Universidade de Brasília

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Departamento de Matemática

Self-similar Representations

of Abelian Groups

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Self-similar Representations of Abelian Groups

por

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Severino, retirante, deixe agora que lhe diga:

eu não sei bem a resposta da pergunta que fazia,

se não vale mais saltar fora da ponte e da vida;

nem conheço essa resposta, se quer mesmo que lhe diga

é difícil defender, só com palavras, a vida,

ainda mais quando ela é esta que vê, Severina

mas se responder não pude à pergunta que fazia,

ela, a vida, a respondeu com sua presença viva.

E não há melhor resposta que o espetáculo da vida:

vê-la desfiar seu fio, que também se chama vida,

ver a fábrica que ela mesma, teimosamente, se fabrica,

vê-la brotar como há pouco em nova vida explodida;

mesmo quando é assim pequena a explosão, como a ocorrida;

como a de há pouco, franzina;

mesmo quando é a explosão

de uma vida Severina.

João Cabral de Melo Neto; Morte e Vida Severina

À memória do meu pai, Silas Cardoso Silva

Abstract

In this work we discuss results concerning faithful self-similar representations of abelian groups as automorphisms of one-rooted trees. In particular, we show the self-similarity of the free abelian group of uncountable rank and of the cartesian product of copies of the integers, the so-called *Baer-Specker group*. Furthermore, we prove that this group does not admit any transitive self-similar representation.

Resumo

Título: Representações Autossimilares de Grupos Abelianos

Neste trabalho discutimos resultados a respeito de representações fiéis autossimilares de grupos abelianos como grupos de automorfismos de árvores unirraiz. Em particular, mostramos a autossimilaridade do grupo abeliano livre de posto não-enumerável e do produto cartesiano de cópias dos inteiros, o chamado *grupo de Baer-Specker*. Ainda, mostramos que esse grupo não admite nenhuma representação autossimilar *transitiva*.

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Milton e Lô, de fato os sonhos não envelhecem.

Contents

Introduction

The paper by Brunner and Sidki [BS10] initiated a systematic study of abelian self-similar groups. In that paper they proved that if A is a transitive self-similar abelian group, then $tor(A)$ is a direct summand of A; also, they explicit transitive self-similar representations for $tor(A)$ and $A/tor(A)$. Additionaly, studying the diagonal-topological closure A^*, they determined important properties when A^* is torsion-free, in which case it can be seen as an $\mathbb{Z}_m$-module and as a pro-m group.

Subsequently, Dantas and Sidki [DS18], considering self-similar wreath products of abelian groups proved that if A and B are abelian, $G = A \wr B$, B is torsion-free, then A is torsion of finite exponent. In particular, the group $\mathbb{Z} \wr \mathbb{Z}$ cannot be transitive self-similar. Also, for a direct sum $G^{(\omega)}$ of an infinite countable number of copies of a self-similar abelian group, $G^{(\omega)} \wr C_2$ is also self-similar.

In [BS20], it was investigated direct sums of self-similar groups, proving in particular that $\mathbb{Z}^{(\omega)}$ is transitive self-similar; however, there is not any finite-state such action. The authors asked if the *Baer-Specker group* $\mathbb{Z}^\omega$, which is the cartesian product of copies of the integers, is self-similar. This question was answered positively in our paper [DLZ24].

It was defined in [DSS21] the notion of *intransitive self-similar groups*, where more than one virtual endomorphism is necessary to represent a given group in the group of automorphisms of a one-rooted tree $\mathcal{A}_m$, for some m; equivalently, this is the case when the induced group of permutations acting on the first level is not transitive. It was proved that, for a self-similar group G of degree m and orbit-type $(m_1, \ldots, m_s)$, $G^{(\omega)}$ admits a self-similar representation of degree $m + 1$ and orbit-type $(m_1, \ldots, m_s, 1)$; in particular, for $G = \mathbb{Z}$, the representation of $\mathbb{Z}^{(\omega)}$ is of orbit-type $(2, 1)$ and is in addition finite-state.

An important example of intransitive self-similar group in this paper is the group $\mathbb{Z} \wr \mathbb{Z}$, which happens to be a 3-state 3-letter automata group.

