty)$, we have

$$\left\|\int_{u_1}^{u_2} \tilde{a}(u_2, s)[\tilde{f}(x_s, s) - \tilde{f}(z_s, s)]\,\mathrm{d}\tilde{g}(s)\right\| \leq \int_{u_1}^{u_2} L(s)\|x_s - z_s\|_\infty\,\mathrm{d}\tilde{g}(s)$$

for all $x, z \in G([t_0 - r, +\infty), \mathbb{R}^n)$ and $\tau_0 \leq u_1 < u_2 \leq \tau_0 + \sigma$.

Let us demonstrate Assertion 5. For this purpose, let $x, z \in G([t_0 - r, +\infty), \mathbb{R}^n)$ and compact interval $[u_1, u_2] \subset [t_0, +\infty)$. Using Lemma 1.3.58 again, we obtain

$$\begin{aligned}
&\int_{u_1}^{u_2} a(u_2, s)[\tilde{f}(x_s, s) - \tilde{f}(z_s, s)]\, \mathrm{d}g(s) \\
= &\int_{u_1}^{u_2} a(u_2, s)[f(x_s, s) - f(z_s, s)]\mathrm{d}g(s) + \sum_{\substack{k \in \{1, \cdots, m\} \\ u_2 \leqslant \tau_k < u_2}} a(u_2, \tau_k)[I_k(x_{\tau_k}(0)) - I_k(z_{\tau_k}(0))]\Delta^+ \tilde{g}(\tau_k) \\
= &\int_{u_1}^{u_2} a(u_2, s)[f(x_s, s) - f(z_s, s)]\mathrm{d}g(s) + \sum_{\substack{k \in \{1, \cdots, m\} \\ u_2 \leqslant \tau_k < u_2}} a(u_2, \tau_k)[I_k(x(\tau_k)) - I_k(z(\tau_k))]\Delta^+ \tilde{g}(\tau_k).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\| \int_{u_1}^{u_2} a(u_2, s)[\tilde{f}(x_s, s) - \tilde{f}(z_s, s)]\, \mathrm{d}g(s) \right\| \\
\leq &\int_{u_1}^{u_2} |a(u_2, s)| L_1(s) \|x_s - z_s\|_\infty \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \cdots, m\} \\ u_2 \leqslant \tau_k < u_2}} |a(u_2, \tau_k)| L_2 \|x(\tau_k) - z(\tau_k)\|_\infty \Delta^+ \tilde{g}(\tau_k).
\end{aligned}$$

Define the function $\tilde{L}(s) = 1 + L_2 + L_1(s)$ for all $s \in [t_0, +\infty)$. Therefore

$$\int_{u_1}^{u_2} |a(u_2, s)| L_1(s) \|x_s - z_s\|_\infty \mathrm{d}g(s) \leq \int_{u_1}^{u_2} |a(u_2, s)| \tilde{L}(s) \|x_s - z_s\|_\infty \mathrm{d}g(s),$$

next, observe that the function

$$\gamma(t) = \int_{t_0}^{t} \tilde{L}(s) |a(u_2, s)| \|x_s - z_s\|_\infty \mathrm{d}\tilde{g}(s), \quad t \in [t_0, +\infty),$$

is nondecreasing and

$$\Delta^+ \gamma(\tau_k) = \tilde{L}(\tau_k) |a(u_2, \tau_k)| \|x_{\tau_k} - z_{\tau_k}\|_\infty \Delta^+ \tilde{g}(\tau_k), \qquad \text{for } k \in \{1, \cdots, m\}.$$

Consequently,

$$\begin{aligned}
&\sum_{\substack{k \in \{1, \cdots, m\} \\ u_2 \leqslant \tau_k < u_2}} |a(u_2, \tau_k)| L_2 \|x(\tau_k) - z(\tau_k)\|_\infty \Delta^+ \tilde{g}(\tau_k) \\
\leq &\sum_{\substack{k \in \{1, \cdots, m\} \\ u_2 \leqslant \tau_k < u_2}} |a(u_2, \tau_k)| \tilde{L}(s) \|x(\tau_k) - z(\tau_k)\|_\infty \Delta^+ \tilde{g}(\tau_k) \leq \gamma(u_2) - \gamma(u_1) \\
= &\int_{u_1}^{u_2} \tilde{L}(s) |a(u_2, s)| \|x_s - z_s\|_\infty \mathrm{d}\tilde{g}(s).
\end{aligned}$$

It follows that

$$\left\| \int_{u_1}^{u_2} a(u_2, s)[\tilde{f}(x_s, s) - \tilde{f}(z_s, s)]\, \mathrm{d}g(s) \right\| \leq 2 \int_{u_1}^{u_2} \tilde{L}(s) |a(u_2, s)| \|x_s - z_s\|_\infty \mathrm{d}\tilde{g}(s).$$

Now, defining $L(t) = 2\tilde{L}(t)$ for all $t \in [t_0, +\infty)$, we get the desired assertion 5. □

The following theorem describes a strong relation between the solutions of impulsive Volterra–Stieltjes integral equations (3.2.7) and the solutions of Volterra–Stieltjes integral equations without impulses (3.1.1).

Theorem 3.2.9. (Lafetá 2022, Theorem 2.1.2) Let $m \in \mathbb{N}$, $\tau_0 \leq \tau_1 < \cdots < \tau_m \leqslant \tau_0 + \sigma$, $I_1, \dots, I_m \colon \mathbb{R}^n \to \mathbb{R}^n$, and $f \colon G([-\tau, 0], \mathbb{R}^n) \times [\tau_0, \tau_0+\sigma] \to \mathbb{R}^n$. Suppose $g \colon [\tau_0, \tau_0+\sigma] \to \mathbb{R}$ is regulated, left-continuous, and continuous at $\tau_1, \dots, \tau_m$; $a \colon [\tau_0, \tau_0 + \sigma]^2 \to \mathbb{R}$ is nondecreasing in the first variable, regulated in the second, locally bounded on $[\tau_0, \tau_0+\sigma]^2$, and continuous in the first variable at $\tau_1, \dots, \tau_m$. Then, $x \colon [\tau_0 - r, \tau_0 + \sigma] \to \mathbb{R}^n$ is a solution of:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^t a(t,s) f(x_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\} \\ \tau_k < t}} I_k(x(\tau_k)) & t \in [\tau_0, \tau_0 + \sigma] \\ x_{\tau_0} = \phi \qquad t \in [\tau_0 - r, \tau_0] \end{cases} \tag{3.2.16}$$

if and only if $x \colon [\tau_0 - r, \tau_0 + \sigma] \to \mathbb{R}^n$ is a solution of:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^t \tilde{a}(t,s) \tilde{f}(x_s, s)\, d\tilde{g}(s), \\ x_{\tau_0} = \phi. \end{cases} \tag{3.2.17}$$

Proof. Define $\tilde{f}, \tilde{g}$ and $\tilde{a}$ as is (3.2.10), (3.2.11) and (3.2.12), respectively. Thus, from the definition of $\tilde{g}$, it is clear that $\Delta^+ \tilde{g}(\tau_k) = 1$ for every $k \in \{1, \dots, m\}$. By hypothesis, the integral

$$\int_{\tau_0}^t a(t,s) f(x_s, s)\,\mathrm{d}g(s)$$

exists for all $t \in [\tau_0, \tau_0 + \sigma]$ and for every $x \in G([\tau_0 - r, \tau_0 + \sigma], \mathbb{R}^n)$. Therefore, by the definition of $\tilde{a}, \tilde{f}$ and $\tilde{g}$, the functions $\tilde{a}, \tilde{f}$ and $\tilde{g}$ inherit the same properties as a, f and g (see Lemma 3.2.8), hence the integral

$$\int_{\tau_0}^t \tilde{a}(t,s) \tilde{f}(x_s, s)\,\mathrm{d}\tilde{g}(s)$$

also exists.

According to Lemma 1.3.58 and the definitions of $\tilde{a}$, $\tilde{f}$, and $\tilde{g}$, we have:

$$\begin{aligned} \int_{\tau_0}^t \tilde{a}(t,s) \tilde{f}(x_s, s)\,\mathrm{d}\tilde{g}(s) &= \int_{\tau_0}^t a(t,s) f(x_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\} \\ \tau_k < t}} \tilde{f}(x_{\tau_k}, \tau_k) \Delta^+ \tilde{g}(\tau_k) \\ &= \int_{\tau_0}^t a(t,s) f(x_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k \in \{1,\dots,m\} \\ \tau_k < t}} I_k(x(\tau_k)). \end{aligned}$$

□

With everything that has been presented, we can consider the following conditions for the equation (3.2.7):

(D1) The function $g \colon [t_0, +\infty) \to \mathbb{R}$ is nondecreasing and left–continuous on (t_0, ∞).

(D2) The function $a\colon [t_0,\infty)^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable and regulated with respect to the second variable.

(D3) The Henstock–Kurzweil–Stieltjes integral

$$\int_{\tau_1}^{\tau_2} a(t,s)f(x_s,s)\mathrm{d}g(s)$$

exists for all $x \in G([t_0 - r, +\infty), \mathbb{R}^n)$, $t \in [t_0, +\infty)$ and all $t_0 \leqslant \tau_1 \leqslant \tau_2 < +\infty$.

(D4) There exists a locally Henstock–Kurzweil–Stieltjes integrable function $M\colon [t_0,+\infty) \to \mathbb{R}^+$ with respect to g such that

$$\left\| \int_{\tau_1}^{\tau_2} \left(c_2 a(\tau_2,s) + c_1 a(\tau_1,s)\right) f(x_s,s)\mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |c_2 a(\tau_2,s) + c_1 a(\tau_1,s)|\, M(s)\mathrm{d}g(s),$$

for all $x \in G([t_0 - r, +\infty), \mathbb{R}^n)$, all $c_1, c_2 \in \mathbb{R}$ and all $t_0 \leqslant \tau_1 \leqslant \tau_2 < \infty$, and there exists a constant $M_2 > 0$ such that

$$\|I_k(x)\| \leqslant M_2,$$

for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}^n$.

(D5) There exists a regulated function $L\colon [t_0,+\infty) \to \mathbb{R}^+$ such that

$$\left\| \int_{\tau_1}^{\tau_2} a(\tau_2,s)[f(x_s,s) - f(z_s,s)]\mathrm{d}g(s) \right\| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\, L(s)\, \|x_s - z_s\|_\infty \mathrm{d}g(s),$$

for all $x, z \in G([t_0 - r, +\infty), \mathbb{R}^n)$, and all $t_0 \leqslant \tau_1 \leqslant \tau_2 < +\infty$, and there exists a constant $L_2 > 0$ such that

$$\|I_k(x) - I_k(y)\| \leqslant L_2 \|x - y\|, \tag{3.2.18}$$

for all $k \in \mathbb{N}$ and all x, $y \in \mathbb{R}^n$.

As an immediate consequence, we obtain a result about existence and uniqueness of solutions of impulsive Volterra–Stieltjes integral equation. We omit its proof, since it follows directly from the correspondence and the analogue result for Volterra–Stieltjes integral equation

Theorem 3.2.10. (Lafetá 2022, Theorem 3.3.1) Let $\{\tau_k\}_{k=0}^{\infty}$ be the moments of impulses in $[t_0,\infty)$, such that $\tau_k < \tau_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to+\infty} \tau_k = +\infty$. Assume that $g\colon [t_0,+\infty) \to \mathbb{R}$ is a regulated left-continuous function which is continuous at $\{\tau_k\}_{k=1}^{\infty}$; $a\colon [t_0,+\infty)^2 \to \mathbb{R}$ is non-decreasing with respect to the first variable, regulated with respect to the second, locally bounded on $[t_0,+\infty)^2$, and continuous with respect to the first variable at $\{\tau_k\}_{k=1}^{\infty}$. Also, suppose that $I_k : \mathbb{R}^n \to \mathbb{R}^n$ for each $k \in \mathbb{N}$, and

$f : G([-\tau, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$ satisfy the following conditions (D3) to (D5). Then, for all $\phi \in G([-\tau, 0], \mathbb{R}^n)$, there exists a unique solution $x : I \to \mathbb{R}^n$ of the initial value problem:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s, s)\, \mathrm{d}g(s) + \sum_{\substack{k\in\mathbb{N} \\ t_k<t}} I_k(x(t_k)), \quad t \geq t_0 \\ x_{t_0} = \phi, \end{cases}$$

where $I = [t_0 - r, \omega)$, $\omega \leq \infty$

Then, for all $\phi \in G([t_0 - r, 0], \mathbb{R}^n)$, there exist $\sigma > 0$ and a unique solution $x : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$ to the initial value problem:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s, s)\, dg(s) + \sum_{\substack{k\in\{1,\dots,m\} \\ t_k<t}} I_k(x(t_k)), \\ x_{t_0} = \phi. \end{cases}$$

3.2.3 Fractional differential equations

The purpose of this subsection is to justify the study of fractional-order equations within the framework of Volterra–Stieltjes impulsive functional integral equations. In particular, we show that impulsive fractional functional differential equations can be rewritten as Volterra-type integral equations, and therefore belong to the class of problems investigated in this work.

Fractional differential equations are widely used to model systems with memory and hereditary effects, which cannot be adequately described by classical integer-order models (Lazarević, Šekara and Rapaic 2014). Such equations naturally lead to integral formulations of Volterra type.

To proof why Volterra-Stieltjes impulsive functional equations encompass impulsive fractional functional differential equations, we consider the following fractional functional differential equation with impulses:

$$\begin{cases} {}_C\mathcal{D}_0^{\rho} x(t) = f(t, x_t), & t \in J := [0, \beta] \setminus \{t_1, \dots, t_m\}, \\ \Delta x(t_k) := x(t_k^+) - x(t_k) = I_k(x(t_k)), & k = 1, 2, \dots, m, \\ x|_{[-\tau,0]} = \varphi, \end{cases} \tag{3.2.19}$$

where f is Lebesgue measurable in t and continuous in x_t, and I_k are continuous impulse functions and ${}_C\mathcal{D}_0^{\rho}$ denotes the Caputo fractional derivative of order $\rho \in (0, 1)$, defined by

$${}_C\mathcal{D}_0^{\rho} x(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t (t-s)^{-\rho} x'(s)\, \mathrm{d}s.$$

In (Guo and Jiang 2011), the authors establish fundamental results of existence, uniqueness, and continuous dependence for an equation (3.2.19) for $0 < \rho < 1$. They also show that the solution to this equation is given by the Volterra system of equations

$$x(t) = \begin{cases} \phi(t), & \text{for } t \in [-\tau, 0], \\ \phi(0) + \dfrac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s, x_s)\, ds, & \text{for } t \in [0, t_1], \\ \phi(0) + I_1(x(\tau_1)) + \dfrac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s, x_s)\, \mathrm{d}s, & \text{for } t \in [t_1, t_2], \\ \vdots \\ \phi(0) + \sum_{k=1}^{m} I_k(x(\tau_k)) + \dfrac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s, x_s)\, \mathrm{d}s, & \text{for } t \in [t_m, T]. \end{cases} \tag{3.2.20}$$

Note that the solution given by (3.2.20) is a special case of the Volterra-Stieltjes impulsive functional integral equations, taking into account that the Henstock-Kurzweil integral generalizes the Lebesgue integral. The similar result of the problem discussed in (Guo and Jiang 2011) is open to a solution in which the integral is given in the Stieltjes sense.

Remark 3.2.11. However, note that the equation in (3.2.20) does not fit those studied in this work, since the fractional order $\rho \in (0, 1)$ causes the kernel to be nondecreasing.

To illustrate a case compatible with our setting, we consider the following boundary value problem with $1 < \rho < 2$ (Wang, Zhou and Fečkan 2012):.

Example 3.2.12. Let u be a function $u\colon [0, \beta] \to \mathbb{R}$. Consider the boundary value problems of order $1 < \rho < 2$, given by

$$\begin{cases} {}_C\mathcal{D}_t^{\rho} u(t) = f(u(t), t), & t \in [0, \beta], \\ \Delta u(t_k) = y_k, \Delta u'(t_k) = \overline{y}_k & k = 1, \dots, m \\ u(0) = 0, u'(1) = 0 \end{cases} \tag{3.2.21}$$

where $y_k, \bar{y}_k \in \mathbb{R}$.

The formula of solutions for problem (3.2.21) should be

$$u(t)=\begin{cases}\dfrac{1}{\Gamma(\rho)}\displaystyle\int_0^t (t-s)^{\rho-1}f(s,u(s))\mathrm{d}s \\ \quad -\left(\dfrac{1}{\Gamma(\rho-1)}\displaystyle\int_0^1 (1-s)^{\rho-2}f(s,u(s))\mathrm{d}s+\sum_{k=1}^m \overline{y}_k\right)t, \qquad t\in[0,t_1) \\ \dfrac{1}{\Gamma(\rho)}\displaystyle\int_0^t (t-s)^{\rho-1}f(s,u(s))\mathrm{d}s+\overline{y}_1(t-t_1)+y_1 \\ \quad -\left(\dfrac{1}{\Gamma(\rho-1)}\displaystyle\int_0^1 (1-s)^{\rho-2}f(s,u(s))\mathrm{d}s+\sum_{k=1}^m \overline{y}_k\right)t, \qquad t\in(t_1,t_2] \\ \vdots \\ \dfrac{1}{\Gamma(\rho)}\displaystyle\int_0^t (t-s)^{\rho-1}f(s,u(s))\mathrm{d}s+\sum_{k=1}^m \overline{y}_k(t-t_k)+\sum_{i=1}^k y_i \\ \quad -\left(\dfrac{1}{\Gamma(\rho-1)}\displaystyle\int_0^1 (1-s)^{\rho-2}f(s,u(s))\mathrm{d}s+\sum_{k=1}^m \overline{y}_k\right)t, \qquad t\in(t_k,t_{k+1}],\ k=1,\dots,m.\end{cases}$$

The solution of (3.2.21) can be written in integral form as a Volterra equation with impulsive terms. Therefore, impulsive fractional differential equations naturally lead to Volterra–Stieltjes integral formulations.

This motivates the study of impulsive functional integral equations of the form

$$\begin{cases} x(t)=\phi(0)+\displaystyle\int_{\tau_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(x_s,s)\,\mathrm{d}s, \quad t\geqslant\tau_0 \\ x_{\tau_0}=\phi, \end{cases} \tag{3.2.22}$$

where Γ is the gamma function, $1<\alpha<2$, $\tau_0\geqslant t_0$, $\phi\in G([-r,0],\mathbb{R})$ and $f\colon G([-r,0],\mathbb{R})\times[t_0,+\infty)\to\mathbb{R}$.

CHAPTER

4

BOUNDEDNESS RESULTS FOR VOLTERRA-STIELJTES EQUATIONS

In this chapter, we establish conditions that guarantee the uniform boundedness, the quasi-uniform boundedness, the final boundedness, and the uniform exponential boundedness of the solutions of the Volterra-Stieltjes functional integral equation (3.1.1), where the integral is in the Perron-Stieltjes sense. Our analysis is inspired by the notions of boundedness developed in (Afonso *et al.* 2021) and (Bonotto *et al.* 2021). Building upon their foundational framework, we extend these concepts to the setting of Volterra–Stieltjes equations, deriving specific criteria that guarantee the various forms of boundedness for solutions of this broader class of functional equations.

As in the previous sections, we assume that the conditions (P1)–(P5) hold and that, for each $\tau_0 \geq t_0$ and every $\phi \in G([-r, 0], \mathbb{R}^n)$, there exists a unique solution

$$x(t) = x(t, \tau_0, \phi),$$

defined on $[\tau_0 - r, +\infty)$, of the functional Volterra–Stieltjes integral equation (3.1.1). This follows from the existence and prolongation results established in Chapter 3 (Theorems 3.1.6 and 3.1.10).

Although the solution is defined on the unbounded interval $[\tau_0, +\infty)$, all assumptions and analytical arguments in this chapter are formulated on finite subintervals $[t_0, t]$, since the required conditions are local. On bounded intervals, the Perron–Stieltjes integral coincides with the Henstock–Kurzweil integral and preserves the structural properties needed for the analysis. Therefore, in what follows, all integrals are understood in the sense of Perron.

For these reasons, in conditions (P1)–(P5), we replace the Henstock–Kurzweil integral by the Perron integral, adopting the latter as the appropriate framework for the analysis developed in this chapter.

4.1 Boundedness results

The following concepts are inspired by (Stamova 2008, Definition 1).

Definition 4.1.1. The functional Volterra–Stieltjes integral equation (3.1.1) is said to be

(i) **Uniformly bounded**, if for all $\delta > 0$, there exists a $c = c(\delta) > 0$ such that if $\phi \in G([-r,0],\mathbb{R}^n)$ with $\|\phi\|_\infty < \delta$ and $\tau_0 \geqslant t_0$, then $\|x(t,\tau_0,\phi)\| < c$, for all $t \in [\tau_0, +\infty)$.

(ii) **Quasi-uniformly ultimately bounded**, if there exists $K > 0$ so that for all $\delta > 0$ exists $\eta = \eta(\delta) > 0$ such that for all $\tau_0 \geqslant t_0$ and $\phi \in G([-r,0],\mathbb{R}^n)$ with $\|\phi\|_\infty < \delta$, we have $\|x(t,\tau_0,\phi)\| < K$, for all $t \geqslant \tau_0 + \eta$.

(iii) **Uniformly ultimately bounded**, if is uniformly bounded and quasi-uniformly ultimately bounded.

(iv) **Uniformly p-exponentially bounded**, if there exists $\lambda \in \mathbb{R}$ and an increasing function $p\colon [t_0,+\infty) \to \mathbb{R}$ so that for all $\delta > 0$, there exists $B = B(\delta) > 0$ such that for all $\phi \in G([-r,0],\mathbb{R}^n)$ with $\|\phi\|_\infty < \delta$ and all $\tau_0 \geqslant t_0$, we have $\|x(t,\tau_0,\phi)\| \leqslant Be^{\lambda(p(t)-p(\tau_0))}$ for all $t \geqslant \tau_0$.

Remark 4.1.2. With respect to Definition 4.1.1–(iv), if $p(t) = t$ for all $t \geqslant t_0$, our definition coincides with the known definitions for exponentially bounded.

We introduce the following notations:

$$H_0 := \{\mu\colon \mathbb{R}^+ \to \mathbb{R}^+ : \mu \text{ is continuous increasing and } \mu(0) = 0\}$$

$$H_\infty := \{l\colon \mathbb{R}^+ \to \mathbb{R}^+ : l \text{ is increasing, } l(0) = 0 \text{ and } l(t) \overset{t\to+\infty}{\longrightarrow} +\infty\},$$

where $\mathbb{R}^+ := [0,+\infty)$.

Definition 4.1.3. We say that $V\colon [t_0,+\infty) \times G([-r,0],\mathbb{R}^n) \to \mathbb{R}$, is a **Lyapunov functional** with respect to the functional Volterra–Stieltjes integral equation (3.1.1), if the following conditions are satisfied:

(i) there exists $l \in H_\infty$ such that

$$l(\|x(t)\|) \leqslant V(t,x_t), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t,\tau_0,\phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1).

(ii) $[\tau_0,+\infty) \ni t \mapsto V(t,x_t)$ is a nonincreasing functional, for all solution $x(t) = x(t,\tau_0,\phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1).

The following result is inspired by the proof of (Gallegos, Grau and Mesquita 2021, Theorem 3.9). We are going to omit its proof, since it is very similar to the one in found (Gallegos, Grau and Mesquita 2021) with some slight modifications.

Lemma 4.1.4. Let $V\colon [t_0,+\infty)\times G([-r,0],\mathbb{R}^n)\to\mathbb{R}$ be a functional that satisfies the condition (ii) from Definition 4.1.3. Moreover, suppose that there exists a constant $\alpha>0$ such that

$$V(t,x_t)-V(s,x_s)\leqslant -\alpha\int_s^t V(\xi,x_\xi)\mathrm{d}\xi,\quad \tau_0\leqslant s\leqslant t,$$

for all solution $x(t)=x(t,\tau_0,\phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1). Then

$$V(t,x_t)\leqslant e^{-\lambda(t-\tau_0)}V(\tau_0,\phi),$$

for all $t\in[\tau_0,+\infty)$.

The next result gives us sufficient conditions to ensure that the functional Volterra–Stieltjes integral equation (3.1.1) is uniformly bounded.

Theorem 4.1.5. Consider the functional Volterra–Stieltjes integral equation (3.1.1). Suppose that the conditions (P1)–(P5) hold. Let $V\colon [t_0,+\infty)\times G([-r,0],\mathbb{R}^n)\to\mathbb{R}$ be a Lyapunov functional with respect to the functional Volterra–Stieltjes integral equation (3.1.1). Suppose the following condition is satisfied:

(B0) There exists $\mu\in H_0$ such that

$$V(t,x_t)\leqslant\mu(\|x(t)\|),\quad t\geqslant\tau_0,$$

for all solution $x(t)=x(t,\tau_0,\phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1).

Then (3.1.1) is uniformly bounded.