The *Baer-Specker group* $\mathbb{Z}^\omega$ is of great interest to topology and to the theory of abelian groups. To underline its importance we cite some of its intriguing and unexpected properties:

(i) It has no basis, that is, it is not free abelian; [Bae37]

(ii) Every countable subgroup is free abelian (called $\aleph_1$-free); [Spe50]

(iii) It contains a pure subgroup isomorphic to the direct sum of $2^{\aleph_0}$ copies of itself; [BI05]

(iv) There are $2^{2^{\aleph_0}}$ basic subgroups in $\mathbb{Z}^\omega$; [BI01]

(v) The quotient $\mathbb{Z}^\omega / \mathbb{Z}^{(\omega)}$ is of the form

$$\mathbb{Q}^{(2^{\aleph_0})} \oplus \prod_p \widehat{\mathbb{Z}_p^{(2^{\aleph_0})}},$$

where $\widehat{\mathbb{Z}_p^{(2^{\aleph_0})}}$ denotes the p-adic completion of $\mathbb{Z}_p^{(2^{\aleph_0})} = \bigoplus_{2^{\aleph_0}} \mathbb{Z}_p$. Also, this quotient is algebraically compact. [GKW04]

Our first result is an answer to a question proposed in [DSS23], concerning the free abelian group of uncountable rank:

Theorem A*: The uncountable free abelian group admits a faithful transitive self-similar representation.*

We use for this proof the idea in [BS20] of choosing a proper transcendental number $\eta \in 2\mathbb{Z}_2^\times$ and write the elements of $\mathbb{Z}_2$ in η-basis. Taking a maximal set of algebraically independent elements in $\mathbb{Z}_2 \setminus \bar{\mathbb{Q}}$, we obtain a basis for our uncountable free abelian group. Then we apply the so-called *shift operator*, here manifested as $a \mapsto a/\eta$, and obtain a trivial f-core. Also its important to notice that this group appears as a subgroup of the Baer-Specker group as the set of bounded integer sequences

$$S = \{ a = (a_n)_{n \in \mathbb{N}} \in \mathbb{Z}^\omega \mid |a_n| \leq N_a, N_a \in \mathbb{N}, \forall n \in \mathbb{N} \}.$$

In Chapter 2 we cite references and make some comments about this fact. Also in Chapter 2 we present a concrete transitive representation of the infinite countable free abelian group using the so-called *Thue-Morse sequence* $(t_n)_{n\in\mathbb{N}}$ and the property that $\tau \in \mathbb{Z}_2$ such that $\tau = \sum_{n\in\mathbb{N}} t_n 2^n$ is transcendental.

The next topic of our work are cartesian products of self-similar groups, obtaining as consequence the self-similarity of the Baer-Specker group:

Theorem B: *If G is a self-similar group of degree m, then the unrestricted direct product $G^\omega = \prod_{n\in\mathbb{N}} G$ admits a faithful self-similar representation of degree $m+1$. For $G = \mathbb{Z}$, there is a faithful self-similar representation of the Baer-Specker group $\mathbb{Z}^\omega$ of degree 3.*

Using the ideas from [DSS21], we obtain that the degree of G^ω increases just by one; we use additionally the shift operator. A natural question to ask is if this group is *transitive* self-similar.

However, we prove it negatively; $\mathbb{Z}^\omega$ does not admit any transitive self-similar representation, thus *intransitive*. Preceding the main proof, we prove two results of having its own interest:

Theorem C: *The Baer-Specker group $\mathbb{Z}^\omega$ does not embed in $\mathbb{Z}_p$, for any prime p.*

The proof of the above theorem has a number-theoretic flavor, whereas the next one is of topological nature:

Theorem D: *There is no embedding of the Baer-Specker group in a finite direct product $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$, where $p_i, 1 \leq i \leq n$ are prime numbers.*

The idea involved here is, if such embedding exists, to induce a topology in $\mathbb{Z}^\omega \leq \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ such that the kernel of the projection to $\mathbb{Z}_{p_1}$ restricted to $\mathbb{Z}^\omega$ is closed. Then, by considerations on the rank of the kernel, we deduce that such embedding can not exist.

Using the previous results, we finally prove that:

Theorem E: *The Baer-Specker group $\mathbb{Z}^\omega$ does not afford a transitive self-similar representation in $\mathcal{A}_m$, for any positive integer m.*

As $\mathbb{Z}^\omega$ would be transitive, the Theorem 1.2.8 would be applied and thus $\mathbb{Z}^\omega$ (or its diagonal-topological closure) could be viewed as a $\mathbb{Z}_m$-module that is also a pro-m group.

But this would imply an embedding of the type

$$\mathbb{Z}^\omega \hookrightarrow \bigoplus_{p|m} \left(\bigoplus_{s(p)} \mathbb{Z}_p \right),$$

which, by the previous theorems, it is not possible.

Our last result was a question that arose motivated by the intransitivity of $\mathbb{Z}^\omega$. In [DSS21] the authors prove the following theorem:

Theorem. Let G be a self-similar group of degree m and orbit-type $(m_1, \ldots, m_s)$, also let K be a regular subgroup of $Sym(\{1, \ldots, s\})$. Then $G \wr K$ admits a faithful *transitive* self-similar representation of degree $m_1.m_2.\ldots.m_s.s$.

Then, there is a method of embedding possibly intransitive self-similar groups into transitive ones. A relevant question we asked was that *if it is possible to embed $\mathbb{Z}^\omega$, or more generally, some self-similar torsion-free group G^ω, possibly intransitive, into a transitive torsion-free self-similar group.* Informally, what was asked is how "wide" a transitive self-similar representation can be. We obtained then:

Theorem F: *Let G^ω be a self-similar group of orbit-type $(m_1, \ldots, m_s)$. Then the group $G^\omega \wr_Y \mathbb{Z}$, where $Y = \{1, \ldots, s\}$, is transitive self-similar of degree $m_1.\ldots.m_s.s^2$. In particular, $\mathbb{Z}^\omega \wr_{\{1,2\}} \mathbb{Z}$ is a torsion-free transitive self-similar metabelian group of degree 8.*

For this result and for others along the text we produce concrete representations to clarify the theory.

At the end, a Section called "Final Comments" lists some open problems we would like to answer in the short-term and present a possible technique to solve questions in the theory and also extend results that appeared in [BS10].

Chapter 1

Automorphism Groups of Trees

In this chapter we define the group $\mathcal{A}_m$ of automorphisms of a tree, its main properties, important subgroups and give some examples that will prepare us for the main results of our work. Also, we define the virtual endomorphisms; they will provide us a method to investigate if a group can be represented as a subgroup of $\mathcal{A}_m$. Some key results on abelian and profinite groups are stated in the last section.

1.1 Trees and their automorphisms

Let m be a positive integer and Y be the set $\{0, ..., m-1\}$. Define $\mathcal{M} = \mathcal{M}(Y)$ by the semigroup consisting of all finite words on the *alphabet* Y. The operation on $\mathcal{M}$ is the concatenation of words and the identity is the empty word $\emptyset$.

Definition 1.1.1. The *one-rooted m-regular tree* is the graph $\mathcal{T}_m = (V(\mathcal{T}_m), E(\mathcal{T}_m))$, where $V(\mathcal{T}_m) = \mathcal{M}$ and for an ordered pair (u, v), we have that $(u, v) \in E(\mathcal{T}_m)$ if and only if $v = uy$, for some $y \in Y$, $u, v \in \mathcal{M}$.

In this definition we have a tree where all vertices have the same number of incident vertices (m-regular), with the exception of one vertex (one-rooted); this vertex is called the *root* of the tree.

Thus, such tree has its vertices labeled by the words in $\mathcal{M}$ with increasing length $|u|, u \in \mathcal{M}$. As an example, with $Y = \{0, 1\}$:
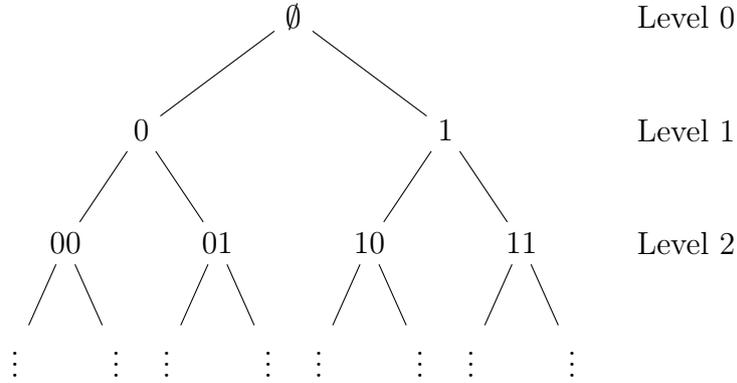
The set of all words of length n is called the *n-th level* of the tree $\mathcal{T}_m$. In the above example we have on level 0 only the root $\emptyset$, on level 1 the words $\{0, 1\}$, on level 2 the words $\{00, 01, 10, 11\}$ and so on; the n-th level is the set $\{u \mid |u| = n, u \in \mathcal{M}\}$.

Definition 1.1.2. An *automorphism* of the tree $\mathcal{T}_m$ is a graph bijection that preserves vertex incidence. The set of all such automorphisms is a group with respect to function composition, and will be denoted by $\mathcal{A}_m$.

Example 1.1.3. Let γ be a permutation of the alphabet Y. We can extend γ to an automorphism σ of the entire tree by setting:

$$(\emptyset)\sigma = \emptyset$$
$$(yu)\sigma = y^\gamma u, \text{ for all } y \in Y, u \in \mathcal{M}.$$

On the other side, every automorphism α of $\mathcal{T}_m$ induces a permutation $\sigma(\alpha)$ on Y just by considering $\sigma(\alpha)$ to be the restriction $\alpha\big|_Y : Y \to Y$.

Now, considering $\sigma(\alpha)$ to be the restriction on the above example (that is, we can see the permutation as an automorphism of the tree), we have that the composition $\alpha\sigma(\alpha)^{-1}$ has trivial action on the first level of the tree, i.e.,

$$(y)\alpha\sigma(\alpha)^{-1} = y, \text{ for all } y \in Y.$$

In this way, we can write the composition $\alpha\sigma(\alpha)^{-1}$ as

$$\alpha\sigma(\alpha)^{-1} = (\alpha_1, ..., \alpha_m), \quad (*)$$

where each α_y, $y = 1, ..., m$, is an automorphism of the tree rooted on y, which is

$$y\mathcal{T}_m = (yV(\mathcal{T}_m), yE(\mathcal{T}_m)),$$

where $yV(\mathcal{T}_m) = \{yu \mid u \in \mathcal{M}\}$ and $yE(\mathcal{T}_m) = \{(yu, yv) \mid (u, v) \in E(\mathcal{T}_m)\}$.