Proof. According to Definition 4.1.3, there exists $l\in H_\infty$ such that

$$l(\|x(t)\|)\leqslant V(t,x_t),\quad t\geqslant\tau_0,$$

for all solution $x(t)=x(t,\tau_0,\phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1).

Let $\delta>0$ be given. Since $l(t)\overset{t\to+\infty}{\longrightarrow}+\infty$ and $\mu(\delta)>0$, there exists $c=c(\delta)>0$ such that

$$\mu(\delta)<l(c). \tag{4.1.1}$$

Now, let $\tau_0\geqslant t_0$, $\phi\in G([-r,0],\mathbb{R}^n)$ with $\|\phi\|_\infty<\delta$ and $x(t)=x(t,\tau_0,\phi)$ be the solution of the functional Volterra–Stieltjes integral equation (3.1.1).

We show that $\|x(t)\| = \|x(t, \tau_0, \phi)\| < c$, for all $t \geqslant \tau_0$.

Indeed, assume by contradiction that there exists $\xi \in [\tau_0, +\infty)$ such that

$$\|x(\xi)\| \geqslant c.$$

Hence,

$$l(c) \leqslant l(\|x(\xi)\|) \leqslant V(\xi, x_\xi).$$

On the other hand, by condition (ii) from Definition 4.1.3, (B0) and (4.1.1), we have

$$V(\xi, x_\xi) \leqslant V(\tau_0, x_{\tau_0}) \leqslant \mu(\|x(\tau_0)\|) = \mu(\|\phi(0)\|) \leqslant \mu(\|\phi\|_\infty) < \mu(\delta) < l(c).$$

It implies that $V(\xi, x_\xi) < l(c)$, which is a contradiction. Hence, $\|x(t)\| = \|x(t, \tau_0, \phi)\| < c$, for all $t \geqslant \tau_0$, and therefore, (3.1.1) is uniformly bounded. □

The following theorem provides us with a criterion to guarantee that the functional Volterra–Stieltjes integral equation (3.1.1) is quasi-uniformly ultimately bounded.

Theorem 4.1.6. Consider the functional Volterra–Stieltjes integral equation (3.1.1). Suppose that the conditions (P1)–(P5) hold. Let $V\colon [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ be a functional such that V is bounded on each set $[t_0, +\infty) \times B$, where B is a bounded subset of $G([-r, 0], \mathbb{R}^n)$. Moreover assume that:

(B1) There exists a continuous increasing function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $h(0) = 0$ such that

$$h(\|\varphi(0))\|) \leqslant V(t, \varphi),$$

for all $t \geqslant t_0$ and all $\varphi \in G([-r, 0], \mathbb{R}^n)$.

(B2) There exists a real number $\alpha > 0$ such that

$$V(t, x_t) - V(s, x_s) \leqslant -\alpha \int_s^t V(\xi, x_\xi) \mathrm{d}\xi, \quad \tau_0 \leqslant s \leqslant t,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1).

Then, (3.1.1) is quasi-uniformly ultimately bounded.

Proof. Let $\delta > 0$ be given. Define $B_\delta := \{\varphi \in G([-r, 0], \mathbb{R}^n) : \|\varphi\|_\infty < \delta\}$. Note that B_δ is a bounded subset of $G([-r, 0], \mathbb{R}^n)$, and therefore, according to the hypothesis, we have that $b_\delta := \sup\{V(t, \varphi) : (t, \varphi) \in [t_0, +\infty) \times B_\delta\}$ is well-defined. It is easy to check from condition (B1) that $b_\delta > 0$. On the other hand, take $\eta > 0$ such that $\eta > \frac{\ln(b_\delta)}{\alpha}$.

Now, let $\tau_0 \geqslant t_0$, $\phi \in G([-r, 0], \mathbb{R}^n)$ with $\|\phi\|_\infty < \delta$ and $x(t) = x(t, \tau_0, \phi)$ be the solution of the functional Volterra–Stieltjes integral equation (3.1.1).

Since $(\tau_0, \phi) \in [t_0, +\infty) \times B_\delta$, it follows from the definition of b_δ that

$$V(\tau_0, \phi) \leqslant b_\delta. \tag{4.1.2}$$

On the other hand, according to the conditions (B1)-(B2), V satisfies condition (ii) from Definition 4.1.3. Using this fact together with (B2), Lemma 4.1.4 guarantees that

$$V(t, x_t) \leqslant e^{-\alpha(t-\tau_0)} V(\tau_0, \phi), \text{ for all } t \geqslant \tau_0. \tag{4.1.3}$$

Note that for each $t \geqslant \tau_0 + \eta$, we have $-\alpha(t - \tau_0) \leqslant -\alpha\eta < -\ln(b_\delta)$, which implies

$$e^{-\alpha(t-\tau_0)} < \frac{1}{b_\delta}, \quad \text{for all } t \geqslant \tau_0 + \eta. \tag{4.1.4}$$

Hence, by (B1), (4.1.2), (4.1.3) and (4.1.4), we obtain

$$h(\|x(t)\|) = h(\|x_t(0)\|) \leqslant V(t, x_t) < 1, \text{ for all } t \geqslant \tau_0 + \eta.$$

Since h is an increasing function, it follows that $\|x(t)\| = \|x(t, \tau_0, \phi)\| < 1$, for all $t \geqslant \tau_0 + \eta$, obtaining the desired result.

□

The next theorem ensures us that the functional Volterra–Stieltjes integral equation (3.1.1) is uniformly ultimately bounded.

Theorem 4.1.7. Consider the functional Volterra–Stieltjes integral equation (3.1.1). Suppose that the conditions (P1)–(P5) hold. Let $V\colon [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ be a functional so that V satisfies condition (i) from Definition 4.1.3 and condition (B0) from Theorem 4.1.5. Moreover, assume that

(B3) there exists a continuous and increasing function $\gamma\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$ such that
$$V(t, x_t) - V(s, x_s) \leqslant -\int_s^t \gamma(\|x(\xi)\|)\mathrm{d}\xi, \quad \tau_0 \leqslant s \leqslant t,$$
for all solution $x(t) = x(t, \tau_0, \phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1).

Then, (3.1.1) is uniformly ultimately bounded.

Proof. According to the condition (B0) and (i) from Definition 4.1.3, there are $l \in H_\infty$ and $\mu \in H_0$ such that

$$l(\|x(t)\|) \leqslant V(t, x_t) \leqslant \mu(\|x(t)\|), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1).

On the other hand, it is easy to see that (B3) implies that condition (ii) from Definition 4.1.3 is satisfied. Thus, all hypotheses from Theorem 4.1.5 are satisfied, which implies that (3.1.1) is uniformly bounded.

Now, we shall show that (3.1.1) is quasi-uniformly ultimately bounded.

Indeed, according to uniform boundedness of the functional Volterra–Stieltjes integral equation (3.1.1), there exists $K = K(\varrho) > 0$ such that for all $s_0 \geqslant \tau_0$ and $\phi \in G([-r,0],\mathbb{R}^n)$ with $\|\phi\|_\infty < \varrho$, we have $\|x(t)\| = \|x(t,s_0,\phi)\| < K$, for all $t \geqslant s_0$.

Let $\delta > 0$ be given. By the continuity of μ and the fact that $\mu(0) = 0$, we can choose $\nu = \nu(K) > 0$ so that $\mu(\nu) < l(K)$. Define $\eta := \eta(\delta)$ be such that $\eta > \frac{\mu(\delta)}{\gamma(\nu)}$.

Let $\tau_0 \geqslant t_0$, $\phi \in G([-r,0],\mathbb{R}^n)$ with $\|\phi\|_\infty < \delta$ and $x(t) = x(t,\tau_0,\phi)$ be the functional Volterra–Stieltjes integral equation (3.1.1). In order to prove that $\|x(t)\| < K$, for all $t \geqslant \tau_0 + \eta$, we claim that there exists $s \in [\tau_0, \tau_0 + \eta]$ such that

$$\|x(s)\| < \nu. \tag{4.1.5}$$

For contradiction, assume that $\|x(\xi)\| \geqslant \nu$ for all $\xi \in [\tau_0, \tau_0 + \eta]$. Using this fact together with (B3) and $\|x(\tau_0)\| = \|\phi(0)\| < \delta$, we obtain

$$V(\tau_0+\eta, x_{\tau_0+\eta}) \leqslant V(\tau_0, x_{\tau_0}) - \int_{\tau_0}^{\tau_0+\eta} \gamma(\|x(\xi)\|)\mathrm{d}\xi \leqslant \mu(\|x(\tau_0)\|) - \gamma(\nu)\eta < \mu(\delta) - \gamma(\nu)\eta < 0,$$

which contradicts condition (i) from Definition 4.1.3, getting the claim. Hence, for $t \geqslant \tau_0 + \eta$, by (4.1.5), we obtain

$$l\left(\|x(t)\|\right) \leqslant V(t, x_t) \leqslant V(s, x_s) \leqslant \mu(\|x(s)\|) < \mu(\nu) < l(K).$$

Since l is an increasing function, we have $\|x(t)\| = \|x(t,\tau_0,\phi)\| < K$, for all $t \geqslant \tau_0 + \eta$.

□

To conclude this section, we will present two criteria that allow us to guarantee that the functional Volterra–Stieltjes integral equation (3.1.1) is uniformly p-exponentially bounded.

Theorem 4.1.8. Consider the functional Volterra–Stieltjes integral equation (3.1.1). Suppose that the conditions (P1)–(P5) hold. Moreover, suppose that:

(B4) a is bounded on $[t_0, +\infty)^2$.

(B5) There exists a constant $\beta > 0$ such that

$$\|f(x_t, t)\| \leqslant \beta \|x(t)\|, \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1).

Then, (3.1.1) is uniformly p-exponentially bounded.

Proof. Since a is bounded on $[t_0, +\infty)^2$, there exists $c > 0$ such that $|a(t,s)| \leqslant c$ for all $(t,s) \in [t_0, +\infty)^2$.

On the other hand, define $\lambda := c\beta > 0$ and $p : [t_0, +\infty) \to \mathbb{R}$ given by $p(t) := t + g(t)$ for $t \geqslant t_0$. Note that p is an increasing function, since g is a nondecreasing function. Now, let $\delta > 0$ be given. Consider $\phi \in G([-r, 0], \mathbb{R}^n)$ with $\|\phi\|_\infty < \delta$, $\tau_0 \geqslant t_0$ and $x(t) = x(t, \tau_0, \phi)$ is the solution of the functional Volterra–Stieltjes integral equation (3.1.1). Then, by (B5) and properties of the integral and the Theorems 1.3.44 and 1.3.52, we have

$$\begin{aligned}
\|x(t)\| &\leqslant \|\phi(0)\| + \| \int_{\tau_0}^{t} a(t,s) f(x_s, s)\, \mathrm{d}g(s) \| \\
&\leqslant \|\phi\|_\infty + \int_{\tau_0}^{t} |a(t,s)|\, \|f(x_s, s)\|\, \mathrm{d}g(s) \\
&\leqslant \delta + \int_{\tau_0}^{t} \lambda \|x(s)\|\, \mathrm{d}g(s).
\end{aligned}$$

By Gronwall's inequality for the Perron–Stieltjes integral, see lemma 1.3.60, we have

$$\|x(t)\| \leqslant \delta e^{\lambda(g(t) - g(\tau_0))}, \quad \text{for all } t \geqslant t_0.$$

Choosing $B := \delta$ and taking into account that $g(t) - g(\tau_0) \leqslant p(t) - p(\tau_0)$, for all $t \geqslant \tau_0$, we obtain the desired result. □

Theorem 4.1.9. Assume that in Theorem 4.1.8 the condition (B5) is replaced by the condition:

(B6) (i) $f(0,t) = 0$ for all $t \geqslant t_0$.

(ii) the function L (from the condition (P5)) is bounded on $[t_0, +\infty)$.

Then (3.1.1) is uniformly p-exponentially bounded.

Proof. Due to (B4) and (B6)–(ii), there are constants $c, \beta > 0$ such that

$$|a(t,s)| \leqslant c \quad \text{and} \quad L(s) \leqslant \beta, \text{ for all } t, s \geqslant t_0.$$

Now, given a solution $x(t) = x(t, \tau_0, \phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1), by conditions (P5), (B6)–(i), and properties of the integral and by Theorems 1.3.44 and 1.3.52, we have

$$\begin{aligned}
\|x(t)\| &\leqslant \|\phi(0)\| + \left\| \int_{\tau_0}^{t} a(t,s) f(x_s, s)\, \mathrm{d}g(s) \right\| \\
&= \|\phi\|_\infty + \int_{\tau_0}^{t} |a(t,s)|\, \|f(x_s, s) - f(0,s)\|\, \mathrm{d}g(s) \\
&\leqslant \|\phi\|_\infty + \int_{\tau_0}^{t} cL(s) \|x_s\|_\infty\, \mathrm{d}g(s) \\
&\leqslant \|\phi\|_\infty + \int_{\tau_0}^{t} c\beta \|x_s\|_\infty\, \mathrm{d}g(s).
\end{aligned}$$

Under these conditions, the rest of the proof follows in an analogous way to the proof of Theorem 4.1.8. □

4.1.1 Converse Lyapunov Theorem

In this section, our goal is to prove that uniform boundedness implies the existence of functional $V\colon [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ that satisfies some similar properties described in Theorem 4.1.5.

Theorem 4.1.10. Consider the functional Volterra–Stieltjes integral equation (3.1.1). Suppose that the conditions (P1)–(P5) hold. If the functional Volterra–Stieltjes integral equation (3.1.1) is uniformly bounded, then there exists a functional $V\colon [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ satisfying:

(a) There exists an increasing function $l\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $l(0) = 0$, $l(t) \overset{t\to+\infty}{\longrightarrow} +\infty$ and

$$l(\|\varphi(0)\|) \leqslant V(\tau, \varphi),$$

for all $(\tau, \varphi) \in [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n)$.

(b) There exists an increasing function $\mu\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(0) = 0$ and

$$V(\tau, \varphi) \leqslant \mu(\|\varphi\|_\infty),$$

for all $(\tau, \varphi) \in [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n)$.

In particular, V satisfies that $V(\tau, 0) = 0$ for all $\tau \geqslant t_0$.

Proof. Suppose that the functional Volterra–Stieltjes integral equation (3.1.1) is uniformly bounded. For all $\tau \geqslant t_0$ and all $\varphi \in G([-r, 0], \mathbb{R}^n)$, define

$$V(\tau, \varphi) := \sup_{t\in[\tau,+\infty)} \|x(t, \tau, \varphi)\|. \tag{4.1.6}$$

Notice that V is well–defined. In fact, let $\delta := \|\varphi\|_\infty + 1 > 0$. Since the functional Volterra–Stieltjes integral equation (3.1.1) is uniformly bounded, there exists $B = B(\delta) > 0$ such that

$$\|x(t, \tau, \varphi)\| < B, \quad \text{for all } t \geqslant \tau,$$

getting the statement.

Now, let $\tau \geqslant t_0$ and $\varphi \in G([-r, 0], \mathbb{R}^n)$ be given. Note that

$$\|\varphi(0)\| = \|x(\tau, \tau, \varphi)\| \leqslant \sup_{t\in[\tau,+\infty)} \|x(t, \tau, \varphi)\| = V(\tau, \varphi).$$

Hence, the condition (a) follow, by taking $l(\xi) := \xi$ for $\xi \in \mathbb{R}^+$.

On the other hand, we show that condition (b) holds. Indeed, let $\delta > 0$. By the uniform boundedness of the functional Volterra–Stieltjes integral equation (3.1.1), there exists $B = B(\delta) > 0$ such that for all $s_0 \geqslant t_0$ and $\phi \in G([-r,0],\mathbb{R}^n)$ with $\|\phi\|_\infty < \delta$, we have $\|x(t,s_0,\phi)\| < B$, for all $t \geqslant s_0$. Thus, the set

$$\{\|x(t,s_0,\phi)\| : s_0 \geqslant t_0, t \geqslant s_0, \phi \in B_G(\delta)\},$$

is bounded, where $B_G(\delta) := \{\phi \in G([-r,0],\mathbb{R}^n) : \|\phi\|_\infty < \delta\}$. Then, the function $\mu_0 \colon \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$\mu_0(\delta) := \begin{cases} \sup\{\|x(t,s_0,\phi)\| : s_0 \geqslant t_0, t \geqslant s_0, \phi \in B_G(\delta)\}, & \delta > 0 \\ 0, & \delta = 0, \end{cases} \tag{4.1.7}$$

is well–defined.

We claim that μ_0 is nondecreasing. In fact, let $0 \leqslant \delta_1 < \delta_2$. If $\delta_1 = 0$, the result is immediate. Thus, assume that $0 < \delta_1 < \delta_2$. This last inequality implies that

$$\{\|x(t,s_0,\phi)\| : s_0 \geqslant t_0, t \geqslant s_0, \phi \in B_G(\delta_1)\} \subset \{\|x(t,s_0,\phi)\| : s_0 \geqslant t_0, t \geqslant s_0, \phi \in B_G(\delta_2)\},$$

and therefore, $\mu_0(\delta_1) \leqslant \mu_0(\delta_2)$, getting the claim. Now, since μ_0 may not be increasing we can choose a function $\rho \colon \mathbb{R}^+ \to \mathbb{R}^+$ (for example, $\rho(t) := t + \mu_0(t)$) which is increasing, $\rho(0) = 0$ and $\mu_0(t) \leqslant \rho(t)$ for $t \geqslant 0$.

Now, define $\mu \colon \mathbb{R}^+ \to \mathbb{R}^+$ by $\mu(0) := 0$ and $\mu(s) := \rho(s^+) = \lim_{\varepsilon \to 0^+} \rho(s+\varepsilon)$ for all $s > 0$. Note that, for all $s_1, s_2 \in \mathbb{R}^+$ with $s_1 < s_2$, we have

$$\mu(s_1) = \rho(s_1^+) < \rho(s_2) \leqslant \rho(s_2^+) = \mu(s_2),$$

from which it follows that μ is an increasing function.

Finally, let $\tau \geqslant t_0$ and $\varphi \in G([-r,0],\mathbb{R}^n)$ be given. Let $\varepsilon > 0$ arbitrary and consider $\delta := \|\varphi\|_\infty + \varepsilon$. By the definition of μ_0 and μ, we have

$$\|x(t,\tau,\varphi)\| \leqslant \mu_0(\delta) \leqslant \mu_0(\|\varphi\|_\infty + \varepsilon) \leqslant \rho(\|\varphi\|_\infty + \varepsilon), \text{ for all } t \geqslant \tau.$$

Hence,

$$V(\tau,\varphi) = \sup_{t \in [\tau,+\infty)} \|x(t,\tau,\varphi)\| \leqslant \rho(\|\varphi\|_\infty + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$V(\tau,\varphi) \leqslant \lim_{\varepsilon \to 0^+} \rho(\|\varphi\|_\infty + \varepsilon) = \mu(\|\varphi\|_\infty),$$

obtaining the desired result.

Now, to finish the proof, notice that $V(\tau,0) = 0$ for all $\tau \geqslant t_0$, once $0 = l(0) \leqslant V(t,0) \leqslant \mu(0) = 0$.

□

4.2 Applications

4.2.1 Application to functional fractional integral equations

Consider the functional fractional integral equation:

$$\begin{cases} x(t) = \phi(0) + \int_{\tau_0}^{t} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(x_s, s)\,\mathrm{d}s, & t \geqslant \tau_0 \\ x_{\tau_0} = \phi, \end{cases} \tag{4.2.1}$$

where Γ is the gamma function, $1 < \alpha < 2$, $\tau_0 \geqslant t_0$, $\phi \in G([-r,0],\mathbb{R})$ and $f\colon G([-r,0],\mathbb{R}) \times [t_0,+\infty) \to \mathbb{R}$. For more details on this type of equations, the reader can consult (Chen, Nieto and Zhou 2012) and the references therein.

We assume the following conditions:

(F1) The function $[\tau_1,\tau_2] \ni s \mapsto \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(x_s,s)$ is Perron–Stieltjes integrable, for each compact interval $[\tau_1,\tau_2] \subset [t_0,+\infty)$ and all $x \in G([t_0-r,+\infty),\mathbb{R}^n)$.

(F2) There exists a locally Perron integrable function $M\colon [t_0,+\infty) \to \mathbb{R}^+$ such that for each compact interval $[\tau_1,\tau_2] \subset [t_0,+\infty)$, we have

$$\left| \int_{\tau_1}^{\tau_2} \beta \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(x_s,s)\mathrm{d}s \right| \leqslant \int_{\tau_1}^{\tau_2} |\beta| \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} M(s)\mathrm{d}s,$$

for all $x \in G([t_0-r,+\infty),\mathbb{R})$ and all $\beta \in \mathbb{R}$.

(F3) There exists a locally regulated function $L\colon [t_0,+\infty) \to \mathbb{R}^+$ such that for each compact interval $[\tau_1,\tau_2] \subset [t_0,+\infty)$, we have

$$\left| \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} [f(x_s,s) - f(z_s,s)]\mathrm{d}s \right| \leqslant \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} L(s)\|x_s - z_s\|_\infty \mathrm{d}s,$$

for all $x, z \in G([t_0-r,+\infty),\mathbb{R})$.

Taking $g : [t_0,+\infty) \to \mathbb{R}$ and $a\colon [t_0,+\infty)^2 \to \mathbb{R}$ given by $g(s) = s$ and

$$a(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & t \geqslant s, \\ 0, & t < s. \end{cases},$$

we have that (4.2.1) is in the form of (3.1.1).

Clearly g satisfies condition (P1). To verify that $a(t,s)$ satisfies condition (P2), we analyze its two conditions : $a(\cdot,s)$ is nondecreasing $\forall s \in [t_0,+\infty)$ and $a(t,\cdot)$ is regulated $\forall t \in [t_0,+\infty)$.

First, let $s \in [t_0, +\infty)$ be fixed. For $t_1, t_2 \in [t_0, +\infty)$ with $t_1 < t_2$, we have that, if $t_2 < s$, then $a(t_1, s) = a(t_2, s) = 0$. If $t_1 < s \leq t_2$, then $a(t_1, s) = 0 \leq \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} = a(t_2, s)$, since $\Gamma(\alpha) > 0$ and $t_2 - s \geq 0$. If $s \leq t_1 < t_2$, then $0 \leq (t_1 - s) < (t_2 - s)$. Since the mapping $x \mapsto x^{\alpha-1}$ is nondecreasing for $x \geq 0$ and $\alpha > 1$, it follows that $0 \leq (t_1 - s)^{\alpha-1} < (t_2 - s)^{\alpha-1}$. Dividing by $\Gamma(\alpha) > 0$, we obtain $a(t_1, s) < a(t_2, s)$. Therefore, the kernel $t \mapsto \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ is nondecreasing.

Furthermore, $a(t, \cdot)$ is regulated for all $t \in [t_0, +\infty)$. For a fixed $t \in [t_0, +\infty)$, the mapping $s \mapsto a(t, s)$ is continuous for all s and the left-hand limit is $\lim_{s \to t^-} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} = 0$ and the right-hand limit is $\lim_{s \to t^+} 0 = 0$. Since all lateral limits exist and are finite for all t, $a(t, \cdot)$ is a regulated function. Therefore, condition (P2) is satisfied.

Moreover, by conditions (F1)–(F3), f satisfies (P3)–(P5).

Now, let $\tau_0 \geqslant t_0$, $\phi \in G([-r, 0], \mathbb{R}^n)$ and $x(t) = x(t, \tau_0, \phi)$ be the solution of the functional Volterra–Stieltjes integral equation (4.2.1) for $t \geqslant \tau_0$. Consider $V\colon [t_0, +\infty) \times G([-r, 0], \mathbb{R}) \to \mathbb{R}$ defined by $V(t, \varphi) := \varphi^2(0)$. Now, define $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ by $\mu(\xi) := 2\xi^2$, for all $\xi \geqslant 0$. It is clear that $\mu \in H_0$ and

$$V(t, x_t) = x_t^2(0) = x^2(t) = |x(t)|^2 \leqslant 2\,|x(t)|^2 = \mu(|x(t)|),$$

obtaining condition (B0) from Theorem 4.1.5.

On the other hand, define $l\colon \mathbb{R}^+ \to \mathbb{R}^+$ by $l(\xi) = \frac{1}{4}\xi^2$, for all $\xi \geqslant 0$. Note that $l \in H_\infty$ and

$$l(|x(t)|) = \frac{1}{4}\,|x(t)|^2 \leqslant |x(t)|^2 = x^2(t) = V(t, x_t).$$

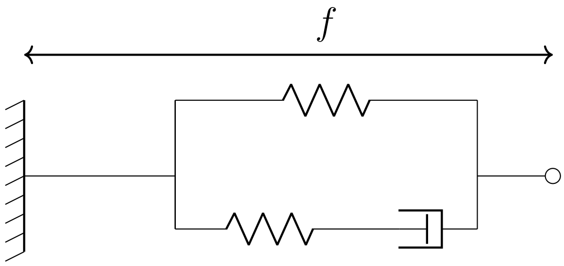
Assuming further that, $t \mapsto V(t, x_t)$ is nonincreasing, we have that all hypotheses of Theorem 4.1.5 are satisfied, and therefore, the (4.2.1) is uniformly bounded.

4.2.2 Poynting-Thompson solid model

In this section, we will work with the Poynting–Thomson solid model, this model was created to treat more accurately the viscoelastic behavior of solids, since the Maxwell and Kelvin-Voigt models offer only a simplified description. It describes the mechanical response of viscoelastic solids, such as polymers, and this response is governed by the interaction between elastic and viscous dynamics, which occurs when stresses are applied, producing stresses and strain rates. See (Phan-Thien 2002, Chapter 5, pp. 95–111) and (Prüss 1993) for more details.

In mechanical engineering, linear viscoelastic materials are often represented by finite mechanical networks composed of springs and dashpots. Within this framework, the Poynting–Thomson solid consists of a dashpot and two springs connected in parallel Figure 7.

Figure 7 – Poynting-Thomson model



Source: Author's compilation

For this section, we will consider the class of kernel defined by $a(t,s) = k(t-s)$, with $k(t) = \mu_0^{-1}(1 - \mu(\mu_0+\mu)^{-1}e^{-\frac{\mu\mu_0 t}{\nu(\mu_0+\mu)}})$, where the constants μ_0, μ, ν are real positive numbers. This class of kernels naturally appears in the theory of integral equations of convolution type, more precisely, in Poynting–Thompson models (see (Prüss 1993, Chapter 5, p. 131)).

Now, we consider as external non-linear force the function $f(\varphi, s) = e^{-\|\varphi\|_\infty}\arctan(\varphi(0))$ for all $\varphi \in G([-r,0],\mathbb{R})$.

Consider the integral equation

$$\begin{cases} x(t) = \displaystyle\int_0^t \mu_0^{-1}(1-\mu(\mu_0+\mu)^{-1}e^{-\frac{\mu\mu_0(t-s)}{\nu(\mu_0+\mu)}})\, e^{-\|x_s\|_\infty}\arctan(x(s))\,\mathrm{d}s \quad t \geqslant 0 \\ x(t) = \phi(t), \quad t \in [-r,0], \end{cases} \tag{4.2.2}$$

where $\phi \in G([-r,0],\mathbb{R})$. Taking $a\colon [0,+\infty)^2 \to \mathbb{R}$ and $g\colon [0,+\infty) \to \mathbb{R}$, given, respectively, by

$$a(t,s) := \begin{cases} \mu_0^{-1}(1-\mu(\mu_0+\mu)^{-1}e^{-\frac{\mu\mu_0(t-s)}{\nu(\mu_0+\mu)}}), & \text{if } \ s \leqslant t \\ 0, & \text{if } \ t < s, \end{cases}$$

$g(s) = s$ for $s \geqslant 0$, we have that (4.2.2) is in the form of (3.1.1). Note that $x_s(0) = x(s)$ and, therefore, $f(x_s, s) = e^{-\|x_s\|_\infty}\arctan(x(s))$.

It is not difficult to see that the assumptions (P1)–(P3) are satisfied.

Let us see that assumptions (P4) is satisfied. In fact, by properties of the integral, we have

$$\left|\int_{\tau_1}^{\tau_2} b_{\tau_1,\tau_2}(s) f(x_s,s)\mathrm{d}g(s)\right| \leqslant \int_{\tau_1}^{\tau_2} |b_{\tau_1,\tau_2}(s)|\,|f(x_s,s)|\,\mathrm{d}g(s) \leqslant \int_{\tau_1}^{\tau_2} |b_{\tau_1,\tau_2}(s)|\,\frac{\pi}{2}\,\mathrm{d}g(s),$$

for all $[\tau_1,\tau_2] \subset [0,+\infty)$, all $c_1, c_2 \in \mathbb{R}$, $b_{\tau_1,\tau_2}(s) := c_1 a(\tau_2,s) + c_2 a(\tau_1,s)$ and all $x \in G([-r,+\infty),\mathbb{R})$, proving (H4) (with $M(s) := \frac{\pi}{2}$).

Now, we show that (P5) also holds. Indeed, define $L(s) = 1 + \frac{\pi}{2}$ for $s \in [0,+\infty)$.

For $x, y \in G([-r, +\infty), \mathbb{R})$ and $[\tau_1, \tau_2] \subset [0, +\infty)$, we have

$$\left|\int_{\tau_1}^{\tau_2} a(\tau_2, s)\,[f(x_s, s) - f(y_s, s)]\,\mathrm{d}g(s)\right| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\,|f(x_s, s) - f(y_s, s)|\,\mathrm{d}g(s)$$

$$\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \left(e^{-\|x_s\|_\infty} |\arctan(x(s)) - \arctan(y(s))|\right) \mathrm{d}g(s) +$$

$$+ \int_{\tau_1}^{\tau_2} \Big|a(\tau_2, s)\,|\arctan(x(s))|\,\left|e^{-\|x_s\|_\infty} - e^{-\|y_s\|_\infty}\right|\Big)\,\mathrm{d}g(s)$$

$$\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \left(|x(s) - y(s)| + \frac{\pi}{2}\,|\|x_s\|_\infty - \|y_s\|_\infty|\right) \mathrm{d}g(s)$$

$$= \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \left(|x_s(0) - y_s(0)| + \frac{\pi}{2}\,|\|x_s\|_\infty - \|y_s\|_\infty|\right) \mathrm{d}g(s)$$

$$\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2, s)| \left(1 + \frac{\pi}{2}\right) \|x_s - y_s\|_\infty \mathrm{d}g(s)$$

getting (P5), where the third inequality follows from the estimates given by the Mean Value Theorem.

On the other hand, note that

$$|a(t, s)| \leqslant \mu_0^{-1}(1 + \mu(\mu_0 + \mu)^{-1}), \text{ for all } (t, s) \in [0, +\infty)^2,$$

proving (B4). Moreover, it is easy to check that

$$|f(x_t, t)| \leqslant |x(t)| = |x_t(0)| \leqslant \|x_t\|_\infty,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of the functional Volterra–Stieltjes integral equation (4.2.2). This proves that (B5) is satisfied setting $\beta := 1$.

Hence, all the assumptions of Theorem 4.1.8 are verified, and therefore, (4.2.2) is uniformly p-exponentially bounded.

4.2.3 A nonlinear network model

We consider a nonlinear network model of type:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -c_0^{-1}\mathcal{G}(y_t)y(t), \tag{4.2.3}$$

with initial condition $y_{\tau_0} = \phi$. Here,

$y(t)$: denotes the voltage across the capacitor.

$c_0 > 0$: denotes the capacitance.

$\mathcal{G}(\cdot)$: is considered as the nonlinear version of the conductance function.

For more details, see for instance (Vidyasagar 1993). Note that (4.2.3) can be rewritten as the following integral equation

$$\begin{cases} y(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} -c_0^{-1}\mathcal{G}(y_s)y(s)\,\mathrm{d}s \\ y_{\tau_0} = \phi. \end{cases} \tag{4.2.4}$$

We assume the following conditions:

(A1) $\mathcal{G}\colon G_0 \to (0,+\infty)$ is a bounded function such that $\sigma := \inf\limits_{\psi\in G_0} \mathcal{G}(\psi) > 0$, where $G_0 := G([-r,0],\mathbb{R})$.

(A2) There exists $L_0 > 0$ such that $|\mathcal{G}(x_s) - \mathcal{G}(y_s)| \leqslant L_0|x_s - z_s|_\infty$ for all $x, y \in G_\infty$ and all $s \geqslant 0$, where $G_\infty := \{z : [-r,+\infty) \to \mathbb{R} : z$ is regulated and $|z(t)| \leqslant m,$ for all $t \geqslant -r\}$.

Defining $a(t,s) := 1$ for $(t,s) \in [0,+\infty)^2$, $f(\psi,s) := -c_0^{-1}\mathcal{G}(\psi)\psi(0)$ for $\psi \in G_0$ and $g(s) := s$ for $s \geqslant 0$, we have that (4.2.4) is in the integral form given by (3.1.1). Notice that $y_s(0) = y(s)$ and, therefore, $f(y_s,s) = -c_0^{-1}\mathcal{G}(y_s)y(s)$.

It is easy to see that conditions (P1)-(P3) are satisfied. Furthermore, (P4) is satisfied by taking $M(s) := \beta m c_0^{-1}$, where $\beta := \sup_{\psi\in G_0}\mathcal{G}(\psi)$. Now, for $x, y \in G_\infty$ and $0 \leqslant \tau_1 \leqslant \tau_2$, by properties of the integral, we have

$$\begin{aligned} &\left|\int_{\tau_1}^{\tau_2} [f(x_s,s) - f(z_s,s)]\,\mathrm{d}g(s)\right| \\ \leqslant\; &\int_{\tau_1}^{\tau_2} \left(c_0^{-1}|\mathcal{G}(x_s)||x(s) - z(s)| + c_0^{-1}|\mathcal{G}(x_s) - \mathcal{G}(z_s)||z(s)|\right)\mathrm{d}g(s) \\ \leqslant\; &\int_{\tau_1}^{\tau_2} \left(\beta c_0^{-1}|x(s) - z(s)| + mL_0c_0^{-1}|x_s - z_s|_\infty\right)\mathrm{d}g(s) \\ =\; &\int_{\tau_1}^{\tau_2} \left(\beta c_0^{-1}|x_s(0) - z_s(0)| + mL_0c_0^{-1}|x_s - z_s|_\infty\right)\mathrm{d}g(s) \\ \leqslant\; &\int_{\tau_1}^{\tau_2} (\beta c_0^{-1} + mL_0c_0^{-1})\,|x_s - z_s|_\infty\,\mathrm{d}g(s), \end{aligned}$$

for all $x, y \in G_\infty$, which implies that (P5) is fulfilled with $L(s) := \beta c_0^{-1} + mL_0c_0^{-1}$.

On the other hand, let $\tau_0 \geqslant 0$, $\phi \in G_0$, $y(t) = y(t,\tau_0,\phi)$ be the solution of (4.2.3) and consider $V(t,y_t) := (y_t(0))^2 = y^2(t)$.

Now, consider $l(s) := \frac{1}{2}s^2$ and $\mu(s) := 2s^2$ for $s \in [0,+\infty)$. Notice that $l \in H_\infty$, $\mu \in H_0$ and

$$l(|y(t)|) = \frac{1}{2}\,|y(t)|^2 = \frac{1}{2}y^2(t) \leqslant y^2(t) = V(t,y_t) \leqslant 2y^2(t) = 2\,|y(t)|^2 = \mu(|y(t)|),$$

getting condition (i) from Definition 4.1.3 and condition (B0), respectively. Also,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2y(t)\frac{\mathrm{d}y}{\mathrm{d}t} = -2c_0^{-1}\mathcal{G}(y_t)(y(t))^2 \leqslant -2\sigma c_0^{-1}(y(t))^2 = -2\sigma c_0^{-1}|y(t)|^2.$$

Thus
$$V(t,x_t) - V(s,x_s) \leqslant -\int_s^t \gamma(|x(\xi)|)\,\mathrm{d}\xi,$$
for all $s_0 \leqslant s \leqslant t$, where $\gamma(\xi) := 2\sigma c_0^{-1}\xi^2$ for all $\xi \geqslant 0$. It is clearly that γ is a continuous and increasing function with $\gamma(0) = 0$. So, condition (B5) is satisfied.

Since all the hypotheses of Theorem 4.1.7 are satisfied, it follows that (4.2.3) is uniformly ultimately bounded.

4.2.4 Functional differential equations

We consider the functional differential equation of type, adapted from FDE of (Bonotto *et al.* 2021, Chapter 3):
$$\begin{cases} y(t) &= \varphi(0) + \displaystyle\int_{t_0}^t f(y_s,s)\,\mathrm{d}s, \\ y_{t_0} &= \varphi, \end{cases} \tag{4.2.5}$$
where $\varphi \in G([-\tau,0],\mathbb{R}^n)$, $\tau > 0$ and $f\colon G([-\tau,0],\mathbb{R}^n) \times [t_0,+\infty) \to \mathbb{R}^n$.

We assume the following conditions:

(O1) The Perron integral $\int_{\tau_1}^{\tau_2} f(y_s,s)\mathrm{d}s$ exists, for each compact interval $[\tau_1,\tau_2] \subset [t_0,+\infty)$ and all $y \in G([t_0-\tau,+\infty),\mathbb{R}^n)$.

(O2) There exists a locally Perron integrable function $M\colon [t_0,+\infty) \to \mathbb{R}^+$ such that for each compact interval $[\tau_1,\tau_2] \subset [t_0,+\infty)$, we have
$$\left\| \int_{\tau_1}^{\tau_2} f(y_s,s)\,\mathrm{d}s \right\| \leqslant \int_{\tau_1}^{\tau_2} M(s)\,\mathrm{d}s,$$
for all $y \in G([t_0-\tau,+\infty),\mathbb{R}^n)$.

(O3) There exists $L \in G([t_0,+\infty),\mathbb{R})$ such that for each compact interval $[\tau_1,\tau_2] \subset [t_0,+\infty)$, we have
$$\left\| \int_{\tau_1}^{\tau_2} [f(y_s,s) - f(z_s,s)]\,\mathrm{d}s \right\| \leqslant \int_{\tau_1}^{\tau_2} L(s)\|y_s - z_s\|_\infty\,\mathrm{d}s,$$
for all $x,z \in G([t_0-\tau,+\infty),\mathbb{R}^n)$.

Theorem 4.2.1. Suppose that the conditions (O1)-(O3) hold. Let $W\colon [t_0,+\infty)\times G([-\tau,0],\mathbb{R}^n) \to \mathbb{R}$ be a Lyapunov functional with respect to the functional differential equation (4.2.5). Suppose that:

(T0) there exists $\mu \in H_0$ such that
$$V(t,y_t) \leqslant \mu(\|y(t)\|), \quad t \geqslant \tau_0,$$
for all solution $y(t) = y(t,\tau_0,\varphi)$ of the functional differential equation (4.2.5).

Then (4.2.5) is uniformly bounded.

Proof. Note that the functional differential equation (4.2.5) can be written in the form of (3.1.1) by setting $a(t,s) = 1$ and $g(s) = s$ for all $t, s \in [t_0, +\infty)$.

In view of the definition of the functions a and g and the conditions (O1)-(O3), it follows that (P1)-(P5) are satisfied. On the other hand, (T0) guarantees that condition (B0) from Theorem 4.1.5 is satisfied. Hence, all hypotheses of Theorem 4.1.5 are satisfied, and therefore, the (4.2.5) is uniformly bounded. □

Theorem 4.2.2. Suppose that (O1)–(O3) hold. Let $W\colon [t_0, +\infty) \times G([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}$ be a function such that W is bounded on each set $[t_0, +\infty) \times D$, where D is a bounded subset of $G([-\tau, 0], \mathbb{R}^n)$. Also assume that

(T1) there exists a continuous increasing function $v\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $v(0) = 0$ such that

$$v(\|\varphi(0))\|) \leqslant W(t, \varphi),$$

for all $t \geqslant t_0$ and all $\varphi \in G([-\tau, 0], \mathbb{R}^n)$.

(T2) there exists a real number $\beta > 0$ such that

$$W(t, y_t) - W(s, y_s) \leqslant -\beta \int_s^t W(\xi, y_\xi)\mathrm{d}\xi, \quad \tau_0 \leqslant s \leqslant t,$$

for all solution $y(t) = y(t, \tau_0, \varphi)$ of the functional differential equation (4.2.5).

Then, (4.2.5) is quasi-uniformly ultimately bounded.

Proof. Define $a\colon [t_0, +\infty) \times [t_0, +\infty) \to \mathbb{R}$ and $g\colon [t_0, +\infty) \to \mathbb{R}$ by $a(t,s) = 1$ and $g(s) = s$ for all $t, s \in [t_0, +\infty)$. Using the same arguments as in the proof of Theorem 4.2.1, we can prove that conditions (P1)-(P5) are satisfied.

Now, according to the hypotheses, it is not difficult to see that W satisfies all the hypotheses from Theorem 4.1.6. Hence, (4.2.5) is quasi-uniformly ultimately bounded. □

Theorem 4.2.3. Suppose that the conditions (O1)–(O3) hold. Let $W\colon [t_0, +\infty) \times G([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}$ be a function so that W satisfies condition (i) from Definition 4.1.3 and condition (T0) from Theorem 4.2.1. Assume that

(T3) there exists a continuous and increasing function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $h(0) = 0$ such that

$$W(t, y_t) - W(s, y_s) \leqslant -\int_s^t h(\|y(\xi)\|)\mathrm{d}\xi, \quad \tau_0 \leqslant s \leqslant t,$$

for all solution $y(t) = y(t, \tau_0, \varphi)$ of the functional differential equation (4.2.5).

Then, (4.2.5) is uniformly ultimately bounded.

Proof. Taking $a(t,s) = 1$ and $g(s) = s$ for all $t, s \geqslant t_0$, we can see that (4.2.5) is in the form of (3.1.1).

It is easy to check that (P1) and (P2) are satisfied. Also, since (O1)–(O3) holds, it follows that (P3)–(P5) are satisfied.

On the other hand, according to the hypotheses, W satisfies all the hypotheses from Theorem 4.1.7. Thus, the (4.2.5) is uniformly ultimately bounded. □

Theorem 4.2.4. Suppose that the conditions (O1)–(O3) hold. Suppose that:

(H) There exists a constant $\alpha > 0$ such that

$$\|f(y_t, t)\| \leqslant \alpha \|y(t)\|, \quad t \geqslant \tau_0,$$

for all solution $y(t) = y(t, \tau_0, \varphi)$ of the functional differential equation (4.2.5).

Then, (4.2.5) is uniformly p-exponentially bounded.

Proof. Define $a(t,s) = 1$ and $g(s) = s$ for all $t, s \geqslant t_0$. Using the same arguments as in the proof of Theorem 4.2.1, we can prove that conditions (P1)-(P5) are satisfied.

Since a is a constant function, it follow that a satisfies the condition (B4) from Theorem 4.1.8. Also, by hypotheses f satisfies (B5).

Thus, the (4.2.5) is is uniformly p-exponentially bounded. □

4.2.5 Impulsive functional Volterra-Stieltjes integral equations

In this section, we will show that we can obtain the analogues of Section 3 for impulsive functional Volterra–Stieltjes integral equations

$$\begin{cases} x(t) & = & \phi(0) + \displaystyle\int_{t_0}^{t} a(t,s) f(x_s, s) \mathrm{d}g(s) + \sum_{\substack{k \in \mathbb{N} \\ t_k < t}} I_k(x(t_k)) \\ x_{\tau_0} & = & \phi. \end{cases} \tag{4.2.6}$$

as discussed in Chapter 3, here we will consider the existence and uniqueness theorems, as well as their correspondences with the Volterra-Stieltjes integral equation.

Definition 4.2.5. The impulsive functional Volterra–Stieltjes integral equation (4.2.6) is said to be

(i) **Uniformly bounded**, if for all $\delta > 0$, there exists a $b = b(\delta) > 0$ such that if $\phi \in G([-r,0], \mathbb{R}^n)$ with $\|\phi\|_\infty < \delta$ and $\tau_0 \geqslant t_0$, then $\|x(t, \tau_0, \phi)\| < b$, for all $t \in [\tau_0, +\infty)$.

(ii) **Quasi-uniformly ultimately bounded**, if thete exists $C > 0$ so that for all $\delta > 0$ exists $\eta = \eta(\delta) > 0$ such that for all $\tau_0 \geqslant t_0$ and $\phi \in G([-r, 0], \mathbb{R}^n)$ with $\|\phi\|_\infty < \delta$, we have $\|x(t, \tau_0, \phi)\| < C$, for all $t \geqslant \tau_0 + \eta$.

(iii) **Uniformly ultimately bounded**, if (i) and (ii) hold together.

(iv) **Uniformly p-exponentially bounded**, if there exists $\alpha \in \mathbb{R}$ and an increasing function $p\colon [t_0, +\infty) \to \mathbb{R}$ so that for all $\delta > 0$, there exist $K = K(\delta) > 0$ such that for all $\phi \in G([-r, 0], \mathbb{R}^n)$ with $\|\phi\|_\infty < \delta$ and all $\tau_0 \geqslant t_0$, we have $\|x(t, \tau_0, \phi)\| \leqslant Ke^{\alpha(p(t)-p(\tau_0))}$ for all $t \geqslant \tau_0$.

Definition 4.2.6. We said that $V\colon [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n) \to \mathbb{R}$, is a **Lyapunov function with respect to the impulsive functional Volterra–Stieltjes integral equation** (4.2.6), if the following conditions are satisfied:

(i) there exists $\zeta \in H_\infty$ such that

$$\zeta(\|x(t)\|) \leqslant V(t, x_t), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of the impulsive functional Volterra–Stieltjes integral equation (4.2.6).

(ii) $t \mapsto V(t, x_t)$ is nonincreasing functions for all and all solution $x(t) = x(t, \tau_0, \phi)$ of the impulsive functional Volterra–Stieltjes integral equation (4.2.6).

Theorem 4.2.7. Let $\{t_k\}_{k=1}^{\infty}$ be the moments of impulses in $[t_0, \infty)$, such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Assume that the conditions (D1)–(D5) hold. Moreover, suppose that $g(\cdot)$ is continuous at $\{t_k\}_{k=1}^{\infty}$, $a(\cdot, \cdot)$ is continuous with respect to first variable at $\{t_k\}_{k=1}^{\infty}$. Let $U\colon [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ be a Lyapunov function with respect to the impulsive functional Volterra–Stieltjes integral equation (4.2.6). Suppose the following condition:

(A0) there exists $\vartheta \in H_0$ such that

$$U(t, x_t) \leqslant \vartheta(\|x(t)\|), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of the impulsive functional Volterra–Stieltjes integral equation (4.2.6).

Then (4.2.6) is uniformly bounded.

Proof. Suppose that $\tilde{f}$, $\tilde{a}$ and $\tilde{g}$ are given by (3.2.10), (3.2.12) and (3.2.11), respectively. Now, proceeding as in (Alvarez *et al.* 2021, Theorem 4.3), we can prove that $a, \tilde{f}, \tilde{g}$ satisfy conditions (P1)–(P5).

Now, according to Theorem 3.2.9 and from the hypotheses, it follows that $U\colon [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ satisfies all hypotheses of Theorem 4.2.7. Thus, (3.2.17) is uniformly bounded.

Again, due to Theorem 3.2.9, it follows (4.2.6) is also uniformly bounded, getting the desired result. □

Theorem 4.2.8. Let $\{t_k\}_{k=1}^{\infty}$ be the moments of impulses in $[t_0, \infty)$, such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Assume that the conditions (D1)-(D5) hold. Moreover, suppose that $g(\cdot)$ is continuous at $\{t_k\}_{k=1}^{\infty}$, $a(\cdot, \cdot)$ is continuous with respect to first variable at $\{t_k\}_{k=1}^{\infty}$. Let $U\colon [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ be a function such that U is bounded on each set $[t_0, +\infty) \times B$, where B is a bounded subset of $G([-r, 0], \mathbb{R}^n)$. Moreover assume that

(A1) there exists a continuous increasing function $q\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $q(0) = 0$ such that

$$q(\|\varphi(0))\|) \leqslant U(t, \varphi),$$

for all $t \geqslant t_0$ and all $\varphi \in G([-r, 0], \mathbb{R}^n)$.

(A2) there exists a real number $\alpha > 0$ such that

$$U(t, x_t) - U(s, x_s) \leqslant -\alpha \int_s^t U(\xi, x_\xi) \mathrm{d}\xi, \quad \tau_0 \leqslant s \leqslant t,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of the impulsive functional Volterra–Stieltjes integral equation (4.2.6).

Then, (4.2.6) is quasi-uniformly ultimately bounded.

Proof. Define $\tilde{f}\colon G([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$, $\tilde{a}\colon [t_o, +\infty)^2 \to \mathbb{R}$ and $\tilde{g}\colon [t_0, +\infty) \to \mathbb{R}$ by (3.2.10), (3.2.12) and (3.2.11), respectively. Analogously as in the proof of Theorem 4.2.7 we can verify that $a, \tilde{f}, \tilde{g}$ satisfy conditions (P1)–(P5).

On the other hand, due to Theorem 3.2.9 and by hypotheses, it is clear that U satisfies all assumptions from Theorem 4.1.6. Hence, (3.2.17) is quasi-uniformly ultimately bounded. Hence, using Theorem 3.2.9 again, we have that (4.2.6) is quasi-uniform ultimately bounded. □

The next result ensures that (4.2.6) is uniform ultimately bounded.