We can establish an isomorphism between $\mathcal{T}_m$ and $y\mathcal{T}_m$ by setting $yu \mapsto u$ (simply deleting the prefix y); so we can consider α_y itself an automorphism of $\mathcal{T}_m$. From this fact and $(*)$ we conclude that α can be expressed as

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)\sigma(\alpha),$$

where $\alpha_i \in \mathcal{A}_m$, $i = 1, ..., m$. As the entries α_i run over $\mathcal{A}_m$, we can identify $\mathcal{A}_m$ as the semidirect product

$$\mathcal{A}_m = (\mathcal{A}_m \times ... \times \mathcal{A}_m) \rtimes S_m,$$

where the action of S_m on $(\mathcal{A}_m \times ... \times \mathcal{A}_m)$ is given by the permutation of the indexes. Then, given $\sigma \in S_m$ (again, σ is seen as an automorphism of the tree, $\sigma = (e, e, \ldots, e)\sigma$) and $(\alpha_1, \alpha_2, ..., \alpha_m) \in (\mathcal{A}_m \times ... \times \mathcal{A}_m)$ it follows that

$$\sigma(\alpha_1, \alpha_2, ..., \alpha_m) = (\alpha_{1^\sigma}, \alpha_{2^\sigma}, ..., \alpha_{m^\sigma})\sigma$$

thus, for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)\sigma(\alpha)$ and $\beta = (\beta_1, \beta_2, ..., \beta_m)\sigma(\beta)$ in $\mathcal{A}_m$, the product and the inverses are given by

$$\alpha\beta = (\alpha_1\beta_{1^{\sigma(\alpha)}}, \alpha_2\beta_{2^{\sigma(\alpha)}}, ..., \alpha_m\beta_{m^{\sigma(\alpha)}})\sigma(\alpha)\sigma(\beta)$$
$$\alpha^{-1} = (\alpha_{1^{\sigma(\alpha)^{-1}}}^{-1}, \alpha_{2^{\sigma(\alpha)^{-1}}}^{-1}, ..., \alpha_{m^{\sigma(\alpha)^{-1}}}^{-1})\sigma(\alpha)^{-1}.$$

The previous semidirect product has great importance in the theory, so we make its definition precise.

Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of groups, where Λ is an index set. We denote the *Cartesian Product (or Direct Product)* of this family by

$$\prod_{\lambda \in \Lambda} G_\lambda = \{(g_\lambda)_{\lambda \in \Lambda} \mid g_\lambda \in G_\lambda\},$$

endowed with coordinatewise multiplication $(g_\lambda)_{\lambda \in \Lambda}(h_\lambda)_{\lambda \in \Lambda} = (g_\lambda h_\lambda)_{\lambda \in \Lambda}$. This is a group with identity denoted by $(e_\lambda)_{\lambda \in \Lambda}$, where e_λ is the identity element in G_λ. Now, the *Direct Sum* $\bigoplus_{\lambda \in \Lambda} G_\lambda$ is the subgroup of $\prod_{\lambda \in \Lambda} G_\lambda$ given by all elements $(x_\lambda)_{\lambda \in \Lambda}$ where $x_\lambda \neq e_\lambda$ for a finite number of indexes λ. Notice that if Λ is finite we have $\prod_{\lambda \in \Lambda} G_\lambda = \bigoplus_{\lambda \in \Lambda} G_\lambda$. In our text, when taking direct products (resp. sums) of the same group G, we will denote it by G^μ (resp. $G^{(\mu)}$), where μ is some cardinal.

Consider a group K, Λ an index set and H a group acting on Λ. Denote by $\varphi : H \to S_\Lambda$ the action of H on Λ, where S_Λ is the set of all bijections of Λ. The *Unrestricted Wreath Product* of K by H with respect to φ is defined by

$$Kwr_\varphi H = (\prod_{\lambda \in \Lambda} K) \rtimes_\varphi H,$$

where $(k_\lambda)_{\lambda \in \Lambda}^h = (k_{\lambda^{h\varphi}})_{\lambda \in \Lambda}$, for all $h \in H$ and $\lambda \in \Lambda$.

Analogously, the *Restricted Wreath Product* of K by H with respect to φ is defined by

$$K \wr_\varphi H = (\bigoplus_{\lambda \in \Lambda} K) \rtimes_\varphi H,$$

where $(k_\lambda)_{\lambda \in \Lambda}^h = (k_{\lambda^{h\varphi}})_{\lambda \in \Lambda}$, for all $h \in H$ and $\lambda \in \Lambda$.

The following example is classic in the theory and is known as the *binary adding machine*:

Example 1.1.4. *(Binary Adding Machine)* Let $\alpha = (e, \alpha)\sigma$ be an automorphism in $\mathcal{A}_2$, where e is the identity automorphism in $\mathcal{A}_2$ and $\sigma = (01)$, the transposition in S_2. Let $010 \in \mathcal{M}(\{0, 1\})$. Then

$$(101)^\alpha = (101)^{(e,\alpha)\sigma} = 1^\sigma (01)^{\alpha_1} = 0(01)^\alpha = 0(0)^\sigma (1)^{\alpha_0} = 01(1)^e = 011$$

Definition 1.1.5. Given $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)\sigma(\alpha) \in \mathcal{A}_m$, the *set of states* of α is defined recursively by

$$Q(\alpha) = \{\alpha\} \cup Q(\alpha_1) \cup ... \cup Q(\alpha_m).$$

In the previous example we can see that the set of states of α is $\{\alpha, e\}$.

Additionally, a group $G \leq \mathcal{A}_m$ will be called *finite-state* if $Q(\alpha)$ is finite, for all $\alpha \in G$.

1.2   Some Important Subgroups

There are some standard subgroups of automorphisms of one-rooted trees that are essential to the theory. No less important, some closure operations will be defined and will help us to study the groups under discussion.

Definition 1.2.1. Let G be a subgroup of $\mathcal{A}_m$. Then we define

$$Stab_G(n) = \{\alpha \in G \mid u^\alpha = u, \forall u \in \mathcal{M}, |u| = n\};$$
$$Fix_G(u) = \{\alpha \in G \mid u^\alpha = u, \text{ for a fixed } u \in \mathcal{M}\};$$
$$P(G) = \{\sigma(\alpha) \in S_m \mid \alpha \in G\}.$$

They are respectively the *stabilizer* of the level n, the *fixator* of the word u and the subgroup of the permutations induced by the elements of G. We say that G is *transitive* if $P(G)$ is a transitive subgroup of S_m.

The topological closure

Given $\alpha \in Stab_{\mathcal{A}_m}(1)$, it follows that $\sigma(\alpha) = e$ (α acts trivially on the first level). Then,

$$\alpha = (\alpha_1, ..., \alpha_m), \ \alpha_i \in \mathcal{A}_m.$$

Thus, given $\beta = (\beta_1, ..., \beta_m)\sigma(\beta) \in \mathcal{A}_m$, we have

$$\alpha^\beta = (\alpha_1^{\beta_1}, ..., \alpha_m^{\beta_m})^{\sigma(\beta)} \in Stab_{\mathcal{A}_m}(1),$$

where we write $(\gamma_1, ..., \gamma_m)^\sigma = (\gamma_{1^\sigma}, ..., \gamma_{m^\sigma})$ to ease the notation. Then, we have that $Stab_G(1) = G \cap Stab_{\mathcal{A}_m}(1)$ is a normal subgroup of G, for all $G \leq \mathcal{A}_m$. We can see that $Stab_G(n)$ is a normal subgroup of G in a similar way; also, we notice that $Stab_{\mathcal{A}_m}(n)$ has finite index in $\mathcal{A}_m$. Thus $\mathcal{A}_m$ is the inverse limit of its quotients by the i-th level stabilizers, that is

$$\mathcal{A}_m \simeq \varprojlim \frac{\mathcal{A}_m}{Stab_{\mathcal{A}_m}(n)},$$

where each stabilizer is a closed and open subgroup.

With the above observation, we take the *topological closure* of G, denoted by $\overline{G}$, as the closure of G in the profinite topology of $\mathcal{A}_m$. It will coincide with the set of all infinite products of the form $\beta = \alpha_0 \alpha_1 \alpha_2 \ldots$, where each α_i belongs to $Stab_{\mathcal{A}_m}(i)$.

The diagonal closure

Let $\alpha \in \mathcal{A}_m$ be an automorphism. We define recursively the *diagonal map* by

$$\alpha^{(0)} = \alpha, \ \alpha^{(1)} = (\alpha, \alpha, ..., \alpha), \ \alpha^{(n+1)} = (\alpha^{(n)})^{(1)}, \text{ for } i \geq 0.$$

The *diagonal closure*, denoted by $\widetilde{G}$, will be the group $\widetilde{G} = \langle G^{(i)} \mid i \geq 0 \rangle$, where $G^{(i)} = \{g^{(i)} \mid g \in G\}$, for a fixed positive integer i. Intuitively, we are taking an automorphism of the tree $\mathcal{T}_m$ and making it act on every subtree, in order to "exhaust all possibilities" for its actions.