Theorem 4.2.9. Let $\{t_k\}_{k=1}^{\infty}$ be the moments of impulses in $[t_0, \infty)$, such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Assume that the conditions (D1)-(D5) hold. Moreover, suppose that $g(\cdot)$ is continuous at $\{t_k\}_{k=1}^{\infty}$, $a(\cdot, \cdot)$ is continuous with respect to first variable at $\{t_k\}_{k=1}^{\infty}$. Let $U\colon [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ be a function so that U satisfies condition (i) from Definition 4.2.6 and condition (A0) from Theorem 4.2.7. Also, assume that

(A3) there exists a continuous and increasing function $\beta\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta(0) = 0$ such that
$$U(t, x_t) - U(s, x_s) \leqslant -\int_s^t \beta(\|x(\xi)\|)\mathrm{d}\xi, \quad \tau_0 \leqslant s \leqslant t,$$
for all solution $x(t) = x(t, \tau_0, \phi)$ of the impulsive functional Volterra–Stieltjes integral equation (4.2.6).

Then, (4.2.6) is uniformly ultimately bounded.

Proof. Define the functions $\tilde{f}\colon G([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$ given by
$$\tilde{f}(y, \tau) := \begin{cases} f(y, \tau), & \tau \in [t_0, \infty)\backslash\{t_k\}_{k=1}^{\infty}, \\ I_k(y(0)), & \tau = t_k, \quad k \in \mathbb{N}, \end{cases}$$
$\tilde{g}\colon [t_0, \infty) \to \mathbb{R}$ by
$$\tilde{g}(\tau) := \begin{cases} g(\tau), & \tau \in [t_0, t_1], \\ g(\tau) + k, & \tau \in (t_k, t_{k+1}], \quad k \in \mathbb{N}, \end{cases}$$
and $\tilde{a}\colon [t_0, \infty)^2 \to \mathbb{R}$ by
$$\tilde{a}(t, s) = \begin{cases} a(t, s), & t \in [t_0, \infty) \text{ and } s \in [t_0, \infty)\backslash\{t_k\}_{k=1}^{\infty}, \\ 1, & t \in [t_0, \infty) \text{ and } s \in \{t_k\}_{k=1}^{\infty}. \end{cases}$$

Using the same arguments as in the proof from Theorem 4.2.7, we can prove that $\tilde{a}, \tilde{f}, \tilde{g}$ satisfy conditions (P1)–(P5).

Also, by Theorem 3.2.9 and by hypotheses, it is clear that U satisfies all hypotheses from Theorem 4.1.7. Hence, (3.2.17) is uniformly ultimately bounded. Now, using Theorem 3.2.9 again, we have that (4.2.6) is uniformly ultimately bounded. □

The following theorem guarantees that the (4.2.6) is uniformly p-exponentially bounded.

Theorem 4.2.10. Let $\{t_k\}_{k=1}^{\infty}$ be the moments of impulses in $[t_0, \infty)$, such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Assume that the conditions (D1)-(D5) hold. Moreover, suppose that $g(\cdot)$ is continuous at $\{t_k\}_{k=1}^{\infty}$, $a(\cdot, \cdot)$ is continuous with respect to the first variable at $\{t_k\}_{k=1}^{\infty}$. Moreover, suppose that:

(A4) a is bounded on $[t_0, +\infty)^2$.

(A5) there exists a constant $c > 0$ such that
$$\|f(x_t, t)\| \leqslant c\|x(t)\|, \quad t \geqslant \tau_0,$$
for all solution $x(t) = x(t, \tau_0, \phi)$ of the impulsive functional Volterra–Stieltjes integral equation (4.2.6).

(A6) $I_k(0) = 0$ for all $k \in \mathbb{N}$.

Then, (4.2.6) is uniformly p-exponentially bounded.

Proof. Define the functions $\tilde{f}\colon G([-r,0],\mathbb{R}^n) \times [t_0,+\infty) \to \mathbb{R}^n$ given by

$$\tilde{f}(y,\tau) := \begin{cases} f(y,\tau), & \tau \in [t_0,\infty)\backslash\{t_k\}_{k=1}^{\infty}, \\ I_k(y(0)), & \tau = t_k, \quad k \in \mathbb{N}, \end{cases}$$

$\tilde{g}\colon [t_0,\infty) \to \mathbb{R}$ by

$$\tilde{g}(\tau) := \begin{cases} g(\tau), & \tau \in [t_0,t_1], \\ g(\tau) + k, & \tau \in (t_k,t_{k+1}], \quad k \in \mathbb{N}, \end{cases}$$

and $\tilde{a}\colon [t_0,\infty)^2 \to \mathbb{R}$ by

$$\tilde{a}(t,s) = \begin{cases} a(t,s), & t \in [t_0,\infty) \text{ and } s \in [t_0,\infty)\backslash\{t_k\}_{k=1}^{\infty}, \\ 1, & t \in [t_0,\infty) \text{ and } s \in \{t_k\}_{k=1}^{\infty}. \end{cases}$$

Proceeding in a similar way as in Theorem 4.2.7 we can prove that $\tilde{a}, \tilde{f}, \tilde{g}$ satisfy conditions (P1)–(P5). Note that condition (A4) implies condition (B4) from Theorem 4.1.8. Now, let $x(t) = x(t,\tau_0,\phi)$ be a solution of the impulsive functional Volterra–Stieltjes integral equation (4.2.6). If $t \in [\tau_0,\infty)\backslash\{t_k\}_{k=1}^{\infty}$, by (A5) and the definition of $\tilde{f}$, we have

$$\|\tilde{f}(x_t,t)\| = \|f(x_t,t)\| \leqslant c\|x(t)\|.$$

On the other hand, if $t = t_k$, $k \in \mathbb{N}$, by (A5), (A6), (3.2.18) and the definition of $\tilde{f}$, we have

$$\|\tilde{f}(x_t,t)\| = \|I_k(x_t(0))\| = \|I_k(x(t))\| \leqslant L_2\|x(t)\|.$$

Taking $\beta := \max\{c, L_2\}$, we obtain $\|\tilde{f}(x_t,t)\| \leqslant \beta\|x(t)\|$, for all $t \geqslant \tau_0$, getting condition (B5) from Theorem 4.1.8. Then, (3.2.17) is uniformly p-exponentially bounded. Now, using Theorem 3.2.9, we have that (4.2.6) is uniformly p-exponentially bounded. □

To conclude this section, we will prove that uniform boundedness implies the existence of function $U\colon [t_0,+\infty)\times G([-r,0],\mathbb{R}^n) \to \mathbb{R}$ that satisfies some similar properties described in Theorem 4.2.7, that is, the nest result is a type of converse Lyapunov theorem.

Theorem 4.2.11. Let $\{t_k\}_{k=1}^{\infty}$ be the moments of impulses in $[t_0,\infty)$, such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Assume that the conditions (D1)-(D5) hold. If the impulsive functional Volterra–Stieltjes equation (4.2.6) is uniformly bounded, then there exists a function $U\colon [t_0,+\infty) \times G([-r,0],\mathbb{R}^n) \to \mathbb{R}$ satisfying:

(i) there exists an increasing function $\zeta\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\zeta(0) = 0$, $\zeta(t) \overset{t\to+\infty}{\longrightarrow} +\infty$ and

$$\zeta(\|\varphi(0)\|) \leqslant U(\tau, \varphi),$$

for all $(\tau, \varphi) \in [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n)$.

(ii) there exists an increasing function $\vartheta\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\vartheta(0) = 0$ and

$$U(\tau, \varphi) \leqslant \vartheta(\|\varphi\|_\infty),$$

for all $(\tau, \varphi) \in [t_0, +\infty) \times G([-r, 0], \mathbb{R}^n)$.

In particular, U satisfies that $U(\tau, 0) = 0$ for all $\tau \geqslant t_0$.

Proof. The proof is an immediate consequence of Theorems 3.2.9 and 4.1.10. □

CHAPTER

5

THEORY OF INSTABILITY

Instability analysis is essential for understanding the qualitative behavior of Volterra–Stieltjes integral functional equations. Instability occurs when small perturbations in initial conditions or parameters lead to large deviations in solutions over time, rather than returning to an equilibrium state. This study is closely related to stability analysis, a fundamental concept in dynamical systems and control theory.

In this chapter, we introduce appropriate notions of instability for this class of equations and establish criteria, using Lyapunov and Chetaev theory, to identify when a given solution (here, $x = 0$) loses stability.

To this end, we work with the Volterra–Stieltjes integral equation presented in (3.1.1), under the conditions (P1)–(P5), understanding the corresponding integral in the Perron–Stieltjes sense.

In the sequel, we recall the concept of stability (in the sense of Lyapunov) in the framework of functional Volterra–Stieltjes integral equations, see (Lyapunov 1992).

Definition 5.0.1. The trivial solution $x \equiv 0$ of the functional integral equation (3.1.1) is said to be

- **Stable**, if for all $\tau_0 \geqslant t_0$ and $\varepsilon > 0$, there exists a $\delta = \delta(\tau_0, \varepsilon) > 0$ such that if $\|\phi\|_\infty < \delta$, then $\|x(t, \tau_0, \phi)\| < \varepsilon$ for all $t \in [\tau_0, \infty)$.

- **Unstable**, if there exist $\varepsilon > 0$ and $\tau_0 \geqslant t_0$ such that for any $\delta > 0$ there exist a ϕ with $\|\phi\|_\infty < \delta$ and a $t_1 > \tau_0$ so that $\|x(t_1, \tau_0, \phi)\| \geqslant \varepsilon$.

Remark 5.0.2. Observe that stability requires all solutions starting close to zero continues close to zero; on the other hand, instability calls for the existence of some solutions starting close to zero and moves away from zero.

Throughout this work, we will use the notation $\mathbb{R}^+ := [0, \infty)$.

Definition 5.0.3. Let $V : [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function and $x(t) = x(t, \tau_0, \phi)$, $t \in [\tau_0 - r, \infty)$, be the solution of the functional Volterra–Stieltjes integral equation (3.1.1). For $\tau_0 \leqslant s \leqslant t$, we will denote by $\Delta_s^t V_{(\tau_0,\phi)}$ **the increment of the function** $\xi \mapsto V(\xi, x_\xi)$ in the interval $[s, t]$, that is,

$$\Delta_s^t V_{(\tau_0,\phi)} := V(t, x_t) - V(s, x_s).$$

This clearly defines a function (of the variables t, s, τ_0, ϕ), which we denote simply by ΔV.

Definition 5.0.4. We say that the function ΔV is **positive definite along the solutions of** (3.1.1) if there exists a strictly increasing continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$, such that for all solution $x(t) = x(t, \tau_0, \phi)$, of (3.1.1), we have

$$\Delta_s^t V_{(\tau_0,\phi)} \geqslant \int_s^t \gamma(\|x(\xi)\|)\, \mathrm{d}\xi, \tag{5.0.1}$$

for all $\tau_0 \leqslant s \leqslant t$, that is,

$$V(t, x_t) - V(s, x_s) \geqslant \int_s^t \gamma(\|x(\xi)\|)\, \mathrm{d}\xi. \tag{5.0.2}$$

However here since we want to allow more general defition of Lyapunov functions, including the ones that are not differentiable, condition (5.0.1) is replaced directly to (5.0.2), allowing less regularity to the Lyapunov function.

Remark 5.0.5. It is important to highlight that Definition 5.0.4 is inspired by the concept of positive definite function presented in (Vidyasagar 1993, Definition 3, p. 147). In fact, according to (Vidyasagar 1993, Definition 3), $\dot{V}$ is positive-definite along the solutions of the ODE $\dot{x} = f(t, x)$ if there exists a strictly increasing continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ with $\alpha(0) = 0$, such that

$$\dot{V}(t, x(t)) \geqslant \alpha(\|x(t)\|), \quad t \geqslant t_0,$$

for all solution $x(t) = x(t, t_0, x_0)$ of the referenced ODE. This implies that

$$V(t, x(t)) - V(s, x(s)) \geqslant \int_s^t \alpha(\|x(\xi)\|)\, \mathrm{d}\xi.$$

The following result is inspired by (Lyapunov 1992, Theorem II, p. 585) and (Kellett 2015, Theorem 8.3).

Theorem 5.0.6 (First Lyapunov-type theorem on instability)**.** Consider the functional Volterra–Stieltjes integral equation (3.1.1). Assume that conditions (P1)–(P5) holds. Let $V : [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function. Moreover, assume the following conditions:

(H1) ΔV is positive definite along the solutions of (3.1.1).

(H2) there exists a continuous strictly increasing function $\mu\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\mu(0) = 0$ such that

$$V(t, x_t) \leqslant \mu(\|x(t)\|), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1).

(H3) For each $\tau \geqslant t_0$, $V(\tau, \cdot)$ assumes positive values at points arbitrarily close to the origin (i.e., for any $\delta > 0$ there is an $\phi \neq 0$ with $\|\phi\|_\infty < \delta$ and $V(\tau, \phi) > 0$).

Then, the solution $x \equiv 0$ of equation (3.1.1) is unstable.

Proof. According to the conditions (H1)-(H2), there are continuous strictly increasing functions $\mu, \gamma\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = \mu(0) = 0$ such that

$$V(t, x_t) \leqslant \mu(\|x(t)\|) \quad \text{and} \quad V(t, x_t) - V(s, x_s) \geqslant \int_s^t \gamma(\|x(\xi)\|)\, \mathrm{d}\xi, \tag{5.0.3}$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of (3.1.1).

Let us assume by contradiction that $x \equiv 0$ is stable. Given $\varepsilon > 0$ and $\tau_0 \geqslant t_0$, there exists $\delta = \delta(\varepsilon, \tau_0) > 0$ such that if $\|\varphi\|_\infty < \delta$, then $\|x(t, \tau_0, \varphi)\| < \varepsilon$ for all $t \in [\tau_0, \infty)$.

Now, by condition (H3), there exists $\phi \neq 0$ such that $\|\phi\|_\infty < \delta$ and $V(\tau_0, \phi) > 0$. Then $\|x(t)\| = \|x(t, \tau_0, \phi)\| < \varepsilon$, for $t \geqslant \tau_0$, which implies that

$$V(t, x_t) \leqslant \mu(\|x(t)\|) < \mu(\varepsilon),$$

for all $t \geqslant \tau_0$, and therefore, $V(t, x_t)$ is bounded above on $[\tau_0, \infty)$.

On the other hand, notice that by condition (5.0.3), we have $\xi \mapsto V(\xi, x_\xi)$ is an increasing function. Consequently,

$$0 < V(\tau_0, \phi) \leqslant V(t, x_t) \leqslant \mu(\|x(t)\|), \qquad \text{for all } t \geqslant \tau_0.$$

In particular,

$$0 < \mu^{-1}(V(t, x_t)) \leqslant \|x(t)\|, \quad t \geqslant \tau_0. \tag{5.0.4}$$

Also, note that the function $\xi \mapsto (\gamma \circ \mu^{-1})(V(\xi, x_\xi))$ is also increasing. Using this fact together with (5.0.3), (5.0.4), properties of integral and the fact that $(\gamma \circ \mu^{-1})(V(\tau_0, \phi)) > 0$, we have

$$\begin{aligned}
V(t,x_t) &\geqslant V(\tau_0,x_{\tau_0}) + \int_{\tau_0}^{t} \gamma(\|x(\xi)\|)\,\mathrm{d}\xi \\
&= V(\tau_0,\phi) + \int_{\tau_0}^{t} \gamma(\|x(\xi)\|)\,\mathrm{d}\xi \\
&\geqslant V(\tau_0,\phi) + \int_{\tau_0}^{t} \gamma(\mu^{-1}(V(\xi,x_\xi)))\,\mathrm{d}\xi \\
&= V(\tau_0,\phi) + \int_{\tau_0}^{t} (\gamma\circ\mu^{-1})(V(\xi,x_\xi))\,\mathrm{d}\xi \\
&\geqslant V(\tau_0,\phi) + \int_{\tau_0}^{t} (\gamma\circ\mu^{-1})(V(\tau_0,x_{\tau_0}))\,\mathrm{d}\xi \\
&= V(\tau_0,\phi) + (\gamma\circ\mu^{-1})(V(\tau_0,\phi))(t-\tau_0) \to \infty,
\end{aligned}$$

on $t\to\infty$, which contradicts that $V(t,x_t)$ is bounded above at $[\tau_0,\infty)$, getting the desired result. □

The following lemma is crucial to prove our second main result.

Lemma 5.0.7. Let $V\colon [t_0,\infty)\times G_0 \to \mathbb{R}$ be a functional. Moreover, suppose that

(ND) the function $[\tau_0,\infty)\ni t\mapsto V(t,x_t)$ is nondecreasing, for all solution $x(t)=x(t,\tau_0,\phi)$ of the equation (3.1.1).

(EX) for all solution $x(t)=x(t,\tau_0,\phi)$ of the equation (3.1.1), we have

$$V(t,x_t)-V(s,x_s) \geqslant \alpha\int_s^t V(\xi,x_\xi)\,\mathrm{d}\xi,$$

for all $t,s\in[\tau_0,\infty)$, $s\leqslant t$, for some $\alpha>0$.

Then

$$V(t,x_t)\geqslant V(\tau_0,\phi)e^{\alpha(t-\tau_0)}, \text{ for all } t\in[\tau_0,\infty).$$

Proof. Let $\tau_0\geqslant t_0$, $\phi\in G_0$ and $x(t)=x(t,\tau_0,\phi)$ the solution of the equation (3.1.1).

By condition (ND), $t\mapsto V(t,x_t)$ is a nondecreasing function. Using this fact together with condition (EX), for all $\tau_0\leqslant t_1\leqslant t_2<\infty$, we have

$$V(t_2,x_{t_2})-V(t_1,x_{t_1}) \geqslant \alpha\int_{t_1}^{t_2} V(\xi,x_\xi)\,\mathrm{d}\xi \geqslant \alpha\int_{t_1}^{t_2} V(t_1,x_{t_1})\,\mathrm{d}\xi = \alpha(t_2-t_1)V(t_1,x_{t_1}).$$

In particular,

$$V(t_2,x_{t_2})\geqslant (1+\alpha(t_2-t_1))\,V(t_1,x_{t_1}), \tag{5.0.5}$$

for all $\tau_0\leqslant t_1\leqslant t_2<\infty$.

Now, we claim that $V(s+\tau_0,x_{s+\tau_0})\geqslant e^{\alpha s}V(\tau_0,\phi)$, for all $s\in[0,\infty)$. Indeed, let $s\in[0,\infty)$ be given and $n\in\mathbb{N}$ be an arbitrary but fixed. Define

$$\gamma_i := \frac{is}{n}+\tau_0 \quad \text{for all } i=0,1,\ldots,n.$$

Note that

$$\gamma_0 = \tau_0 < \gamma_1 < \ldots < \gamma_n = s + \tau_0, \quad \text{and} \quad \gamma_i - \gamma_{i-1} = \frac{s}{n}, \quad \text{for } i = 1, \ldots, n.$$

This gives together with (5.0.5), the following inequality

$$V(\gamma_i, x_{\gamma_i}) \geqslant \left(1 + \frac{\alpha s}{n}\right) V(\gamma_{i-1}, x_{\gamma_{i-1}}),$$

for every $i = 1, \ldots, n$. Therefore, proceeding recursively, we have that

$$V(\gamma_n, x_{\gamma_n}) \geqslant \left(1 + \frac{\alpha s}{n}\right)^n V(\gamma_0, x_{\gamma_0}).$$

Due to $V(\gamma_0, x_{\gamma_0}) = V(\tau_0, \phi)$ and $V(\gamma_n, x_{\gamma_n}) = V(s + \tau_0, x_{s+\tau_0})$, we obtain

$$V(s + \tau_0, x_{s+\tau_0}) \geqslant \left(1 + \frac{\alpha s}{n}\right)^n V(\tau_0, \phi),$$

for all $s \geqslant 0$ and all $n \in \mathbb{N}$. Thus, as $n \to \infty$, we conclude that

$$V(s + \tau_0, x_{s+\tau_0}) \geqslant e^{\alpha s} V(\tau_0, \phi),$$

proving the claim, which implies that

$$V(t, x_t) \geqslant e^{\alpha(t-\tau_0)} V(\tau_0, \phi), \text{ for all } t \in [\tau_0, \infty),$$

obtaining the desired result. □

The following result is inspired by (Lyapunov 1992, Theorem III, p. 588) and (Kellett 2015, Theorem 8.4).

Theorem 5.0.8 (Second Lyapunov-type theorem on instability)**.** Consider the equation (3.1.1). Assume that the conditions (P1)–(P5) hold. Let $V\colon [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function that satisfies condition (ND) from Lemma 5.0.7, and $U\colon [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function. Moreover, assume the following conditions:

(H4) there exists a continuous strictly increasing function $L\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $L(0) = 0$ such that

$$V(t, x_t) \leqslant L(\|x(t)\|), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of the equation (3.1.1).

(H5) $U \geqslant 0$ on $[t_0, \infty) \times G_0$ and $t \mapsto U(t, x_t)$ is Perron integrable for all solution $x(t) = x(t, \tau_0, \phi)$ of the equation (3.1.1).

(H6) for all solution $x(t) = x(t, \tau_0, \phi)$ of the functional Volterra–Stieltjes integral equation (3.1.1), we have

$$V(t, x_t) - V(s, x_s) \geqslant \int_s^t (\lambda V(\xi, x_\xi) + U(\xi, x_\xi))\, \mathrm{d}\xi,$$

for all $t, s \in [\tau_0, \infty)$, $s \leqslant t$, for some $\lambda > 0$.

(H7) For each $\tau \geqslant t_0$, $V(\tau, \cdot)$ assumes positive values at points arbitrarily close to the origin.

Then, the solution $x \equiv 0$ of the equation (3.1.1) is unstable.

Proof. Due to condition (H4), we have

$$V(t, x_t) \leqslant L(\|x(t)\|), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of (3.1.1).

Let us assume by contradiction that $x \equiv 0$ is stable. Given $\varepsilon > 0$ and $\tau_0 \geqslant t_0$, there exists $\delta = \delta(\varepsilon, \tau_0) > 0$ such that if $\|\rho\|_\infty < \delta$, then $\|x(t, \tau_0, \rho)\| < \varepsilon$ for all $t \in [\tau_0, \infty)$.

Now, by condition (H7), there exits $\phi \neq 0$ such that $\|\phi\|_\infty < \delta$ and $V(\tau_0, \phi) > 0$. By simplicity, let $x(t)$ denote $x(t, \tau_0, \phi)$. Thus, $\|x(t)\| = \|x(t, \tau_0, \phi)\| < \varepsilon$ for $t \geqslant \tau_0$, and

$$V(t, x_t) \leqslant L(\|x(t)\|) < L(\varepsilon), \tag{5.0.6}$$

for all $t \geqslant \tau_0$.

On the other hand, by (H5)–(H6) and from the properties of the integral, we have

$$\begin{aligned} V(t, x_t) - V(s, x_s) &\geqslant \int_s^t (\lambda V(\xi, x_\xi) + U(\xi, x_\xi))\, \mathrm{d}\xi \\ &= \lambda \int_s^t V(\xi, x_\xi)\, \mathrm{d}\xi + \int_s^t U(\xi, x_\xi)\, \mathrm{d}\xi \\ &\geqslant \lambda \int_s^t V(\xi, x_\xi)\, \mathrm{d}\xi, \quad \tau_0 \leqslant s \leqslant t, \end{aligned}$$

that is,

$$V(t, x_t) - V(s, x_s) \geqslant \lambda \int_s^t V(\xi, x_\xi)\, \mathrm{d}\xi, \quad \text{for all } \tau_0 \leqslant s \leqslant t.$$

Since the hypotheses from Lemma 5.0.7 are satisfied, we have that

$$V(t, x_t) \geqslant V(\tau_0, \phi) e^{\lambda(t-\tau_0)}, \text{ for all } t \in [\tau_0, \infty).$$

Due to $V(\tau_0, \phi) > 0$ and $e^{\lambda(t-\tau_0)} \to \infty$ as $t \to \infty$, we obtain $V(t, x_t) \to \infty$ as $t \to \infty$, which contradicts (5.0.6), proving the result. □

To state the Chetaev-type theorem on instability, we need to introduce a definition about the region $V > 0$.

Definition 5.0.9. Let $V\colon [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function. We define **the region** $V > 0$, denoted by Ω_{V^+}, as

$$\Omega_{V^+} := \{(t, \varphi) \in [t_0, \infty) \times G_0 : V(t, \varphi) > 0\}.$$

Definition 5.0.10. We say that $V\colon [t_0,\infty)\times G_0 \to \mathbb{R}$ is **radially bounded** on $O \subset G_0$ if there exists a strictly increasing function $\beta\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta(0)=0$, such that for each $\varphi \in O$, we have

$$V(t,\varphi) \leqslant \beta(\|\varphi(0)\|), \text{ for all } t \geqslant t_0.$$

Definition 5.0.11. Let $V\colon [t_0,\infty)\times G_0 \to \mathbb{R}$ be a function and $O \subset G_0$. We will refer to the fact that a point $\varphi \in O$ is in the region Ω_{V^+} if $V(t,\varphi) > 0$ for all $t \geqslant t_0$.