Now, writing the diagonal map as $x : \mathcal{A}_m \to \mathcal{A}_m$, $\alpha \mapsto (\alpha, \alpha, ..., \alpha)$ and $\alpha^{(i)}$ as α^{x^i} we can write

$$\alpha^{a_0}(\alpha^{a_1})^{(1)}(\alpha^{a_2})^{(2)}...(\alpha^{a_n})^{(n)} = \alpha^{a_0 + a_1 x^1 + a_2 x^2 + ... + a_n x^n},$$

where $a_i \in \mathbb{Z}$.

We remark that, despite the above identification, it is possible that these "powers" of a single element alpha do not commute; more precisely, the factors of the polynomial in the exponent, do not commute if the states of alpha also do not commute.

The state closure

A group $G \leq \mathcal{A}_m$ is called *state-closed*, or *self-similar*, if G contains all of its "possible states", i.e., $Q(\alpha)$ is a subset of G, for all $\alpha \in G$. The *state closure* of G, denoted by $\widehat{G}$, is the group generated by all states of all elements of G. We will call *recurrent* a transitive state-closed group such that the projection $\pi_1 : Fix_G(1) \to G$, defined by

$$\alpha^{\pi_1} = ((\alpha_1, ..., \alpha_m)\sigma(\alpha))^{\pi_1} = \alpha_1$$

is surjective, where $\alpha \in G$ and $1^{\sigma(\alpha)} = 1$.

The set of states of an automorphisms can be "tricky", in the sense that a simple automorphism can have infinite states. To illustrate this we have the following

Example 1.2.2. Let $\alpha = (\alpha, \alpha^2)\sigma$ be an automorphism in $\mathcal{A}_2$. Noticing that

$$\alpha^2 = (\alpha^3, \alpha^3), \ \alpha^3 = (\alpha^4, \alpha^5)\sigma, \text{ and so on,}$$

we have

$$\alpha^{2n} = (\alpha^{3n}, \alpha^{3n}) \text{ and } \alpha^{2n+1} = (\alpha^{3n+1}, \alpha^{3n+2})\sigma,$$

and thus α has infinite order and its set of states, $Q(\alpha) = \{\alpha^n \mid i \geq 1\}$ is also infinite.

Another example that the state closure can get more complicated than the original group is the following:

Example 1.2.3. Let A be the group generated by $\alpha = (e, (e, \alpha))\sigma$ in $\mathcal{A}_2$. Its state closure $\widehat{A}$ is the group $\langle \alpha, \beta \rangle$, where $\beta = (e, \alpha)$. This group is known as the *Basilica Group* and it has many interesting properties (see [GŻ02]).

The Diagonal-Topological closure

Now we state important properties about the closures of abelian transitive state-closed groups.

Definition 1.2.4. The *diagonal-topological closure* of A, denoted by A^*, is considered as the diagonal closure applied first and then the topological closure is taken, i.e., $A^* = \overline{\tilde{A}}$.

Notice that these closure operations in general do not commute. For example, consider $\sigma = (12) = (e, e)(12) \in \mathcal{A}_2$. Applying the topological closure first there are no new elements different from σ; then the diagonal closure applied next gives us elements of the form $\sigma^{(n)}$ and their finite products $\sigma^{(i_1)}\sigma^{(i_2)}\dots\sigma^{(i_n)}$. However when we apply the diagonal and then the topological closure, we obtain also *infinite* products of the terms $\sigma^{(n)}$.

Proposition 1.2.5. *[BS10] Let A be an abelian transitive state-closed group of degree m. Then $Stab_A(i) \leq A^{(i)}$ for all $i \geq 0$. The diagonal closure $\widetilde{A}$ is an abelian transitive state-closed group and is a minimal recurrent group containing A. The diagonal-topological closure A^* of A is also an abelian transitive state-closed group.*

Proof. Let $\alpha = (\alpha_1, ..., \alpha_m)\sigma$ and $\beta = (\beta_1, ..., \beta_m) \in A$. The conjugate of β by α is

$$\beta^\alpha = (\beta_1^{\alpha_1}, ..., \beta_m^{\alpha_m})^\sigma.$$

As $\alpha_i, \beta_i \in A$, and A is abelian, it follows that $\beta = (\beta_1, ..., \beta_m)^\sigma$. Furthermore, since A is transitive, $\beta = (\beta_1, ...\beta_1) = (\beta_1)^{(1)}$. Thus, $Stab_A(i) \leq A^{(i)}$, for all i. A similar verification shows that $\tilde{A} = \langle A^{(i)} \mid i \geq 0 \rangle$ is abelian.

Let G be a recurrent group such that $A \leq G \leq \tilde{A}$. Given $\alpha \in G$, as G is recurrent, there exists $\beta \in Stab_G(1)$ such that $\beta = (\beta_1, ..., \beta_m)$ with $\beta_1 = \alpha$. Since G is transitive and abelian, we have $\beta_1 = ... = \beta_m = \alpha$; that is, $\beta = \alpha^{(1)}$. Hence, $A^{(i)} \leq G$ and $G = \tilde{A}$ follows.