The following result is inspired by (Chetaev 1961, Theorem, p. 27).

Theorem 5.0.12 (Chetaev-type theorem on instability)**.** Consider the equation (3.1.1). Assume that the conditions (P1)–(P5) hold. Let $V\colon [t_0,\infty)\times G_0 \to \mathbb{R}$ be a function. Assume that there exists $\varepsilon_0 > 0$ such that:

(H8) each $\varphi \in B_G(\varepsilon_0)$ is in the region Ω_{V^+}.

(H9) V is radially bounded on $B_G(\varepsilon_0)$.

(H10) ΔV is positive definite along the solutions of (3.1.1) in the region $V > 0$, that is, there exists a strictly increasing continuous function $\gamma\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0)=0$, such that for all solution $x(t)=x(t,s_0,\varphi)$, of (3.1.1) with $(t,x_t)\in\Omega_{V^+}$ for all $t \geqslant s_0$, we have

$$V(t,x_t) - V(s,x_s) \geqslant \int_s^t \gamma(\|x(\xi)\|)\,\mathrm{d}\xi, \tag{5.0.7}$$

for all $s_0 \leqslant s \leqslant t$.

Then, the solution $x\equiv 0$ of equation (3.1.1) is unstable.

Proof. According to the condition (H9) there exists a strictly increasing function $\beta\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta(0)=0$ such that for each $\varphi\in B_G(\varepsilon_0)$, we have

$$V(t,\varphi) \leqslant \beta(\|\varphi(0)\|), \text{ for all } t \geqslant t_0. \tag{5.0.8}$$

Let us assume by contradiction that $x\equiv 0$ is stable. Then, for $\tau_0 \geqslant t_0$ and ε_0 there exists $0<\delta<\varepsilon_0$ such that if $\|\varphi\|_\infty < \delta$, then $\|x(t,\tau_0,\varphi)\| < \varepsilon_0$ for all $t\in[\tau_0,\infty)$.

Let ϕ be such that $\|\phi\|_\infty < \delta$. Then, $V(\tau_0,\phi)>0$ and $\|x(t)\| = \|x(t,\tau_0,\phi)\| < \varepsilon_0$ for all $t\in[\tau_0,\infty)$. Using this last inequality and the fact that $\|\phi\|_\infty < \delta < \varepsilon_0$, we obtain $x_t \in B_G(\varepsilon_0)$ for $t \geqslant \tau_0$, and, therefore, by condition (H8), $(t,x_t)\in\Omega_{V^+}$ for all $t\geqslant\tau_0$.

Note that, according to the condition (H10), the function $t\mapsto V(t,x_t)$ is nondecreasing. Using this fact together with (5.0.8), we have

$$0 < V(\tau_0,\phi) = V(\tau_0,x_{\tau_0}) \leqslant V(t,x_t) \leqslant \beta(\|x(t)\|), \text{ for all } t \geqslant \tau_0,$$

which implies that

$$0 < \beta^{-1}(V(\tau_0, \phi)) \leqslant \|x(t)\|, \text{ for all } t \geqslant \tau_0. \tag{5.0.9}$$

Now combining all these inequalities, we have that

$$\begin{aligned}\beta(\varepsilon_0) > \beta(\|x(t)\|) \geqslant V(t, x_t) \geqslant V(\tau_0, \phi) + \int_{\tau_0}^{t} \gamma(\|x(\xi)\|)\, d\xi \\ \geqslant V(\tau_0, \phi) + \int_{\tau_0}^{t} \gamma(\beta^{-1}(V(\tau_0, \phi)))\, d\xi \\ = V(\tau_0, \phi) + \gamma(\beta^{-1}(V(\tau_0, \phi)))(t - \tau_0),\end{aligned}$$

for all $t \geqslant \tau_0$, which is a contradiction, since the right side is an unbounded function of t, while the left side is a fixed constant. □

The following result provides sufficient conditions for instability. This result generalizes the one found in (Ko 1998, Theorem 2.1).

Theorem 5.0.13. Consider the equation (3.1.1). Assume that conditions (P1)–(P5) hold. Let $V\colon [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function such that $\xi \mapsto V(\xi, x_\xi)$ is increasing for all solutions $x(t)$ of (3.1.1). Moreover, assume the following conditions:

(H11) there exists an increasing function $\mu\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\mu(0) = 0$, such that

$$V(t, x_t) \leqslant \mu(\|x(t)\|), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of (3.1.1).

(H12) For each $\tau \geqslant t_0$, $V(\tau, \cdot)$ assumes positive values at points arbitrarily close to the origin.

(H13) there are a strictly increasing continuous function $\mathcal{G}\colon \mathbb{R} \to \mathbb{R}$ with $\mathcal{G}(0) = 0$ and an increasing function $\beta\colon [t_0, \infty) \to \mathbb{R}^+$ with $\int_{t_0}^{\infty} \beta(s)\, \mathrm{d}s = \infty$, such that for all solution $x(t) = x(t, \tau_0, \phi)$, of the equation (3.1.1), we have

$$V(t, x_t) - V(s, x_s) \geqslant \int_{s}^{t} \beta(\xi) \mathcal{G}(V(\xi, x_\xi))\, \mathrm{d}\xi,$$

for all $t, s \in [\tau_0, \infty)$ with $s \leqslant t$.

Then, the solution $x \equiv 0$ of the equation (3.1.1) is unstable.

Proof. Since $V\colon [t_0, \infty) \times G_0 \to \mathbb{R}$ satisfies condition (H11), there exists an increasing function $\mu\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\mu(0) = 0$ such that

$$V(t, x_t) \leqslant \mu(\|x(t)\|),$$

for all solution $x(t)$ of (3.1.1).

Let us assume by contradiction that $x \equiv 0$ is stable. Given $\varepsilon > 0$ and $\tau_0 \geqslant t_0$, there exists $\delta = \delta(\varepsilon, \tau_0) > 0$ such that if $\|\varphi\|_\infty < \delta$, then $\|x(t, \tau_0, \varphi)\| < \varepsilon$ for all $t \in [\tau_0, \infty)$.

Now, by condition (H12), there exits $\phi \neq 0$ such that $\|\phi\|_\infty < \delta$ and $V(\tau_0, \phi) > 0$. Then $\|x(t)\| = \|x(t, \tau_0, \phi)\| < \varepsilon$, for $t \geqslant \tau_0$, which implies that

$$V(t, x_t) \leqslant \mu(\|x(t)\|) \leqslant \mu(\varepsilon), \tag{5.0.10}$$

for all $t \geqslant \tau_0$, and therefore, $V(t, x_t)$ is bounded above on $[\tau_0, \infty)$.

On the other hand, by hypotheses $\xi \mapsto V(\xi, x_\xi)$ is an increasing function, and consequently, the function $\xi \mapsto \mathcal{G}(V(\xi, x_\xi))$ is also increasing. Using this fact together with (H13), $\mathcal{G}(V(\tau_0, \phi)) > 0$ and properties of integral, we have

$$\begin{aligned} V(t, x_t) &\geqslant V(\tau_0, x_{\tau_0}) + \int_{\tau_0}^{t} \beta(\xi)\mathcal{G}(V(\xi, x_\xi))\,\mathrm{d}\xi \\ &\geqslant V(\tau_0, \phi) + \int_{\tau_0}^{t} \beta(\xi)\mathcal{G}(V(\tau_0, x_{\tau_0}))\,\mathrm{d}\xi \\ &= V(\tau_0, \phi) + \mathcal{G}(V(\tau_0, \phi)) \int_{\tau_0}^{t} \beta(\xi)\,\mathrm{d}\xi \to \infty, \end{aligned}$$

as $t \to \infty$, which contradicts that $V(t, x_t)$ is bounded above on $[\tau_0, \infty)$. □

Corollary 5.0.14. Assume that in Theorem 5.0.13 the condition (H13) is replaced by the condition:

(H14) there exists an increasing function $\beta\colon [t_0, \infty) \to \mathbb{R}^+$ with $\int_{t_0}^{\infty} \beta(s)\,\mathrm{d}s = \infty$, such that for all solution $x(t) = x(t, \tau_0, \phi)$, of the equation (3.1.1), we have

$$V(t, x_t) - V(s, x_s) \geqslant \int_{s}^{t} \beta(\xi) V(\xi, x_\xi)\,\mathrm{d}\xi,$$

for all $t, s \in [\tau_0, \infty)$ with $s \leqslant t$.

Then the solution $x \equiv 0$ of the equation (3.1.1) is unstable.

Proof. It is enough to observe that condition (H14) implies condition (H13) by setting $\mathcal{G}(t) := t$ for all $t \in \mathbb{R}$. Now, the result follows from Theorem 5.0.13. □

5.1 Applications

5.1.1 Instability for functional fractional integral equations

In this section, we will present instability criteria for a class of functional fractional integral equation of type:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(x_s, s)\,\mathrm{d}s, \quad t \geqslant \tau_0 \\ x_{\tau_0} = \phi, \end{cases} \tag{5.1.1}$$

where Γ is the gamma function, $1 < \alpha < 2$, $\tau_0 \geqslant t_0$, $\phi \in G_0$ and $f\colon G_0 \times [t_0, \infty) \to \mathbb{R}$. For more details on this type of equations, the reader can consult (Chen, Nieto and Zhou 2012) and references therein.

Throughout this section we will use the notation $G_0 := G([-r, 0], \mathbb{R}^n)$ and $G_\infty := G([t_0 - r, \infty), \mathbb{R}^n)$.

We assume the following conditions:

(FI1) The function $[\tau_1, \tau_2] \ni s \mapsto \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(x_s, s)$ is Perron integrable, for each compact interval $[\tau_1, \tau_2] \subset [t_0, \infty)$ and all $x \in G_\infty$.

(FI2) There exists a locally Perron integrable function $M\colon [t_0, \infty) \to \mathbb{R}$ such that for each compact interval $[\tau_1, \tau_2] \subset [t_0, \infty)$, we have

$$\left| \int_{\tau_1}^{\tau_2} c \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(x_s, s) \mathrm{d}s \right| \leqslant \int_{\tau_1}^{\tau_2} |c| \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} M(s) \mathrm{d}s,$$

for all $x \in G_\infty$ and all $c \in \mathbb{R}$.

(FI3) There exists a locally regulated function $L\colon [t_0, \infty) \to \mathbb{R}$ such that for each compact interval $[\tau_1, \tau_2] \subset [t_0, \infty)$, we have

$$\left| \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} [f(x_s, s) - f(z_s, s)] \mathrm{d}s \right| \leqslant \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} L(s) \|x_s - z_s\|_\infty \mathrm{d}s,$$

for all $x, z \in G_\infty$.

We assume that $f(0, t) \equiv 0$ for all $t \geqslant t_0$. With this condition, $x \equiv 0$ is a solution of the functional fractional integral equation (5.1.1) with initial condition $\phi = 0$.

Next, we present the first Lyapunov-type theorem on instability for the functional fractional integral equation (5.1.1).

Theorem 5.1.1. Assume $f\colon G_0 \times [t_0, \infty) \to \mathbb{R}^n$ satisfies conditions (FI1), (FI2), and (FI3). Let $W\colon [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function. Assume the following conditions:

(FH1) ΔW is positive definite along the solutions of (5.1.1).

(FH2) there exists a continuous strictly increasing function $\mu\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\mu(0) = 0$ such that

$$W(t, x_t) \leqslant \mu(\|x(t)\|), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of (5.1.1).

(FH3) For each $\tau \geqslant t_0$, $W(\tau, \cdot)$ assumes positive values at points arbitrarily close to the origin.

Then, the solution $x \equiv 0$ of the equation (5.1.1) is unstable.

Proof. Define $g : [t_0, \infty) \to \mathbb{R}$ and $a \colon [t_0, \infty)^2 \to \mathbb{R}$ given by $g(s) = s$ and

$$a(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & t \geqslant s, \\ 0, & t < s. \end{cases}$$

Therefore, (5.1.1) is in the form of (3.1.1).

According to the conditions (FI1)–(FI3) and the definitions of g and a, we have that f, g, a satisfy the conditions of Theorem 5.0.6. On the other hand, due to the hypotheses (FH1)–(FH3), W satisfies the conditions (H1)–(H3) from Theorem 5.0.6. Since all the hypotheses from Theorem 5.0.6 are satisfied, we have that the solution $x \equiv 0$ of the equation (5.1.1) is unstable. □

Below we are going to present the second Lyapunov-type theorem on instability for the functional fractional integral equation (5.1.1).

Theorem 5.1.2. Assume $f \colon G_0 \times [t_0, \infty) \to \mathbb{R}^n$ satisfies conditions (FI1), (FI2), and (FI3). Let $W \colon [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function that satisfies condition (ND) from Lemma 5.0.7, and $U \colon [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function. Moreover, assume the following conditions:

(FH4) there exists a continuous strictly increasing function $l \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $l(0) = 0$ such that

$$W(t, x_t) \leqslant l(\|x(t)\|), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of (5.1.1).

(FH5) $U \geqslant 0$ on $[t_0, \infty) \times G_0$ and $t \mapsto U(t, x_t)$ is Perron integrable for all solution $x(t) = x(t, \tau_0, \phi)$ of (5.1.1).

(FH6) for all solution $x(t) = x(t, \tau_0, \phi)$ of (5.1.1), we have

$$W(t, x_t) - W(s, x_s) \geqslant \int_s^t (\lambda W(\xi, x_\xi) + U(\xi, x_\xi))\, \mathrm{d}\xi,$$

for all $t, s \in [\tau_0, \infty)$, $s \leqslant t$, for some $\lambda > 0$.

(FH7) For each $\tau \geqslant t_0$, $W(\tau, \cdot)$ assumes positive values at points arbitrarily close to the origin.

Then, the solution $x \equiv 0$ of the equation (5.1.1) is unstable.

Proof. Note that (5.1.1) can be written in the form of (3.1.1) by setting $g(s) := s$ and

$$a(t,s) := \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & t \geqslant s, \\ 0, & t < s, \end{cases}$$

for all $t, s \in [t_0, \infty)$. Furthermore, using the same arguments as in the proof of Theorem 5.1.1, we can prove that f, g, a satisfy all the conditions of Theorem 5.0.8.

On the other hand, conditions (FH4)–(FH7) imply assumptions (H4)–(H7) from Theorem 5.0.8. Under these conditions, Theorem 5.0.8 guarantees that the solution $x \equiv 0$ of the equation (5.1.1) is unstable. □

To finish this section, we present the Chetaev-type theorem on instability for the equation (5.1.1).

Theorem 5.1.3. Assume $f\colon G_0 \times [t_0, \infty) \to \mathbb{R}^n$ satisfies conditions (FI1), (FI2), and (FI3). Let $W\colon [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function. Assume that there exist $\varepsilon_0 > 0$ such that:

(FH8) each $\varphi \in B_G(\varepsilon_0)$ is in the region Ω_{V^+}.

(FH9) W is radially bounded on $B_G(\varepsilon_0)$.

(FH10) there exists a strictly increasing continuous function $\gamma\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$, such that for all solution $x(t) = x(t, s_0, \varphi)$, of (5.1.1) with $(t, x_t) \in \Omega_{W^+}$ for all $t \geqslant s_0$, we have

$$W(t, x_t) - W(s, x_s) \geqslant \int_s^t \gamma(\|x(\xi)\|)\, \mathrm{d}\xi,$$

for all $s_0 \leqslant s \leqslant t < +\infty$.

Then, the solution $x \equiv 0$ of the equation (5.1.1) is unstable.

Proof. It is an immediate consequence of Theorem 5.0.12 and the fact that the equation (5.1.1) can be written in the form of the equation (3.1.1) taking $g : [t_0, \infty) \to \mathbb{R}$ and $a\colon [t_0, \infty)^2 \to \mathbb{R}$ given, respectively, by $g(s) = s$ and

$$a(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & t \geqslant s, \\ 0, & t < s, \end{cases}$$

for all $t, s \in [t_0, \infty)$. □

5.1.2 Instability for impulsive functional Volterra-Stieltjes integral equations

In this section, we consider the impulsive functional Volterra–Stieltjes integral equation of type:

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{t_0}^t a(t, s) f(x_s, s) \mathrm{d}g(s) + \sum_{\substack{k \in \mathbb{N} \\ t_k < t}} I_k(x(t_k)) \\ x_{t_0} = \phi, \end{cases} \tag{5.1.2}$$

Here, we assume the conditions (D1)–(D5), the existence and uniqueness results, and the correspondence of equation (5.1.2) with equation (3.1.1) presented in Chapter 3.

In the remainder of this section, we assume that $f(0,t) = 0$ for all $t \geqslant t_0$ and $I_k(0) = 0$ for each $k \in \mathbb{N}$. Under these conditions, we have that $x = 0$ is a solution of the equation (5.1.2). Also, the function $\tilde{f}$, given by, (3.2.10), satisfies $\tilde{f}(0,t) = 0$ for $t \in [t_0, +\infty)$.

Definition 5.1.4. The trivial solution $x \equiv 0$ of the equation (5.1.2) is said to be

- **Stable**, if for all $\tau_0 \geqslant t_0$ and all $\varepsilon > 0$, there exists a $\delta = \delta(\tau_0, \varepsilon) > 0$ such that if $\|\phi\|_\infty < \delta$, then $\|x(t, \tau_0, \phi)\| < \varepsilon$ for all $t \in [\tau_0, \infty)$.
- **Unstable**, if there exist $\varepsilon > 0$ and $\tau_0 \geqslant t_0$ such that for any $\delta > 0$ there exist a ϕ with $\|\phi\|_\infty < \delta$ and a $t_1 > \tau_0$ so that $\|x(t_1, \tau_0, \phi)\| \geqslant \varepsilon$.

Next, we present the first Lyapunov-type theorem on instability for the solution $x \equiv 0$ of the equation (5.1.2).

Theorem 5.1.5. Let $\{t_k\}_{k=1}^{\infty}$ be the moments of impulses in $[t_0, \infty)$, such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Assume that the conditions (D1)–(D5) hold. Moreover, suppose that g is continuous at $\{t_k\}_{k=1}^{\infty}$ and a is continuous with respect to first variable at $\{t_k\}_{k=1}^{\infty}$. Let $U\colon [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function. Suppose the following conditions:

(IL1) ΔU is positive definite along the solutions of (5.1.2).

(IL2) there exists a continuous strictly increasing function $\mu\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\mu(0) = 0$ such that
$$U(t, x_t) \leqslant \mu(\|x(t)\|), \quad t \geqslant \tau_0,$$
for all solution $x(t) = x(t, \tau_0, \phi)$ of (5.1.2).

(IL3) each $\tau \geqslant t_0$, $U(\tau, \cdot)$ assumes positive values at points arbitrarily close to the origin.

Then, the solution $x \equiv 0$ of the equation (5.1.2) is unstable.

Proof. Consider the functional Volterra-Stieltjes integral equation:
$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} \tilde{a}(t,s)\tilde{f}(x_s, s)\mathrm{d}\tilde{g}(s) \\ x_{t_0} &= \phi, \end{cases} \tag{5.1.3}$$
where $\tilde{f}$, $\tilde{g}$ and $\tilde{a}$ are given by (3.2.10), (3.2.11) and (3.2.12), respectively.

By conditions (D1)–(D5) and proceeding as in (Alvarez *et al.* 2021, Theorem 4.3), we can prove that $\tilde{f}, \tilde{g}, \tilde{a}$ satisfy the conditions (P1)–(P5) from Theorem 5.0.6. Notice that by Theorem 3.2.9 and by hypotheses (IL1)–(IL3), it is not difficult to see that U satisfies condition (H1)–(H3). Since all the hypotheses of Theorem 5.0.6 are satisfied, it follows that the zero solution of the equation (5.1.3) is unstable. Now, the result follows from Theorem 3.2.9. □

In the sequel, we present the second Lyapunov-type theorem on instability for the solution $x \equiv 0$ of the equation (5.1.2).

Theorem 5.1.6. Let $\{t_k\}_{k=1}^{\infty}$ be the moments of impulses in $[t_0, \infty)$, such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Assume that the conditions (D1)–(D5) hold. Moreover, suppose that g is continuous at $\{t_k\}_{k=1}^{\infty}$ and a is continuous with respect to first variable at $\{t_k\}_{k=1}^{\infty}$. Let $U\colon [t_0,\infty) \times G_0 \to \mathbb{R}$ be a function that satisfies condition (ND) from Lemma 5.0.7 and $Q\colon [t_0,\infty)\times G_0 \to \mathbb{R}$ be a function. Moreover, assume the following conditions:

(IL4) there exists a continuous strictly increasing function $\mathcal{L}\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\mathcal{L}(0) = 0$ such that

$$U(t, x_t) \leqslant \mathcal{L}(\|x(t)\|), \quad t \geqslant \tau_0,$$

for all solution $x(t) = x(t, \tau_0, \phi)$ of (5.1.2).

(IL5) $Q \geqslant 0$ on $[t_0,\infty)\times G_0$ and $t \mapsto Q(t, x_t)$ is Henstock-Kurzweil integrable for all solution $x(t) = x(t, \tau_0, \phi)$ of the impulsive functional Volterra–Stieltjes integral equation (5.1.2).

(IL6) for all solution $x(t) = x(t, \tau_0, \phi)$ of (5.1.2), we have

$$U(t, x_t) - U(s, x_s) \geqslant \int_s^t (\lambda U(\xi, x_\xi) + Q(\xi, x_\xi))\, \mathrm{d}\xi,$$

for all $t, s \in [\tau_0, \infty)$, $s \leqslant t$, for some $\lambda > 0$.

(IL7) for each $\tau \geqslant t_0$, $U(\tau, \cdot)$ assumes positive values at points arbitrarily close to the origin.

Then, the solution $x \equiv 0$ of the equation (5.1.2) is unstable.

Proof. Define $\tilde{f}\colon G_0 \times [t_0,\infty) \to \mathbb{R}^n$, $\tilde{g}\colon [t_0,\infty) \to \mathbb{R}$ and $\tilde{a}\colon [t_0,\infty)^2 \to \mathbb{R}$ by (3.2.10), (3.2.11) and (3.2.12), respectively, and consider the functional Volterra-Stieltjes integral equation:

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^t \tilde{a}(t,s)\tilde{f}(x_s,s)\mathrm{d}\tilde{g}(s) \\ x_{t_0} &= \phi. \end{cases} \tag{5.1.4}$$

Using the same arguments as in the proof of Theorem 5.1.5, we can prove that $\tilde{f}, \tilde{g}, \tilde{a}$ satisfy conditions (P1)–(P5) from Theorem 5.0.8.

By Theorem 3.2.9 and assumptions (IL4), (IL6), U satisfies conditions (H4), (H6) from Theorem 5.0.8. On the other hand, note that (IL7) implies that (H7) holds. Moreover, by Theorem 3.2.9 and hypotheses (IL5), U satisfies condition (H5). Hence, by Theorem 5.0.8, the solution $x \equiv 0$ of (5.1.3) is unstable. Now, by Theorem 3.2.9, the solution $x \equiv 0$ of the impulsive functional Volterra-Stieltjes integral equation (5.1.2) is also unstable, obtaining the desired result. □

Finally, we present a Chetaev-type theorem on instability for the solution $x \equiv 0$ of the impulsive functional Volterra-Stieltjes integral equation (5.1.2).

We recall the reader that $\Omega_{U^+} := \{(t, \varphi) \in [t_0, \infty) \times G_0 : U(t, \varphi) > 0\}$.

Theorem 5.1.7. Let $\{t_k\}_{k=1}^{\infty}$ be the moments of impulses in $[t_0, +\infty)$, such that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \infty$. Assume that the conditions (D1)–(D5) hold. Moreover, suppose that g is continuous at $\{t_k\}_{k=1}^{\infty}$ and a is continuous with respect to first variable at $\{t_k\}_{k=1}^{\infty}$. Let $U\colon [t_0, \infty) \times G_0 \to \mathbb{R}$ be a function. Assume that there are $\eta_0 > 0$ and $\tau_0 \geqslant t_0$ such that

(CIH8) each $\varphi \in B_G(\eta_0)$ is in the region Ω_{U^+}.

(CIH9) U is radially bounded on $B_G(\eta_0)$.

(CIH10) there exists a strictly increasing continuous function $\beta\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta(0) = 0$, such that for all solution $x(t) = x(t, s_0, \varphi)$, of (5.1.2) with $(t, x_t) \in \Omega_{U^+}$ for all $t \geqslant s_0$, we have

$$U(t, x_t) - U(s, x_s) \geqslant \int_s^t \beta(\|x(\xi)\|)\,\mathrm{d}\xi,$$

for all $s_0 \leqslant s \leqslant t < \infty$.

Then, the solution $x \equiv 0$ of the impulsive functional Volterra–Stieltjes integral equation (5.1.2) is unstable.

Proof. Consider the functional Volterra-Stieltjes integral equation:

$$\begin{cases} x(t) &= \phi(0) + \displaystyle\int_{t_0}^{t} \tilde{a}(t,s)\tilde{f}(x_s,s)\mathrm{d}\tilde{g}(s) \\ x_{t_0} &= \phi, \end{cases} \tag{5.1.5}$$

where $\tilde{f}$, $\tilde{g}$ and $\tilde{a}$ are given by (3.2.10), (3.2.11) and (3.2.12), respectively.

Similarly as in the proof of Theorem 5.1.5, we can prove that $\tilde{f}, \tilde{g}, \tilde{a}$ satisfies conditions (P1)–(P5) from Theorem 5.0.12.

On the other hand, it is easy to see that conditions (CIH8)-(CIH9) implies that (H8)-(H9) holds. Also, by Theorem 3.2.9 and (CIH10), U satisfies condition (H10) from Theorem 5.0.12. Under these conditions, Theorem 5.0.12 guarantees that the solution

$x \equiv 0$ of (5.1.5) is unstable. Again, using the Theorem 3.2.9, the solution $x \equiv 0$ of the impulsive functional Volterra-Stieltjes integral equation (5.1.2) is also unstable, getting the result. □

5.1.3 Mechanical mass-spring-damper system with delay and positive feedback

The following example is an adaptation of (Vidyasagar 1993, Problem 3.10, p. 78) for the delay case, and the inclusion of a positive feedback term.

Consider the system

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = x(t) - \mathcal{G}(x(t-r)) - \mathcal{F}(y(t-r)), \end{cases} \tag{5.1.6}$$

with initial condition $x_{\tau_0} = \phi_1$, $y_{\tau_0} = \phi_2$ for $\tau_0 \geqslant 0$ and $\phi_1, \phi_2 \in G([-r, 0], \mathbb{R})$. This system represents a mechanical model consisting of a unit mass, a nonlinear spring (restoring force $F_{\texttt{res}} = -\mathcal{G}(\cdot)$), a nonlinear damper (dissipative force $F_{\texttt{damp}} = -\mathcal{H}(\cdot)$), an active destabilizing term (positive feedback $F_{\text{act}} = +x(\cdot)$), and time delay ($r > 0$) in the restoring and damping forces, see Figure 8. We assume the following condition

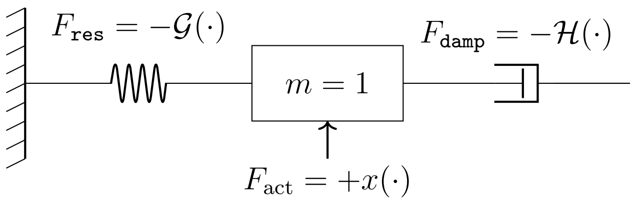


Figure 8 – Mechanical mass-spring-damper system

(M1) The functions $\mathcal{G}, \mathcal{F} \colon \mathbb{R} \to \mathbb{R}$ are bounded and Lipschitz, with respective constants $L_{\mathcal{G}}$ and $L_{\mathcal{F}}$, such that for all $\varphi = (\varphi_1, \varphi_2) \in G_0$ we have $\varphi_1(0)\mathcal{G}(\varphi_1(-r)) \leqslant 0$ and $\varphi_1(0)\mathcal{G}(\varphi_2(-r)) \leqslant 0$. Here, $G_0 := G([-r, 0], \mathbb{R}^2)$.

(M2) $\mathcal{G}(0) = \mathcal{F}(0) = 0$.

For our purposes we will consider $\mathbb{R}^2$ with the norm $\|z\| = \sqrt{x^2 + y^2}$ for $z = (x, y) \in \mathbb{R}^2$.

Note that system (5.1.6) can be transformed into

$$\begin{cases} z(t) = z(\tau_0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(z_s, s)\, \mathrm{d}g(s), \\ z_{\tau_0} = \phi, \end{cases} \tag{5.1.7}$$

when considering $z=(x,y)$, $\phi=(\phi_1,\phi_2)$, $a(t,s)=1$, $g(s)=s$, for $t,s\in[0,\infty)$, and $f(\varphi,t)=(\varphi_2(0),\varphi_1(0)-\mathcal{G}(\varphi_1(-r)))-\mathcal{F}(\varphi_2(-r))$, for $t\in[0,\infty)$, $\varphi=(\varphi_1,\varphi_2)\in G_0$.

By condition (M2), we obtain $f(0,t)=0$ for all $t\geqslant 0$. This implies that $x\equiv 0$ is a solution of (5.1.6) with initial condition $\phi=0$.

It is easy to verify that

$$\left\|\int_{\tau_1}^{\tau_2}\beta f(z_s,s)\,\mathrm{d}g(s)\right\|\leqslant\int_{\tau_1}^{\tau_2}|\beta|\,(2c+b_{\mathcal{G}}+b_{\mathcal{F}})\,\mathrm{d}g(s),$$

and

$$\left\|\int_{\tau_1}^{\tau_2}[f(z_s,s)-f(\overline{z}_s,s)]\,\mathrm{d}g(s)\right\|\leqslant\int_{\tau_1}^{\tau_2}(2+2(L_{\mathcal{G}}+L_{\mathcal{F}}))\,\|z_s-\overline{z}_s\|_\infty\,\mathrm{d}g(s),$$

for all $0\leqslant\tau_1\leqslant\tau_2<\infty$, $\beta\in\mathbb{R}$ and $z,\overline{z}\in G^c_\infty$, where $b_{\mathcal{G}}:=\sup_{u\in\mathbb{R}}|\mathcal{G}(u)|$, $b_{\mathcal{F}}:=\sup_{u\in\mathbb{R}}|\mathcal{F}(u)|$, and $G^c_\infty:=\{z\in G_\infty([-r,\infty),\mathbb{R}^2):\|z(t)\|\leqslant c,\ t\in[-r,\infty)\}$. Hence, (I1) and (I2) are satisfied.

On the other hand, define the functional $V\colon[0,\infty)\times G_0\to\mathbb{R}$ by $V(t,\varphi):=\varphi_1(0)\varphi_2(0)$, for all $\varphi=(\varphi_1,\varphi_2)\in G_0$.

Let $\phi\in G_0$, $\tau_0\geqslant 0$ and $z(t)=z(t,\tau_0,\phi)=(x(t),y(y))$ be the solution of the mechanical system (5.1.6). Then

$$V(t,z_t)=x_t(0)y_t(0)=x(t)y(t).$$

Note that, according to condition (M1), we have

$$\dot{V}=\dot{x}(t)y(t)+x(t)\dot{y}(t)=x^2(t)+y^2(t)-x(t)(\mathcal{G}(x(t-r))+\mathcal{F}(y(t-r)))\geqslant\gamma(\|z(t)\|),$$

where $\gamma(s):=s^2$ for all $s\geqslant 0$. This implies that

$$V(t,z_t)-V(s,z_s)\geqslant\int_s^t\gamma(\|z(\xi)\|)\,\mathrm{d}\xi,$$

and therefore, ΔV is positive definite along the solutions of (5.1.6), getting (H1).

On the other hand, for each $t\geqslant\tau_0$, we have

$$V(t,z_t)\leqslant\frac{1}{2}(x^2(t)+y^2(t))=\mu(\|z(t)\|),$$

where $\mu(s)=\frac{1}{2}s^2$, for all $s\geqslant 0$, and therefore (H2) is satisfied. Finally, it is easy to see that for each $\tau\geqslant 0$, $V(\tau,\cdot)$ assumes positive values at points arbitrarily close to the origin, obtaining the condition (H3).

Since all the conditions of Theorem 5.0.6 are satisfied, we conclude that the solution $x\equiv 0$ of the mechanical system (5.1.6) is unstable.

CHAPTER

6

VOLTERRA-STIELTJES FUNCTIONAL EQUATION WITH INFINITE DELAY

In this is chapter we introduce Volterra-Stieljtes functional equations with infinite delay and an axiomatically described phase space. Here, we show existence and uniqueness of solutions and continuous dependence with respect to initial data.

The following results are auxiliary results, important for the development of the chapter. The first result is an Arzelà–Ascoli-type theorem for regulated functions taking values in a $\mathbb{R}^n$. This result is a special case of (Bonotto, Federson and Mesquita 2021, Corollary 1.19).

Theorem 6.0.1. The following conditions are equivalent:

(i) $\mathcal{A} \subset (G([a,b],\mathbb{R}^n), \|\cdot\|_\infty)$ is relatively compact

(ii) $\{x(a) : x \in \mathcal{A}\}$ is bounded and there exists a nondecreasing function $v\colon [a,b] \to \mathbb{R}$ such that

$$\|x(t_2) - x(t_1)\| \leqslant v(t_2) - v(t_1),$$

for all $x \in \mathcal{A}$ and all $a \leqslant t_1 \leqslant t_2 \leqslant b$.

The next result can be found in (Krasnoselskii 1958).

Theorem 6.0.2 (Krasnosel'skiĭ Fixed Point Theorem)**.** Let X be a Banach Space and Y be a nonempty convex and closed subset of X. Let $\mathcal{Q}_1, \mathcal{Q}_2\colon Y \to X$ be two operators satisfying

(i) if $u, v \in Y$, then $\mathcal{Q}_1 u + \mathcal{Q}_2 v \in Y$.

(ii) $\mathcal{Q}_1$ is a contraction on Y.

(iii) $\mathcal{Q}_2$ is compact and continuous on Y.

Then, there exists $z \in Y$ such that $\mathcal{Q}_1 z + \mathcal{Q}_2 z = z$.

6.0.1 Phase space

When dealing with infinite delay, the crucial problem is the choice of the phase space, i.e., the domain of the first argument of f that contains the delay. In the classical case (3.1.1), i.e. $g(s) = s$, the elements of this phase space are continuous functions. Such a phase space is no longer suitable for a general measure functional differential equation, whose solutions are discontinuous functions. We do not restrict ourselves to a particular phase space; instead, we introduce a certain system of conditions and allow the phase space to be any space that satisfies these conditions. A similar axiomatic approach has been used by several authors (Hino, Murakami and Naito 2025) and (Hale and Kato 1978) to describe the phase space of classical or impulsive functional differential equations with infinite delay.

Inspired by the definition in (Gallegos, Grau and Mesquita 2021, Section 3.1), we consider a linear space $\mathcal{B} \subset G((-\infty, 0], \mathbb{R}^n)$ equipped with a norm denoted by $\|\cdot\|_{\mathcal{B}}$, and that satisfies the following fundamental axioms:

(A1) $\mathcal{B}$ is complete.

(A2) For all $t_0 \in \mathbb{R}$, all $\sigma > 0$, and all $y : (-\infty, t_0 + \sigma] \to \mathbb{R}^n$ such that $y_{t_0} \in \mathcal{B}$ and $y|_{[t_0, t_0+\sigma]}$ is regulated, the following conditions hold: for every $t \in [t_0, t_0 + \sigma]$

(i) $y_t \in \mathcal{B}$

(ii) there exists a locally bounded function $\mu_1 : [0, +\infty) \to (0, +\infty)$ such that

$$\|y(t)\| \leqslant \mu_1(t - t_0)\|y_t\|_{\mathcal{B}}.$$

(iii) there exist locally bounded functions $\mu_2, \mu_3 : [0, +\infty) \to (0, +\infty)$ such that

$$\|y_t\|_{\mathcal{B}} \leqslant \mu_2(t - t_0)\|y_{t_0}\|_{\mathcal{B}} + \mu_3(t - t_0) \sup_{s \in [t_0, t]} \|y(s)\|.$$

(A3) Let $S(t) : \mathcal{B} \to \mathcal{B}$ for $t \geqslant 0$, be the operator defined by

$$[S(t)\varphi](\theta) := \begin{cases} \varphi(t + \theta), & \theta < -t \\ \varphi(0^-), & -t \leqslant \theta < 0 \\ \varphi(0), & \theta = 0. \end{cases}$$

Then there exists a continuous function $k : [0, +\infty) \to (0, +\infty)$ with $k(0) = 0$ such that

$$\|S(t)\varphi\|_{\mathcal{B}} \leqslant (1 + k(t))\|\varphi\|_{\mathcal{B}},$$

for all $\varphi \in \mathcal{B}$.

6.0.2 Existence and uniqueness of solutions and continuous dependence with respect to initial data

Through this section let $\mathcal{B}$ be a phase space as in the Section 6.0.1.

In order to establish our result, we consider the space

$$\mathcal{S} := \{y : (-\infty, t_0 + \sigma] \to \mathbb{R}^n : y_{t_0} \in \mathcal{B} \text{ and } y|_{[t_0,t_0+\sigma]} \text{ is regulated}\} \tag{6.0.1}$$

endowed with the norm

$$\|y\|_{\mathcal{S}} := \|y_{t_0}\|_{\mathcal{B}} + \sup_{t\in[t_0,t_0+\sigma]} \|y(t)\|, \qquad y \in \mathcal{S}. \tag{6.0.2}$$

A complete and detailed proof of the following result is presented in the literature and can be found in (Gallegos, Henríquez and Mesquita 2022, Section 4).

Lemma 6.0.3. The space $\mathcal{S}$ defined by (6.0.1) equipped with the standard norm (6.0.2) is a Banach space.

On the other hand, given $t_0 \in \mathbb{R}$, $\sigma > 0$, $t_0 \leqslant t_1 < t_2 < \ldots < t_m < t_0 + \sigma$, $\phi \in \mathcal{B}$, $g\colon [t_0, t_0 + \sigma] \to \mathbb{R}$, $a\colon [t_0, t_0 + \sigma]^2 \to \mathbb{R}$, $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots, m$ and $f\colon \mathcal{B} \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$, we assume the following condition:

(V1) The function $g\colon [t_0, t_0 + \sigma] \to \mathbb{R}$ is nondecreasing, left–continuous on $(t_0, t_0 + \sigma]$, and continuous at $t_1, \ldots, t_m$.

(V2) The function $a\colon [t_0, t_0 + \sigma]^2 \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable, and continuous with respect to first variable at $t_1, \ldots, t_m$.

(V3) The Henstock–Kurzweil–Stieltjes integral

$$\int_{t_0}^{t_0+\sigma} a(t, s) f(x_s, s) \mathrm{d}g(s)$$

exists for all $x \in \mathcal{S}$ and all $t \in [t_0, t_0 + \sigma]$.

(V4) There exists a locally Henstock–Kurzweil–Stieltjes integrable function $M\colon [t_0, t_0 + \sigma] \to \mathbb{R}^+$ with respect to g such that

$$\left\| \int_\alpha^\beta (c_2 a(\tau_2, s) + c_1 a(\tau_1, s))\, f(x_s, s) \mathrm{d}g(s) \right\| \leqslant \int_\alpha^\beta |c_2 a(\tau_2, s) + c_1 a(\tau_1, s)|\, M(s) \mathrm{d}g(s),$$

for all $x \in \mathcal{S}$, all $c_1, c_2 \in \mathbb{R}$ and all $t_0 \leqslant \alpha \leqslant \beta \leqslant t_0 + \sigma$, and there exists a constant $M_2 > 0$ such that

$$\|I_k(x)\| \leqslant M_2,$$

for all $k = 1, \ldots, m$, and all $x \in \mathbb{R}^n$.

(V5) There exists a regulated function $L\colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left\| \int_{\alpha}^{\beta} a(\tau_2, s)[f(x_s, s) - f(z_s, s)]\mathrm{d}g(s) \right\| \leqslant \int_{\alpha}^{\beta} |a(\tau_2, s)|\, L(s)\, \|x_s - z_s\|_{\mathcal{B}}\, \mathrm{d}g(s),$$

for all $x, z \in \mathcal{S}$, and all $t_0 \leqslant \alpha \leqslant \beta \leqslant t_0 + \sigma$, and there exists a constant $L_2 > 0$ such that

$$\|I_k(x) - I_k(y)\| \leqslant L_2 \, \|x - y\|\,, \tag{6.0.3}$$

for all $k = 1, \ldots, m$, and all $x,\, y \in \mathbb{R}^n$.

Remark 6.0.4. Assume that the function a satisfies condition (V2). Since $a(t_0, \xi) \leqslant a(t, \xi) \leqslant a(t_0 + \sigma, \xi)$ for all $t, \xi \in [t_0, t_0 + \sigma]$ and the functions $a(t_0, \xi)$ and $a(t_0 + \sigma, \xi)$ are regualted with respect to ξ, we obtain a is bounded on $[t_0, t_0 + \sigma]^2$.

In this section, we consider the impulsive functional Volterra-Stieltjes integral equations with infinite delay:

$$\begin{aligned} y(v) - y(u) &= \int_{t_0}^{v} a(v, s) f(y_s, s)\, \mathrm{d}g(s) \\ &\quad - \int_{t_0}^{u} a(u, s) f(y_s, s)\, \mathrm{d}g(s) \;\text{ for } u, v \in l_k,\; k \in \{0, \ldots, m\}, \\ \Delta^+ x(t_k) &= I_k(x(t_k)), \;\; k \in \{1, \ldots, m\}, \\ y(t) &= \phi(t - t_0), \quad t \in (-\infty, t_0], \end{aligned}$$

where $l_0 = [t_0, t_1]$, $l_k = (t_k, t_{k+1}]$ for $k \in \{1, \ldots, m-1\}$, and $l_m = (t_m, t_0 + \sigma]$.

Under assumptions (V1)-(V5), following the ideas of (Alvarez *et al.* 2021, Section 4), our problem can be rewritten as

$$\begin{cases} y(t) = \phi(0) + \displaystyle\int_{t_0}^{t} a(t, s) f(y_s, s) \mathrm{d}g(s) + \sum_{\substack{k \in \{1, \ldots, m\} \\ t_k < t}} I_k(y(t_k)), & t \in [t_0, t_0 + \sigma] \\ y(t) = \phi(t - t_0), \quad t \in (-\infty, t_0] \end{cases}$$

Taking into account the definition of the Heaviside function, $H_\beta\colon [t_0, \infty) \to \mathbb{R}$, concentrated at $\beta \in [t_0, \infty)$,

$$H_\beta(t) := \begin{cases} 0, & t_0 \leqslant t \leqslant \beta, \\ 1, & t > \beta, \end{cases}$$

we can rewrite the previous initial value in the form

$$\begin{cases} y(t) = \phi(0) + \displaystyle\int_{t_0}^{t} a(t, s) f(y_s, s) \mathrm{d}g(s) + \sum_{k=1}^{m} H_{t_k}(t) I_k(y(t_k)) \\ y_{t_0} = \phi. \end{cases} \tag{6.0.4}$$

Next, we are going to present the first main result of this chapter.

Theorem 6.0.5 (Existence and Uniqueness)**.** Consider the impulsive functional Volterra-Stieltjes integral equation with infinite delay (6.0.4). Assume that the conditions (V1)-(V5) are satisfied. Also, suppose that $mL_2 < 1$, where L_2 is given in (V5). Under theses conditions there exits a unique solution $y\colon (-\infty, t_0+\sigma] \to \mathbb{R}^n$ of (6.0.4).

Proof. Define the operators $\mathcal{Q}_1, \mathcal{Q}_2 \colon \mathcal{S} \to \mathcal{S}$ by

$$(\mathcal{Q}_1 y)(t) := \begin{cases} 0, & t \in (-\infty, t_0), \\ \sum_{k=1}^{m} H_{t_k}(t) I_k(y(t_k)), & t \in [t_0, t_0+\sigma], \end{cases}$$

$$(\mathcal{Q}_2 y)(t) := \begin{cases} \phi(t-t_0), & t \in (-\infty, t_0), \\ \phi(0) + \int_{t_0}^{t} a(t,s) f(y_s, s) \mathrm{d}g(s), & t \in [t_0, t_0+\sigma] \end{cases}$$

for all $t \in [t_0, t_0+\sigma]$ and all $y \in \mathcal{S}$.

In view of the condition (V3), the integral on the right–hand side of definition of operator $\mathcal{Q}_2$ is well–defined.

Let us define the following constants:

$$c_0 := \sup_{(t,s)\in[t_0,t_0+\sigma]^2} |a(t,s)|, \quad \|\mu_2\|_0^\sigma := \sup_{\xi\in[t_0,t_0+\sigma]} \mu_2(\xi - t_0) \quad \text{and}$$

$$\|\mu_3\|_0^\sigma := \sup_{\xi\in[t_0,t_0+\sigma]} \mu_3(\xi - t_0).$$

Observe that these constants are well–defined in view of conditions (A2)-(iii), (V2) and Remark 6.0.4.

On the other hand, due to (6.0.2) and condition (A2)-(iii), we have that

$$\|y_t\|_{\mathcal{B}} \leqslant b_0 \|y\|_{\mathcal{S}}, \quad y \in \mathcal{S},\ t \in [t_0, t_0+\sigma], \tag{6.0.5}$$

where $b_0 := \max\{\|\mu_2\|_0^\sigma, \|\mu_3\|_0^\sigma\}$.

Claim 1. $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are well defined.

In fact, by (V2), the definition of $\mathcal{Q}_1$ and the fact that H_{t_k} is a regulated function, the limits $(\mathcal{Q}_1 y)(t+)$ for $t \in [t_0, t_0+\sigma)$ and $(\mathcal{Q}_1 u)(t-)$ for $t \in (t_0, t_0+\sigma]$ exist, and therefore, $(\mathcal{Q}_1 y)|_{[t_0,t_0+\sigma]}$ is regulated. Also, since $(\mathcal{Q}_1 y)_{t_0} \equiv 0$, we have $(\mathcal{Q}_1 y)_{t_0} \in \mathcal{B}$. Hence, $\mathcal{Q}_1 y \in \mathcal{S}$.

On the other hand, for $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0+\sigma$, by (V2), (V4) and properties of integral, we have

$$\begin{aligned}
&\|(\mathcal{Q}_2 y)(\tau_2) - (\mathcal{Q}_2 y)(\tau_1)\| \\
&= \left\| \int_{t_0}^{\tau_2} a(\tau_2, s) f(y_s, s)\, \mathrm{d}g(s) - \int_{t_0}^{\tau_1} a(\tau_1, s) f(y_s, s)\, \mathrm{d}g(s) \right\| \\
&\leqslant \left\| \int_{t_0}^{\tau_1} \left(a(\tau_2, s) - a(\tau_1, s)\right) f(y_s, s)\, \mathrm{d}g(s) \right\| \\
&\qquad\qquad + \left\| \int_{\tau_1}^{\tau_2} a(\tau_2, s) f(y_s, s)\, \mathrm{d}g(s) \right\| \\
&\leqslant \int_{t_0}^{\tau_1} |a(\tau_2, s) - a(\tau_1, s)|\, M(s)\, \mathrm{d}g(s) \\
&\qquad\qquad + \int_{\tau_1}^{\tau_2} |a(\tau_2, s)|\, M(s)\, \mathrm{d}g(s) \\
&\leqslant \int_{t_0}^{\tau_1} (a(\tau_2, s) - a(\tau_1, s)) M(s)\, \mathrm{d}g(s) + \int_{\tau_1}^{\tau_2} c_0 M(s)\, \mathrm{d}g(s) \\
&\leqslant \int_{t_0}^{t_0+\sigma} (a(\tau_2, s) - a(\tau_1, s)) M(s)\, \mathrm{d}g(s) + \int_{\tau_1}^{\tau_2} c_0 M(s)\, \mathrm{d}g(s) \\
&= v(\tau_2) - v(\tau_1),
\end{aligned}$$

where $v(t) := \int_{t_0}^{t_0+\sigma} a(t,s) M(s)\, \mathrm{d}g(s) + \int_{t_0}^{t} c_0 M(s)\, \mathrm{d}g(s)$, for all $t \in [t_0, t_0+\sigma]$. Now, it is easy to see that v is a regulated function on $[t_0, t_0+\sigma]$. Thus, by the Cauchy condition, it follow that $\mathcal{Q}_2 y$ is regulated on $[t_0, t_0+\sigma]$. Also, since $(\mathcal{Q}_2 y)_{t_0} = \phi \in \mathcal{B}$, we have $\mathcal{Q}_2 y \in \mathcal{S}$.