Now, writing the elements of A^* as products of elements of the form

$$\alpha^* = \alpha^{a_0}(\alpha^{a_1})^{(1)}(\alpha^{a_2})^{(2)}\dots = \alpha^{a_0 + a_1 x^1 + a_2 x^2 + \dots}$$

the last assertion is proved.

∎

Definition 1.2.6. Let G be a permutation group on an alphabet X. G is said to be *regular* if it is transitive and $Stab_G(x)$ is trivial, for all $x \in X$.

Proposition 1.2.7. *[BS10] i) Let G be a recurrent abelian group of degree m and let $C_{\mathcal{A}_m}(G)$ be the centralizer of G in $\mathcal{A}_m$. Then $C_{\mathcal{A}_m}(G) = \bar{G}$.*
ii) Let m be a prime number and G be an infinite abelian transitive state-closed group. Then $C_{\mathcal{A}_m}(G) = \bar{G}$.

Proof. i) Let $P = P(G)$ the permutation group on Y induced by G. Since P is an abelian transitive permutation group of degree m, we have that it is also regular; furthermore, the stabilizer in G of any $y \in Y$ is the same as the stabilizer of the first level of the tree, say $H = Stab_G(1)$. By hypothesis the representation of G is recurrent, so the projection $\pi_v : Stab_G(k) \to G$ on any of its coordinates is surjective and therefore produces the group G.

For every $\sigma \in P$, choose $\alpha_0(\sigma) = (\alpha_0(\sigma)_1, ..., \alpha_0(\sigma)_m)\sigma \in G$ which induces σ on Y. Let $h = (h_1, h_2, ..., h_m) \in H$. Then, since h_i and $a_0(\sigma)_i$ are in G, which is abelian,

$$h^{\alpha_0(\sigma)} = \left((h_1)^{\alpha_0(\sigma)_1}, (h_2)^{\alpha_0(\sigma)_2}, ..., (h_m)^{\alpha_0(\sigma)_m}\right)^\sigma = (h_1, h_2, ..., h_m)^\sigma.$$

By varying $\sigma \in P$ we find that $h = (h_1, ..., h_1)$. Now, for every $\sigma \in P$, there exists $\alpha_1(\sigma) = (\alpha_0(\sigma), ..., \alpha_0(\sigma)) \in H$, which induces $\sigma^{(1)}$ modulo $Stab_G(2)$. Thus, we produce a sequence $\alpha_i(\sigma) \in Stab_G(i)$ of elements in G such that $\alpha_i(\sigma) = \sigma^{(i)}$ modulo $Stab_G(i+1)$. Let $\gamma \in C = C_{\mathcal{A}_m}(G)$. Then,

$$\gamma = (\gamma_1, ..., \gamma_m)\sigma,$$
$$\gamma' = \gamma.\alpha_0(\sigma)^{-1} = (\gamma'_1, ..., \gamma'_m) \in Stab_C(1),$$

and $\gamma'_1 = ... = \gamma'_m$; say γ'_1 induces a permutation σ' on Y. Thus,

$$\gamma.\alpha_0(\sigma)^{-1}.\alpha_1(\sigma')^{-1} \in Stab_C(2).$$

We produce in this manner a sequence

$$\alpha_0(\sigma), \alpha_1(\sigma'), \alpha_2(\sigma''), ...$$

of elements of G such that γ is equal to the infinite product of these elements. Hence, $C_{\mathcal{A}_m}(G) = \widehat{G}$.

ii) Let $m = p$ be a prime number. The permutation group P induced on $Y = \{1, ..., p\}$ is cyclic, say generated by σ. Since G is infinite, there exists an $h = (h_1, ..., h_1) \in H$ such that $h_1 \notin H$ and therefore we may assume h_1 induces σ on Y. We produce elements $a_i \in G$ such that $a_i = \sigma^{(i)}$ modulo $Stab_{\mathcal{A}}(i+1)$ and the proof follows as in the first item.

$\blacksquare$

The following theorem is a key result to obtain the last theorem of this thesis.

Theorem 1.2.8. *([BS10], Theorem 1) Let A be an abelian transitive self-similar group of degree m. Then*

> *i) the group A^* is isomorphic to a finitely presented $\mathbb{Z}[[x]]$-module;*

> *ii) if A^* is torsion-free then it is a finitely generated $\mathbb{Z}_m$-module that is also a pro-m group.*

Now, we define the groups $D_m(j)$ to be the group generated by the states of the generalized adding machine $\alpha = (e, e, \ldots, \alpha^{x^{j-1}})\sigma$, acting on the m-ary tree , where $\sigma = (1 \ldots m)$.