Claim 2. We will show that if $u, v \in B_{\mathcal{S}}(r)$, then $\mathcal{Q}_1 u + \mathcal{Q}_2 v \in B_{\mathcal{S}}(r)$, where

$$B_{\mathcal{S}}(r) := \{z \in \mathcal{S} : \|z\|_{\mathcal{S}} \leqslant r\} \quad \text{and} \quad r \geqslant \|\phi\|_{\mathcal{B}} + \|\phi(0)\| + \int_{t_0}^{t_0+\sigma} c_0 M(s)\, \mathrm{d}g(s) + m c_0 M_2.$$

In fact, given $t \in [t_0, t_0+\sigma]$, by (V4) and properties of integral, we obtain

$$\begin{aligned}
\|(\mathcal{Q}_1 u)(t) + (\mathcal{Q}_2 v)(t)\| &\leqslant \|\phi(0)\| + \left\| \int_{t_0}^{t} a(t,s) f(v_s, s)\, \mathrm{d}g(s) \right\| \\
&\qquad + \left\| \sum_{k=1}^{m} H_{t_k}(t) I_k(u(t_k)) \right\| \\
&\leqslant \|\phi(0)\| + \int_{t_0}^{t} |a(t,s)| M(s)\, \mathrm{d}g(s) + m M_2 \\
&\leqslant \|\phi(0)\| + \int_{t_0}^{t_0+\sigma} c_0 M(s)\, \mathrm{d}g(s) + m M_2,
\end{aligned}$$

and therefore,

$$\sup_{t \in [t_0, t_0+\sigma]} \|(\mathcal{Q}_1 u)(t) + (\mathcal{Q}_2 v)(t)\| \leqslant \|\phi(0)\| + \int_{t_0}^{t_0+\sigma} c_0 M(s)\, \mathrm{d}g(s) + m M_2.$$

Also,

$$\|(\mathcal{Q}_1 u + \mathcal{Q}_2 v)_{t_0}\|_{\mathcal{B}} = \|\phi\|_{\mathcal{B}}.$$

Therefore, combining these inequalities

$$\|\mathcal{Q}_1 u + \mathcal{Q}_2 v\|_{\mathcal{S}} = \|(\mathcal{Q}_1 u + \mathcal{Q}_2 v)_{t_0}\|_{\mathcal{B}} + \sup_{t \in [t_0, t_0+\sigma]} \|(\mathcal{Q}_1 u)(t) + (\mathcal{Q}_2 v)(t)\| \leqslant r,$$

that is, $\mathcal{Q}_1 u + \mathcal{Q}_2 v \in B_{\mathcal{S}}(r)$.

Claim 3. $\mathcal{Q}_1$ is a contraction on $B_{\mathcal{S}}(r)$.

Indeed, let $u, v \in B_{\mathcal{S}}(r)$. Notice that $\|(\mathcal{Q}_1 u - \mathcal{Q}_1 v)_{t_0}\|_{\mathcal{B}} = 0$. On the other hand, for $t \in [t_0, t_0+\sigma]$, by (V5), we have

$$\begin{aligned}
\|(\mathcal{Q}_1 u)(t) - (\mathcal{Q}_1 v)(t)\| &\leqslant \sum_{k=1}^{m} |H_{t_k}(t)| \, \|I_k(u(t_k)) - I_k(v(t_k))\| \\
&\leqslant \sum_{k=1}^{m} L_2 \|u(t_k) - v(t_k)\| \\
&\leqslant L_2 \sum_{k=1}^{m} \sup_{s\in[t_0,t_0+\sigma]} \|u(s) - v(s)\| \\
&\leqslant L_2 \sum_{k=1}^{m} (\|(u-v)_{t_0}\|_{\mathcal{B}} + \sup_{s\in[t_0,t_0+\sigma]} \|u(s) - v(s)\|) \\
&= L_2 \sum_{k=1}^{m} \|u - v\|_{\mathcal{S}} \\
&= m L_2 \|u - v\|_{\mathcal{S}}\,.
\end{aligned}$$

Thus,

$$\|\mathcal{Q}_1 u - \mathcal{Q}_1 v\|_{\mathcal{S}} = \|(\mathcal{Q}_1 u - \mathcal{Q}_1 v)_{t_0}\|_{\mathcal{B}} + \sup_{t\in[t_0,t_0+\sigma]} \|(\mathcal{Q}_1 u)(t) - (\mathcal{Q}_1 v)(t)\| \leqslant m L_2 \|u - v\|_{\mathcal{S}}\,.$$

Since by hypothesis $mL_2 < 1$, the claim follows.

Claim 4. $\mathcal{Q}_2$ is continuous on $B_{\mathcal{S}}(r)$.

In fact, let $u \in B_{\mathcal{S}}(r)$ and $\{u_n\}_{n\in\mathbb{N}} \subset B_{\mathcal{S}}(r)$ such that $u_n \overset{\|\cdot\|_{\mathcal{S}}}{\longrightarrow} u$ as $n \to \infty$. Then, for all $t \in [t_0, t_0+\sigma]$, by (6.0.5), (V5) and properties of integral, we have

$$\begin{aligned}
\|(\mathcal{Q}_2 u_n)(t) - (\mathcal{Q}_2 u)(t)\| &= \left\| \int_{t_0}^{t} a(t,s) \left(f((u_n)_s, s) - f(u_s, s) \right) \mathrm{d}g(s) \right\| \\
&\leqslant \int_{t_0}^{t} |a(t,s)| \, L(s) \, \|(u_n)_s - u_s\|_{\mathcal{B}} \, \mathrm{d}g(s) \\
&\leqslant c_0 b_0 \|u_n - u\|_{\mathcal{S}} \int_{t_0}^{t} L(s) \, \mathrm{d}g(s) \\
&\leqslant \left(c_0 b_0 \int_{t_0}^{t_0+\sigma} L(s) \, \mathrm{d}g(s) \right) \|u_n - u\|_{\mathcal{S}}\,.
\end{aligned}$$

In particular,

$$\sup_{t\in[t_0,t_0+\sigma]} \|(\mathcal{Q}_2 u_n)(t) - (\mathcal{Q}_2 u)(t)\| \leqslant \left(c_0 b_0 \int_{t_0}^{t_0+\sigma} L(s) \, \mathrm{d}g(s) \right) \|u_n - u\|_{\mathcal{S}}\,.$$

Also, $\|(\mathcal{Q}_2 u_n - \mathcal{Q}_2 u)_{t_0}\|_{\mathcal{B}} = 0$. Thus,

$$\begin{aligned}\|\mathcal{Q}_2 u_n - \mathcal{Q}_2 u\|_{\mathcal{S}} &= \|(\mathcal{Q}_2 u_n - \mathcal{Q}_2 u)_{t_0}\|_{\mathcal{B}} + \sup_{t\in[t_0,t_0+\sigma]} \|(\mathcal{Q}_2 u_n)(t) - (\mathcal{Q}_2 u)(t)\| \\ &\leqslant \left(c_0 b_0 \int_{t_0}^{t_0+\sigma} L(s)\,\mathrm{d}g(s)\right) \|u_n - u\|_{\mathcal{S}}\,.\end{aligned}$$

Since $\|u_n - u\|_{\mathcal{S}} \overset{n\to\infty}{\to} 0$, we have $\|(\mathcal{Q}_2 u_n) - (\mathcal{Q}_2 u)\|_{\mathcal{S}} \to 0$ as $n \to \infty$, proving the claim.

Step 5. $\mathcal{Q}_2$ is compact on $B_{\mathcal{S}}(r)$. We affirm that $\mathcal{Q}_2(B_{\mathcal{S}}(r)) = \{\mathcal{Q}_2 x : x \in B_{\mathcal{S}}(r)\}$ is relatively compact (with respect to the topology generated by $\|\cdot\|_{\mathcal{S}}$). Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{Q}_2(B_{\mathcal{S}}(r))$. Then there exists $(x_n)_{n\in\mathbb{N}} \subset B_{\mathcal{S}}(r)$ such that $y_n = \mathcal{Q}_2 x_n$ for all $n \in \mathbb{N}$. Now, consider $\mathcal{A} := \{y_n|_{[t_0,t_0+\sigma]}\}_{n\in\mathbb{N}}$. We show that $\mathcal{A}$ is relatively compact in $(G([t_0, t_0 + \sigma], \mathbb{R}^n), \|\cdot\|_\infty)$. In fact, using the same arguments as in the proof of Claim 1, we can prove that

$$\left\|y_n|_{[t_0,t_0+\sigma]}(\tau_2) - y_n|_{[t_0,t_0+\sigma]}(\tau_1)\right\| = \|(\mathcal{Q}_2 x_n)(\tau_2) - (\mathcal{Q}_2 x_n)(\tau_1)\| \leqslant v(\tau_2) - v(\tau_1),$$

for all $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0 + \sigma$, where $v(t) := \int_{t_0}^{t_0+\sigma} a(t,s)M(s)\,\mathrm{d}g(s) + \int_{t_0}^{t} c_0 M(s)\,\mathrm{d}g(s)$. It is easy see that v is a nondecreasing function. On the other hand, notice that the set $\{y_n|_{[t_0,t_0+\sigma]}(t_0) : n \in \mathbb{N}\} = \{(\mathcal{Q}_2 x_n)(t_0) : n \in \mathbb{N}\} = \{\phi(0)\}$ is bounded. Hence, we see that the assumption (ii) from Theorem 6.0.1 is satisfied and it follows that $\{y_n|_{[t_0,t_0+\sigma]}\}_{n\in\mathbb{N}}$ contains a subsequence convergent (with respect to the norm $\|\cdot\|_\infty$) which converges to a function $y_0 \in G([t_0, t_0+\sigma], \mathbb{R}^n)$. Without loss of generality, we can denote this subsequence again by $\{y_n|_{[t_0,t_0+\sigma]}\}_{n\in\mathbb{N}}$.

Define $z\colon (-\infty, t_0 + \sigma] \to \mathbb{R}^n$ given by

$$z(t) := \begin{cases} \phi(t - t_0), & t \in (-\infty, t_0) \\ y_0(t), & t \in [t_0, t_0 + \sigma]. \end{cases}$$

Notice that $z_{t_0} = \phi \in \mathcal{B}$ and $z|_{[t_0,t_0+\sigma]} = y_0$ is regulated, and therefore, $z \in \mathcal{S}$. Finally, since $(y_n - z)_{t_0} = 0_{\mathcal{B}}$ and $\sup_{t\in[t_0,t_0+\sigma]} \|y_n(t) - z(t)\| = \|y_n|_{[t_0,t_0+\sigma]} - y_0\|_\infty$, we have

$$\begin{aligned}\|y_n - z\|_{\mathcal{S}} &= \|(y_n - z)_{t_0}\|_{\mathcal{B}} + \sup_{t\in[t_0,t_0+\sigma]} \|y_n(t) - z(t)\| \\ &= \|y_n|_{[t_0,t_0+\sigma]} - y_0\|_\infty.\end{aligned}$$

Due to $\|y_n|_{[t_0,t_0+\sigma]} - y_0\|_\infty \overset{n\to\infty}{\to} 0$, we get $\|y_n - z\|_{\mathcal{S}} \to 0$ as $n \to \infty$.

Since all hypotheses from Theorem 6.0.2 are satisfied, there exists $x \in B_{\mathcal{S}}(r)$ such that $\mathcal{Q}_1 x + \mathcal{Q}_2 x = x$, which implies that x is a solution of the impulsive functional Volterra-Stieltjes integral equation with infinite delay (6.0.4).

The uniqueness of the solution remains to be demonstrated.

In fact, let $x, z\colon (-\infty, t_0+\sigma] \to \mathbb{R}^n$ be solutions of the impulsive functional Volterra-Stieltjes integral equation with infinite delay (6.0.4). Then, $x_{t_0} = z_{t_0} = \phi$, and therefore,

$$x(t) = z(t), \quad \text{for all } t \in (-\infty, t_0].$$

It only remains for us to verify that $x(t) = z(t)$ for $t \in [t_0, t_0+\sigma]$. Indeed, let $t \in [t_0, t_0+\sigma]$. For each $\tau \in [t_0, t]$, we have

$$\begin{aligned}
\|x(\tau) - z(\tau)\| &\leqslant \left\| \int_{t_0}^{\tau} a(\tau, s)[f(x_s, s) - f(z_s, s)]\, \mathrm{d}g(s) \right\| \\
&\qquad + \left\| \sum_{\substack{k \in \{1,\ldots,m\} \\ t_k < \tau}} [I_k(x(t_k)) - I_k(z(t_k))] \right\| \\
&\leqslant \int_{t_0}^{\tau} L(s)\, \|x_s - z_s\|_{\mathcal{B}}\, \mathrm{d}g(s) \\
&\qquad + \sum_{\substack{k \in \{1,\ldots,m\} \\ t_k < \tau}} L_2 \|x(t_k) - z(t_k)\| \\
&\leqslant \int_{t_0}^{\tau} c_0 \|L\|_\infty\, \|x_s - z_s\|_{\mathcal{B}}\, \mathrm{d}g(s) + mL_2 \sup_{\xi \in [t_0, \tau]} \|x(\xi) - z(\xi)\|.
\end{aligned} \tag{6.0.6}$$

Note that, according to condition (A2)-(iii) and the fact that $(x-z)_{t_0} = 0_{\mathcal{B}}$, we have

$$\begin{aligned}
\|x_s - z_s\|_{\mathcal{B}} &\leqslant \mu_2(s - t_0) \|(x-z)_{t_0}\|_{\mathcal{B}} + \mu_3(s - t_0) \sup_{\xi \in [t_0, s]} \|x(\xi) - z(\xi)\| \\
&\leqslant \|\mu_3\|_0^\sigma \sup_{\xi \in [t_0, s]} \|x(\xi) - z(\xi)\|, \quad s \in [t_0, t_0+\sigma].
\end{aligned}$$

In particular,

$$\|x_s - z_s\|_{\mathcal{B}} \leqslant \|\mu_3\|_0^\sigma \sup_{\xi \in [t_0, s]} \|x(\xi) - z(\xi)\|, \quad s \in [t_0, t_0+\sigma]. \tag{6.0.7}$$

Thus, by (6.0.6) and (6.0.7), we have

$$\|x(\tau) - z(\tau)\| \leqslant \int_{t_0}^{\tau} c_0\, \|L\|_\infty\, \|\mu_3\|_0^\sigma \sup_{\xi \in [t_0, s]} \|x(\xi) - z(\xi)\|\, \mathrm{d}g(s) + mL_2 \sup_{\xi \in [t_0, \tau]} \|x(\xi) - z(\xi)\|,$$

for all $\tau \in [t_0, t]$. Now, since the right hand side of the above inequality is non-decreasing, we obtain

$$\sup_{\tau \in [t_0, t]} \|x(\tau) - z(\tau)\| \leqslant \int_{t_0}^{t} c_0\, \|L\|_\infty\, \|\mu_3\|_0^\sigma \sup_{\xi \in [t_0, s]} \|x(\xi) - z(\xi)\|\, \mathrm{d}g(s) + mL_2 \sup_{\xi \in [t_0, t]} \|x(\xi) - z(\xi)\|.$$

This gives

$$\sup_{\tau \in [t_0, t]} \|x(\tau) - z(\tau)\| \leqslant \epsilon + \lambda \int_{t_0}^{t} \sup_{\xi \in [t_0, s]} \|y(\xi) - z(\xi)\|\, \mathrm{d}g(s),$$

for all $\epsilon > 0$, where $\lambda := \frac{\|L\|_\infty \|\mu_3\|_0^\sigma}{1 - mL_2}$. Note that λ is well-defined, since $mL_2 < 1$. Therefore, according to the Gronwall's inequality for the Henstock–Kurzweil integral, we get

$$\sup_{\tau \in [t_0, t]} \|x(\tau) - z(\tau)\| \leqslant \epsilon e^{\lambda(g(t) - g(t_0))}.$$

In particular,

$$\|x(t) - z(t)\| \leqslant \epsilon e^{\lambda(g(t_0+\sigma)-g(t_0))}.$$

Now, since $\epsilon > 0$ arbitrary, it follows that $x(t) = z(t)$ for all $t \in [t_0, t_0 + \sigma]$. □

Remark 6.0.6. Notice that, under conditions (V1)–(V5) and Theorem 6.0.5, for all $\phi \in \mathcal{B}$, there exists a unique solution $y\colon (-\infty, t_0 + \sigma] \to \mathbb{R}^n$ of the impulsive functional Volterra-Stieltjes integral equation with infinite delay (6.0.4). This solution will be denoted by $y(t, t_0, \phi)$. Also, for each $t \in [t_0, t_0 + \sigma]$, we use the notation $y_t(\cdot, t_0, \phi) := (y(\cdot, t_0, \phi))_t$ to indicate the element of $\mathcal{B}$ define by $y_t(\cdot, t_0, \phi)(\theta) := y(t + \theta, t_0, \phi)$ for all $\theta \in (-\infty, 0]$.

The following definition is inspired by (Santos 2015, Section 3).

Definition 6.0.7. We will say that the solutions of the impulsive functional Volterra-Stieltjes integral equation with infinite delay (6.0.4) depend continuously on the initial condition if given $\epsilon > 0$, there exists $\delta > 0$ such that if $\|\phi - \varphi\|_{\mathcal{B}} < \delta$, then

$$\|y(\cdot, t_0, \phi) - y(\cdot, t_0, \varphi)\|_{\mathcal{S}} < \epsilon.$$

In the following we present the second main result of the chapter.

Theorem 6.0.8 (Continuous dependence with respect to initial condition)**.** Assume that the conditions (V1)-(V5) are satisfied. Also, suppose that $mL_2 < 1$, where L_2 is given in (V5). Then, for each $\phi, \varphi \in \mathcal{B}$, we obtain

$$\|y(\cdot, t_0, \phi) - y(\cdot, t_0, \varphi)\|_{\mathcal{S}} \leqslant (1 + \mu_1(0))\|\phi - \varphi\|_{\mathcal{B}}\, e^{\lambda(g(t_0+\sigma)-g(t_0))}. \tag{6.0.8}$$

In particular, the solutions of the impulsive functional Volterra-Stieltjes integral equation with infinite delay (6.0.4) depend continuously on the initial condition. Here, $\lambda := \frac{\|L\|_\infty \|\mu_3\|_0^\sigma}{1-mL_2}$ and the function μ_1 is given in (A2)-(ii).

Proof. Let $\phi, \varphi \in \mathcal{B}$ be given. By definition of solution and the fact that $\lambda(g(t_0 + \sigma) - g(t_0)) \geqslant 0$, we obtain

$$\|y_{t_0}(\cdot, t_0, \phi) - y_{t_0}(\cdot, t_0, \varphi)\|_{\mathcal{B}} = \|\phi - \varphi\|_{\mathcal{B}} \leqslant \|\phi - \varphi\|_{\mathcal{B}}\, e^{\lambda(g(t_0+\sigma)-g(t_0))}. \tag{6.0.9}$$

On the other hand, using the same arguments as in the proof of the Theorem 6.0.5, we can prove that

$$\sup_{\tau\in[t_0,t]} \|y(\tau, t_0, \phi) - y(\tau, t_0, \varphi)\| \leqslant \|\phi(0) - \varphi(0)\| + \lambda \int_{t_0}^{t} \sup_{\xi\in[t_0,s]} \|y(\xi, t_0, \phi) - y(\xi, t_0, \varphi)\|\, \mathrm{d}g(s),$$

for all $t \in [t_0, t_0 + \sigma]$. Thus, according to the Gronwall's inequality for the Henstock–Kurzweil, we have

$$\sup_{\tau\in[t_0,t]} \|y(\tau, t_0, \phi) - y(\tau, t_0, \varphi)\| \leqslant \|\phi(0) - \varphi(0)\| e^{\lambda(g(t)-g(t_0))}.$$

In particular, choosing $t = t_0 + \sigma$ and taking into account condition (A2)-(ii), we get

$$
\begin{aligned}
&\sup_{\tau\in[t_0,t_0+\sigma]} \|y(\tau,t_0,\phi) - y(\tau,t_0,\varphi)\| \\
&\leqslant \|\phi(0) - \varphi(0)\| e^{\lambda(g(t_0+\sigma)-g(t_0))} \\
&= \|y(t_0,t_0,\phi) - y(t_0,t_0,\varphi)\| e^{\lambda(g(t_0+\sigma)-g(t_0))} \\
&\leqslant \mu_1(0)\|y_{t_0}(\cdot,t_0,\phi) - y_{t_0}(\cdot,t_0,\varphi)\|_{\mathcal{B}}\, e^{\lambda(g(t_0+\sigma)-g(t_0))} \\
&= \mu_1(0)\|\phi - \varphi\|_{\mathcal{B}}\, e^{\lambda(g(t_0+\sigma)-g(t_0))}.
\end{aligned} \tag{6.0.10}
$$

Now, by (6.0.9) and (6.0.10), we have

$$
\begin{aligned}
\|y(\cdot,t_0,\phi) - y(\cdot,t_0,\varphi)\|_{\mathcal{S}} &= \|y_{t_0}(\cdot,t_0,\phi) - y_{t_0}(\cdot,t_0,\varphi)\|_{\mathcal{B}} + \sup_{\tau\in[t_0,t_0+\sigma]} \|y(\tau,t_0,\phi) - y(\tau,t_0,\varphi)\| \\
&\leqslant (1+\mu_1(0))\|\phi - \varphi\|_{\mathcal{B}}\, e^{\lambda(g(t_0+\sigma)-g(t_0))},
\end{aligned}
$$

getting the desired result.

□

6.1 Applications

6.1.1 Dynamic equations on time scales

In this section, we are going to investigate the impulsive functional Volterra-Stieltjes integral equations on times scales with infinite delay. In particular, we explain the relation between this type of equations and impulsive functional Volterra-Stieltjes integral equations with infinite delay.

A *time scale* is a closed and nonempty subset $\mathbb{T}$ of $\mathbb{R}$. For $a, b \in \mathbb{T}$, $a \leq b$, we define the time scale interval by $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$. For $t \in \mathbb{T}$, we define the *forward jump operator* and the *backward jump operator* by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

respectively. We set that $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\sigma(t) = t$, we say that t is right-dense and if $\sigma(t) > t$ we say that t is a right-scattered point. Analogously, if $\rho(t) = t$, then t is called left-dense left-scattered, otherwise, t is said to be left-scattered.

We recall the reader that a function $f : \mathbb{T} \to \mathbb{R}^n$ is called rd-continuous, if it is regulated on $\mathbb{T}$ and continuous at right-dense points of $\mathbb{T}$.

In time scale calculus, the usual derivative $f'(t)$ and integral $\int_\alpha^\beta f(t)dt$ of a function $f : [\alpha,\beta] \to \mathbb{R}^n$ are replaced by the Δ-derivative $f^\Delta(t)$ and Δ-integral $\int_\alpha^\beta f(t)\Delta t$, where $f : [\alpha,\beta]_{\mathbb{T}} \to \mathbb{R}^n$. The definitions and basic properties on time scale, the Δ-derivative and Δ-integral, can be found in (Bohner and Peterson 2001, Bohner and Peterson 2003, Peterson 2006) and Chapter 2 this is work.

Given a real number $t \leqslant \sup \mathbb{T}$, define

$$t^* = \inf \{ s \in \mathbb{T} : s \geqslant t \} .$$

Since $\mathbb{T}$ is a closed set, we have $t^* \in \mathbb{T}$. Further, define the extension of $\mathbb{T}$ by

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty) & \text{otherwise.} \end{cases} \tag{6.1.1}$$

On the other hand, given a function $f\colon \mathbb{T} \to \mathbb{R}^n$, we define its extension $f^*\colon \mathbb{T}^* \to \mathbb{R}^n$ by

$$f^*(t) = f(t^*), \quad t \in \mathbb{T}^*.$$

Analogously, given a function $a\colon \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, consider its extension $a^{**}\colon \mathbb{T}^* \times \mathbb{T}^* \to \mathbb{R}$ by:

$$a^{**}(t,s) = a(t^*, s^*), \quad (t,s) \in \mathbb{T}^* \times \mathbb{T}^*.$$

Let $[t_0, t_0+\delta]_{\mathbb{T}}$ be a time scale interval. Let us consider the following moments of impulses $t_1, \dots, t_m \in \mathbb{T}$ with $t_0 \leqslant t_1 < t_2 < \ldots < t_m < t_0 + \delta$, and the impulse operators $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots, m$. Typically, an impulse condition is given by

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad \text{for } k = 1, \dots, m.$$

In the theory of time scales, the following assumptions are usually adopted, $y(t_k^+) = y(t)$ in the case in which $t \in \mathbb{T}$ is a right-scattered point and $y(t_k^-) = y(t)$ when $t \in \mathbb{T}$ is a right-scattered point.

Notice that if y is a left-continuous function and t_k is right-scattered, then $I_k(y(t_k^-)) = I_k(y(t_k)) = 0$. In this way, it makes sense to consider impulses at right-dense points only, in (Benchohra *et al.* 2004, Chang 2006) the authors assume the same conditions), which leads us to consider the following impulsive functional Volterra-Stieltjes integral equations on times scales with infinite delay:

$$\begin{cases} y(t) &= y(t_0) + \displaystyle\int_{t_0}^{t} a(t,s) f(y_s^*, s) \Delta s, \quad t \in [t_0, t_0+\delta]_{\mathbb{T}} \setminus \{t_1, \dots, t_m\} \\ \Delta^+ y(t_k) &= I_k(y(t_k)) \quad k = 1, \dots m, \\ y(t) &= \phi(t), \quad t \in (-\infty, t_0]_{\mathbb{T}}, \end{cases} \tag{6.1.2}$$

where $\phi \in G((-\infty, t_0], \mathbb{R}^n)$ is such that $\phi_{t_0}^* \in \mathcal{B}$, $a\colon [t_0, t_0+\delta]^2 \to \mathbb{R}$, $f\colon \mathcal{B} \times [t_0, t_0+\delta]_{\mathbb{T}} \to \mathbb{R}^n$, $t_1, \dots, t_m \in \mathbb{T}$ are right-dense points of impulse effects satisfying $t_0 \leqslant t_1 < t_2 < \ldots < t_m < t_0 + \delta$, $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots, m$, are the impulse operators, and the symbol y_s^* should be understood as $(y^*)_s$. Here we are assuming that the solution is left-continuous. It is not difficult to check that problem (6.1.2) can be rewritten in the form

$$\begin{cases} y(t) &= y(t_0) + \displaystyle\int_{t_0}^{t} a(t,s) f(y_s^*, s) \Delta s + \sum_{\substack{k \in \{1, \dots, m\} \\ t_k < t}} I_k(y(t_k)), \quad t \in [t_0, t_0+\delta]_{\mathbb{T}} \\ y(t) &= \phi(t), \quad t \in (-\infty, t_0]_{\mathbb{T}}, \end{cases} \tag{6.1.3}$$

The next result concerns a relation between solutions of a impulsive functional Volterra-Stieltjes integral equation with infinite delay and the solutions of a impulsive functional Volterra-Stieltjes integral equation on times scales with infinite delay.

Theorem 6.1.1. Let $\mathbb{T}$ be a time scale with $t_0, t_1, \ldots, t_m, t_0+\delta \in \mathbb{T}$ such that $t_0 \leqslant t_1 < t_2 < \ldots < t_m < t_0+\delta$. Let $\phi \in G((-\infty, t_0]_{\mathbb{T}}, \mathbb{R}^n)$ such that $\phi^*_{t_0} \in \mathcal{B}$, $f\colon \mathcal{B}\times[t_0, t_0+\delta]_{\mathbb{T}} \to \mathbb{R}^n$, $a\colon [t_0, t_0+\delta]^2_{\mathbb{T}} \to \mathbb{R}$ and $g(s) := s^*$ for all $s \in [t_0, t_0+\delta]$ and $I_1, \ldots, I_m\colon \mathbb{R}^n \to \mathbb{R}^n$. If $x\colon (-\infty, t_0+\delta]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of the impulsive functional Volterra-Stieltjes integral equation on times scales with infinite delay

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^{t} a(t,s) f(x^*_s, s)\Delta s + \sum_{\substack{k\in\{1,\ldots,m\} \\ t_k<t}} I_k(x(t_k)), \quad t \in [t_0, t_0+\sigma]_{\mathbb{T}} \\ x(t) = \phi(t), \quad t \in (-\infty, t_0]_{\mathbb{T}}, \end{cases} \tag{6.1.4}$$

then $x^*\colon (-\infty, t_0+\delta] \to \mathbb{R}^n$ is a solution of the impulsive functional Volterra-Stieltjes integral equation with infinite delay

$$\begin{cases} y(t) = y(t_0) + \int_{t_0}^{t} a^{**}(t,s) f^*(y_s, s)\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\} \\ t_k<t}} I_k(y(t_k)), \quad t \in [t_0, t_0+\sigma] \\ y_{t_0} = \phi^*_{t_0}. \end{cases} \tag{6.1.5}$$

Conversely, if $y\colon (-\infty, t_0+\delta] \to \mathbb{R}^n$ is a solution of (6.1.5), then it must have the form $y = x^*$, where $x\colon (-\infty, t_0+\delta]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of (6.1.4).

Proof. Assume that $x\colon (-\infty, t_0+\delta]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution of the functional Volterra-Stieltjes integral equation on time scales with infinite delay (6.1.4). Then

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} a(t,s) f(x^*_s, s)\Delta s + \sum_{\substack{k\in\{1,\ldots,m\} \\ t_k<t}} I_k(x(t_k)) \\ &= x^*(t_0) + \int_{t_0}^{t} a(t^*,s) f(x^*_s, s)\Delta s + \sum_{\substack{k\in\{1,\ldots,m\} \\ t_k<t}} I_k(x(t_k)), \quad t \in [t_0, t_0+\delta]_{\mathbb{T}}. \end{aligned}$$

According to (Federson, Mesquita and Slavík 2013, Theorem 4.5), we have

$$x^*(t) = x^*(t_0) + \int_{t_0}^{t} a(t^*, s^*) f(x^*_{s^*}, s^*)\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\} \\ t_k<t^*}} I_k(x(t_k)), \quad t \in [t_0, t_0+\delta].$$

Notice that $x^*(t_k) = x(t_k)$ for $k = 1, \ldots, m$ (since $t_k \in \mathbb{T}$), and $t_k < t^*$ if and only if $t_k < t$. Also, observe that $a(t^*, s) f^*(x^*_{s^*}, s^*) = a(t^*, s) f^*(x^*_s, s^*)$ for all $s \in [t_0, t] \cap \mathbb{T}$. Hence, by (Federson, Mesquita and Slavík 2013, Theorem 5.1), we have

$$\begin{aligned} x^*(t) &= x^*(t_0) + \int_{t_0}^{t} a(t^*, s^*) f(x^*_s, s^*)\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\} \\ t_k<t}} I_k(x^*(t_k)) \\ &= x^*(t_0) + \int_{t_0}^{t} a^{**}(t,s) f^*(x^*_s, s)\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\} \\ t_k<t}} I_k(x^*(t_k)), \quad t \in [t_0, t_0+\delta]. \end{aligned}$$

Moreover, for each $\theta \in (-\infty, 0]$, we have

$$x^*_{t_0}(\theta) = x^*(t_0+\theta) = x((t_0+\theta)^*) = \phi((t_0+\theta)^*) = \phi^*_{t_0},$$

which shows that x^* is a solution of (6.1.5).

Now, assume that $y\colon (-\infty, t_0+\delta] \to \mathbb{R}^n$ is a solution of (6.1.5). Note that if $t \in \mathbb{T}$, $t_0 < \tau < t \leqslant t_0+\delta$ and $[\tau, t) \cap \mathbb{T} = \emptyset$, then $g|_{[\tau,t]}$ is constant and, therefore, $y(\tau) = y(t)$. Hence, $y = x^*$, for some $x\colon (-\infty, t_0+\delta]_{\mathbb{T}} \to \mathbb{R}^n$. Using the same arguments as in the previous part, we can show that x is a solution of the functional Volterra-Stieltjes integral equation on time scales with infinite delay (6.1.4). □

We assume the following conditions

(S1) The function $a\colon [t_0, t_0+\delta]^2_{\mathbb{T}} \to \mathbb{R}$ is nondecreasing with respect to the first variable, regulated with respect to the second variable and rd–continuous with respect to the first variable.

(S2) The Henstock–Kurzweil Δ–integral

$$\int_{t_0}^{t_0+\delta} a(\tau, s) f(x_s, s)\Delta s$$

exists for each $x \in \mathcal{S}$.

(S3) There exists a Henstock–Kurzweil Δ–integrable function $M\colon [t_0, t_0+\delta]_{\mathbb{T}} \to \mathbb{R}^+$ such that

$$\left\| \int_{s_1}^{s_2} (c_1 a(s_2, s) + c_2 a(s_1, s)) f(x_s, s)\Delta s \right\| \leqslant \int_{s_1}^{s_2} M(s) \left| c_1 a(s_2, s) + c_2 a(s_1, s) \right| \Delta s,$$

for all $x \in \mathcal{S}$, $c_1, c_2 \in \mathbb{R}$, $s_1, s_2 \in [t_0, t_0+\delta]_{\mathbb{T}}$, $s_1 \leqslant s_2$, and there exists a constant $M_2 > 0$ such that

$$\|I_k(x)\| \leqslant M_2,$$

for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}^n$.

(S4) There exists a regulated function $L\colon [t_0, t_0+\delta]_{\mathbb{T}} \to \mathbb{R}^+$ such that

$$\left\| \int_{s_1}^{s_2} a(s_2, s)[f(x_s, s) - f(z_s, s)]\Delta s \right\| \leqslant \int_{s_1}^{s_2} L(s) \left| a(s_2, s) \right| \|x_s - z_s\|_\infty \Delta s,$$

for all $x, z \in \mathcal{S}$, $s_1, s_2 \in [t_0, t_0+\delta]_{\mathbb{T}}$, $s_1 \leqslant s_2$, and there exists a constant $L_2 > 0$ such that

$$\|I_k(x) - I_k(y)\| \leqslant L_2 \|x - y\|, \tag{6.1.6}$$

for all $k \in \mathbb{N}$ and all x, $y \in \mathbb{R}^n$.

The following result is a combination of two results. The first one can be found in (Bonotto, Federson and Mesquita 2021), see Corollary 3.21. The second part is inspired by (Lafetá 2022, Lemma 2.2.2). We omit its proof here, since it is similar to the proof presented in (Lafetá 2022).

Lemma 6.1.2. Let $[t_0, t_0+\delta]_{\mathbb{T}}$ be a time scale interval, $t_1, t_2, \ldots, t_m \in [t_0, t_0+\delta]$ are right-dense points such that $t_0 \leqslant t_1 < t_2 < \ldots < t_m < t_0+\delta$, and $f\colon \mathcal{B} \times [t_0, t_0+\delta]_{\mathbb{T}} \to \mathbb{R}^n$ be a arbitrary function. Consider $g(t) = t^*$, $f^*(\varphi, t) = f(\varphi, t^*)$ and $a^{**}(t,s) = a(t,s)$ for all $\varphi \in \mathcal{B}$ and $t, s \in [t_0, t_0+\delta]$.

1. g is nondecreasing, left–continuous on $(t_0, t_0+\delta]$, and continuous at $t_1, \ldots, t_m$.

2. If $f, a, \{I_k\}_{k=1}^m$ satisfies conditions (S1)-(S4), then $f^*, a^{**}, g, \{I_k\}_{k=1}^m$ satisfies conditions (V2)-(V5) with f^*, a^{**} respectively in the place of f, a.

Next, our goal is to prove the analogue of Theorem 6.0.5 for the following impulsive functional Volterra-Stieltjes integral equations on times scales with infinite delay.

Theorem 6.1.3. Consider the impulsive functional Volterra-Stieltjes integral equation on times scales with infinite delay (6.1.3). Assume that the conditions (S1)–(S4) hold. Suppose that $mL_2 < 1$. Then there exists a function $x\colon (-\infty, t_0+\delta]_{\mathbb{T}} \to \mathbb{R}^n$ which is the unique solution of (6.1.3).

Proof. Consider the impulsive functional Volterra-Stieltjes integral equation with infinite delay:

$$\begin{cases} y(t) &= y(t_0) + \displaystyle\int_{t_0}^{t} a^{**}(t,s) f^*(y_s, s)\,\mathrm{d}g(s) + \sum_{\substack{k\in\{1,\ldots,m\}\\ t_k<t}} I_k(y(t_k)), \quad t \in [t_0, t_0+\sigma] \\ y_{t_0} &= \phi^*_{t_0}, \end{cases} \tag{6.1.7}$$

where $g(s) = s^*$ for $s \in [t_0, t_0+\delta]$, $f^*(\psi, s) = f(\psi, s^*)$ for $s \in [t_0, t_0+\delta]$, $\psi \in \mathcal{B}$ and $a^{**}(t,s) = a(t^*, s^*)$ for $t, s \in [t_0, t_0+\delta]$. By the hypotheses and Lemma 6.1.2, we have that f^*, a^{**}, $\phi^*_{t_0}$ and g satisfy all conditions of Theorem 6.0.5. Hence, there exists a unique solution $y\colon (-\infty, t_0+\delta] \to \mathbb{R}^n$ of the impulsive functional Volterra-Stieltjes integral equation with infinite delay (6.1.7).

Now, by Theorem 6.1.1, y must have the form $y = x^*$, where $x\colon (-\infty, t_0+\delta] \to \mathbb{R}^n$ is a solution of the impulsive functional Volterra-Stieltjes integral equation on times scales with infinite delay (6.1.3). Again, by Theorem 6.1.1, the solution x is unique.

□

6.1.2 A class of impulsive functional fractional integral equations

Consider the impulsive functional fractional integral equations with infinite delay:

$$\begin{cases} y(t) = \varphi(0) + \int_{t_0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathcal{G}(y_s, s)\,\mathrm{d}s + \sum\limits_{\substack{k\in\{1,\dots,m\}\\ t_k<t}} I_k(y(t_k)), \quad t \in [t_0, t_0+\sigma], \\ y(t) = \varphi(t-t_0), \quad t \in (-\infty, t_0], \end{cases} \tag{6.1.8}$$

where Γ is the gamma function, $1 < \beta < 2$, where $t_0 \in \mathbb{R}$, $\sigma > 0$, $t_0 \leqslant t_1 < t_2 < \ldots < t_m < t_0 + \sigma$, $\varphi \in \mathcal{B}$, $I_k \colon \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \dots, m$ and $\mathcal{G} \colon \mathcal{B} \times [t_0, t_0+\sigma] \to \mathbb{R}^n$. Here, $\mathcal{B}$ is phase space as in Section 6.0.1.

We recall the reader that the space $\mathcal{S}$ is given by (6.0.1).

We assume the following conditions:

(P1) The Henstock–Kurzweil–Stieltjes integral

$$\int_{t_0}^{t_0+\sigma} a(t,s) f(y_s, s) \mathrm{d}g(s)$$

exists for all $y \in \mathcal{S}$ and all $t \in [t_0, t_0+\sigma]$.

(P2) There exists a Henstock–Kurzweil integrable function $M \colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left| \int_{\tau_1}^{\tau_2} c \frac{(\tau_2 - s)^{\beta-1}}{\Gamma(\beta)} \mathcal{G}(y_s, s) \mathrm{d}s \right| \leqslant \int_{\tau_1}^{\tau_2} |c| \, \frac{(\tau_2 - s)^{\beta-1}}{\Gamma(\beta)} M(s) \mathrm{d}s,$$

for all $y \in \mathcal{S}$, $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0 + \sigma$, and all $c \in \mathbb{R}$, and there exists a constant $M_2 > 0$ such that

$$\|I_k(x)\| \leqslant M_2,$$

for all $k \in \{1, \dots, m\}$ and all $x \in \mathbb{R}^n$.

(P3) There exists a regulated function $L \colon [t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left| \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\beta-1}}{\Gamma(\beta)} [\mathcal{G}(y_s, s) - \mathcal{G}(z_s, s)] \mathrm{d}s \right| \leqslant \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\beta-1}}{\Gamma(\beta)} L(s) \|y_s - z_s\|_\infty \mathrm{d}s,$$

for all $y, z \in \mathcal{S}$ and all $t_0 \leqslant \tau_1 \leqslant \tau_2 \leqslant t_0 + \sigma$, and there exists a constant $L_2 > 0$ such that

$$\|I_k(x) - I_k(y)\| \leqslant L_2 \|x - y\|, \tag{6.1.9}$$

for all $k = 1, \dots, m$, and all x, $y \in \mathbb{R}^n$.

Consider $g : [t_0, t_0+\sigma] \to \mathbb{R}$ and $a \colon [t_0, t_0+\sigma]^2 \to \mathbb{R}$ given by $g(s) = s$ and

$$a(t,s) = \begin{cases} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}, & t \geqslant s, \\ 0, & t < s. \end{cases},$$

we have that (6.1.8) is in the form of (6.0.4).

By definition g is nondecreasing, left–continuous on $(t_0, t_0 + \sigma]$, and continuous at $t_1, \ldots, t_m$, obtaining condition (V1). On the other hand, by simple calculations we can show that $a(\cdot, s)$ is nondecreasing for all $s \in [t_0, t_0 + \sigma]$, $a(t, \cdot)$ is regulated for all $t \in [t_0, t_0 + \sigma]$, and continuous with respect to first variable at $t_1, \ldots, t_m$, getting (V2). Moreover, due to conditions (P1)–(P3), f satisfies (V3)–(V5). Assuming further that, $mL_2 < 1$, we have that all hypotheses of Theorem 6.0.5 are satisfied, and therefore, there exist a unique solution $y\colon (-\infty, t_0 + \sigma] \to \mathbb{R}^n$ of the impulsive functional fractional integral equations with infinite delay (6.1.8).

6.1.3 Maxwell Fluid model

We consider the class of kernel defined by $a(t, s) = k(t - s)$, with $k(t) = \nu(1 - e^{-\frac{\mu t}{\nu}})$, where the parameter μ, ν are real positive numbers. This class of kernels naturally appears in the theory of integral equations of convolution type, more precisely, in Maxwell Fluid models, see (Prüss 1993, Chapter 5, p. 130) with several applications. It is known that a Maxwell fluid can be viewed as a spring and a dashpot in series (see Figure 9).

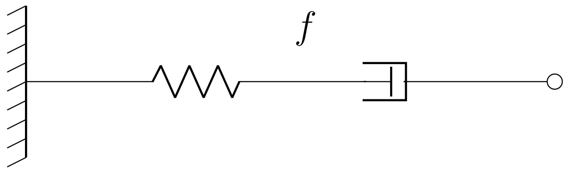


Figure 9 – Maxwell fluid

We consider the phase space $\mathcal{B} := BG((-\infty, 0], \mathbb{R})$ endowed with the norm

$$\|\varphi\|_{\mathcal{B}} := \sup_{t \in (-\infty, 0]} |\varphi(t)|, \qquad \varphi \in \mathcal{B}.$$

Here, $BG((-\infty, 0], \mathbb{R})$ denote the space of all regulated and bounded functions from $(-\infty, 0]$ to $\mathbb{R}$. It is not difficult to verify that conditions (A1)-(A2) from Section 6.0.1 are satisfied with $\mu_i(t) := 1$, for all $t \geqslant 0$ and $i = 1, 2, 3$. It is clear also that $\|S(t)\varphi\|_{\mathcal{B}} = \|\varphi\|_{\mathcal{B}}$, and therefore, (A3) is satisfied by setting $k(t) = 0$ for $t \geqslant 0$.

We consider the space:

$$\mathcal{S} := \{y : (-\infty, \sigma] \to \mathbb{R} : y_0 \in \mathcal{B} \text{ and } y|_{[0,\sigma]} \text{ is regulated}\},$$

where $\sigma > 0$.

Now, we consider as an external nonlinear force the function

$$f(\varphi, s) := 2\gamma_0 e^{-\beta\|\varphi\|_{\mathcal{B}}} \sin(x(s))$$

for all $\varphi \in \mathcal{B}$ and all $s \in [0, \sigma]$, where $\beta, \gamma_0 > 0$.

Consider the following integral equation

$$\begin{cases} y(t) = \phi(0) + \int_0^t \nu(1 - e^{-\frac{\mu(t-s)}{\nu}}) 2\gamma e^{-\beta\|y_s\|_{\mathcal{B}}} \sin(y(s))\, \mathrm{d}g(s) + \sum_{\substack{k\in\{1,\dots,m\}\\ t_k<t}} I_k(y(t_k)), \quad t \in [0,\sigma] \\ y(t) = \phi(t), \quad t \in (-\infty, 0], \end{cases} \tag{6.1.10}$$

where $\phi \in \mathcal{B}$, $0 \leqslant t_1 < t_2 < \ldots < t_m < \sigma$ are the moments of impulses, $g\colon [0,\sigma] \to \mathbb{R}$ is a nondecreasing, left–continuous function and continuous at $t_1, t_2, \dots, t_m$, and $I_1, \dots, I_m \colon \mathbb{R}^n \to \mathbb{R}^n$ are the impulse operators.

We assume the following conditions with respect to the impulse operators.

($\mathcal{I}_1$) there exists a constant $M_2 > 0$ such that

$$\|I_k(x)\| \leqslant M_2,$$

for all $k = 1, \dots, m$, and all $x \in \mathbb{R}^n$.

($\mathcal{I}_2$) there exists a constant $L_2 > 0$ such that

$$\|I_k(x) - I_k(y)\| \leqslant L_2 \|x - y\|, \tag{6.1.11}$$

for all $k = 1, \dots, m$, and all x, $y \in \mathbb{R}^n$.

By setting $a\colon [0,\sigma]^2 \to \mathbb{R}$ by

$$a(t,s) := \begin{cases} \nu(1 - e^{-\frac{\mu(t-s)}{\nu}}), & \text{if } s \leqslant t \\ 0, & \text{if } t < s, \end{cases}$$

we have that (6.1.10) is in the form of (6.0.4).

It is not difficult to verify that conditions (V1)–(V3) from Theorem 6.0.5 are satisfied.

On the other hand, for $y \in \mathcal{S}$, $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \sigma$, $c_1, c_2 \in \mathbb{R}$, $c_{\tau_2,\tau_1}(s) := c_1 a(\tau_2, s) + c_2 a(\tau_1, s)$, and properties of the integral, we get

$$\left| \int_{\tau_1}^{\tau_2} c_{\tau_2,\tau_1}(s) f(x_s, s)\, \mathrm{d}g(s) \right| \leqslant \int_{\tau_1}^{\tau_2} |c_{\tau_2,\tau_1}(s)|\, 2\gamma_0\, \mathrm{d}g(s),$$

obtaining that (V4) is satisfied by setting $M(s) := 2\gamma_0$.

In order to prove (V5), define $L(s) := 2\gamma_0\,(1+\beta)$ for $s \in [0,\sigma]$. It is clear that L is a regulated function and for $x, y \in \mathcal{S}$ and $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \sigma$, we have

$$\left|\int_{\tau_1}^{\tau_2} a(\tau_2,s)\,[f(x_s,s)-f(y_s,s)]\,\mathrm{d}g(s)\right| \leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,|f(x_s,s)-f(y_s,s)|\,\mathrm{d}g(s)$$

$$\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,2\gamma_0\left(e^{-\beta\|x_s\|_{\mathcal{B}}}\,|\sin(x(s))-\sin(y(s))| + |\sin(x(s))|\left|e^{-\beta\|x_s\|_{\mathcal{B}}} - e^{-\beta\|y_s\|_{\mathcal{B}}}\right|\right)\mathrm{d}g(s)$$

$$\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,2\gamma_0\,(|x(s)-y(s)| + \beta\,|\|x_s\|_{\mathcal{B}} - \|y_s\|_{\mathcal{B}}|)\,\mathrm{d}g(s)$$

$$= \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,2\gamma_0\,(|x_s(0)-y_s(0)| + \beta\,|\|x_s\|_{\mathcal{B}} - \|y_s\|_{\mathcal{B}}|)\,\mathrm{d}g(s)$$

$$\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,2\gamma_0\left(\sup_{\xi\in(-\infty,0]} |x_s(\xi)-y_s(\xi)| + \beta\,|\|x_s\|_{\mathcal{B}} - \|y_s\|_{\mathcal{B}}|\right)\mathrm{d}g(s)$$

$$= \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,2\gamma_0\,(\|x_s-y_s\|_{\mathcal{B}} + \beta\,|\|x_s\|_{\mathcal{B}} - \|y_s\|_{\mathcal{B}}|)\,\mathrm{d}g(s)$$

$$\leqslant \int_{\tau_1}^{\tau_2} |a(\tau_2,s)|\,2\gamma_0\,(1+\beta)\,\|x_s-y_s\|_{\mathcal{B}}\mathrm{d}g(s),$$

obtaining (V5). Assuming further that, $mL_2 < 1$, we have that all the hypotheses of Theorem 6.0.5 are satisfied. Therefore, the equation has a unique solution.

CHAPTER 7

CONCLUSION AND OPEN PROBLEMS

This chapter aims to encourage further study of the Volterra–Stieltjes functional equation considered in this work. A substantial body of theory has already been developed, and for readers who are interested in pursuing this topic further, we present a set of open problems related to this equation.

Our goal is to stimulate future research through a solid theoretical framework that allows these problems to be formulated and analyzed in a clear and rigorous way. Since the equation is considered here in its most general form, interested readers may, for instance, specialize it by assuming that the integral is of Riemann or Lebesgue type, or by requiring the solutions to be continuous instead of regulated.

7.1 Open problems

Regarding the functional Volterra–Stieltjes integral equation of type

$$\begin{cases} x(t) = \phi(0) + \displaystyle\int_{\tau_0}^{t} a(t,s) f(x_s, s)\, \mathrm{d}g(s), & t \geqslant \tau_0 \\ x_{\tau_0} = \phi, \end{cases}$$

where $\tau_0 \geqslant t_0$, $f\colon G([-r,0],\mathbb{R}^n) \times [t_0,+\infty) \to \mathbb{R}^n$, $g\colon [t_0,+\infty) \to \mathbb{R}$, $a\colon [t_0,+\infty)^2 \to \mathbb{R}$, $\phi \in G([-r,0],\mathbb{R}^n)$ and $r > 0$, have not yet been studied.

We list below several open problems related to equation:

1. Existence and uniqueness of (ω, Q)-affine-periodic solutions, where $\omega > 0$ and $Q \in GL_n(\mathbb{R})$.

2. Existence of a bifurcation point with respect to the trivial solution of a boundary value problem for equation, depending upon a parameter.

3. Criteria for the existence of oscillatory and nonoscillatory solutions of the functional Volterra-Stieltjes integral equation.

4. Criteria for permanence of solutions of the Volterra-Stieltjes functional integral equation.

5. Existence and uniqueness of an almost automorphic solution.

6. Existence of global attractors and extreme stability.

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