Proposition 1.2.9. *[BS10] The topological closure of $D_m(j)$, seen as a $\mathbb{Z}_m$-module is isomorphic to the ring $\frac{\mathbb{Z}_m[[x]]}{(r)}$, $r = m - x^j$.*

The authors of the above paper make an interesting characterization of the diagonal-topological closure for abelian transitive self-similar groups:

Theorem 1.2.10. *[BS10] Let A be an abelian transitive self-similar group of prime degree m and let σ be the m-cycle automorphism. If $tor(A)$ is nontrivial then A^* is a torsion group conjugate to $\langle\sigma\rangle^* \simeq \frac{\mathbb{Z}}{m\mathbb{Z}}[[x]]$. If A is torsion-free then A^* is a torsion-free group conjugate to the topological closure of $D_m(j)$ for some j.*

In [DSS23], the authors generalized the notion of diagonal-topological closure for intransitive induced permutation groups as the following.

Let $A \leq \mathcal{A}_m$ abelian group and $B(A)$ denote the projection of the actions of elements of A in the orbits of $P(A)$. Let us define the *partial diagonal monomorphisms* from $\mathcal{A}_m$ to $Stab_{\mathcal{A}_m}(1)$ as

$$x_i : \alpha \mapsto (e, \ldots, e, \alpha, \ldots, \alpha, e, \ldots, e)$$

with α ocurring in the coordinates of the orbit O_i and the trivial automorphism occurring in the other positions. Let Δ be the free monoid generated by such monomorphisms; the closure of A under Δ is defined by

$$\Delta(A) = \langle A^{\delta} \mid \delta \in \Delta \rangle.$$

We define then the *diagonal-topological closure* as $A^* = \overline{\Delta(B(A))}$, where the bar represents topological closure.

The authors prove that, given A a self-similar abelian group, $\Delta(A)$, $\overline{\Delta(A)}$, $\Delta(B(A))$ and A^* are again self-similar abelian groups of the same permutation-type as A. Notice the resemblance of the construction on the transitive case; here we changed the monoid $\langle x \rangle$ for Δ.

1.3 Representations of groups as automorphism groups of the tree

One of the main goals of the theory of groups acting on trees is to investigate if a given group G can be represented in $\mathcal{A}_m$, that is, if G can be seen as a group of automorphisms of a tree. One of the central tools for this purpose is the notion of *virtual endomorphism*, that we will define in this section.

Definition 1.3.1. We say that a group G has a *representation of degree m* if exists an homomorphism $\varphi : G \to \mathcal{A}_m$. If φ is a monomorphism, then the representation is called *faithful*.

We will call both φ and G^{φ} a *representation of degree m* of G. So, if G^{φ} is state-closed,

finite-state or transitive, we say the same about the representation.

The coset tree

Let G be a group and a chain of subgroups such that

$$G = G_0 \geq G_1 \geq G_2 \geq ... \geq G_n \geq ...$$

with $\bigcap G_i = \{1\}$. Now, we take each of these subgroups as partitions of next subgroup in the chain:

$$G = \bigcup G_1 h_{j,1} \ , \ G_1 = \bigcup G_2 h_{j,2}, \text{ and so on.}$$

So, for some G_s, we can write its cosets in G as $G_s h_{j_s,s} h_{j_{s-1},s-1}...h_{j_1,s}$, which will be the vertices of the tree where the edges will be determined by set inclusion. Then G acts (faithfully) on this resulting tree by right multiplication, say $h : G_i k \mapsto G_i k h$.

In this action, the set of vertices fixed by h is a subtree, although it can be irregular. But in the case G_i is a normal subgroup of G and h fixes some coset $G_i k$, we have that h fixes all such cosets of G_i on G (as in this case $h \in G_i k$), and then it fixes all the vertices of the tree down to the i-th level.

If we require a bound m for the indexes $|G_i : G_{i+1}|$, we can embed the coset tree into the m-ary tree $\mathcal{T}_m$; as the coset tree is possibly smaller than $\mathcal{T}_m$, we can extend the action of G fixing pointwise all the extra subtrees that may appear. In particular, if the indexes $|G_i : G_{i+1}|$ are constant, the coset tree is regular.

The following is an example with constant indexes $|G_i : G_{i+1}| = 2$:
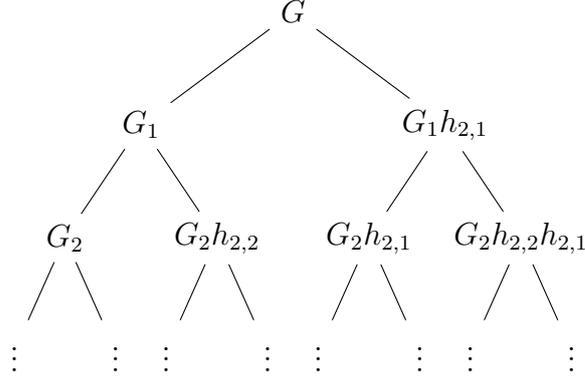
Lemma 1.3.2. *Let G be a state-closed group of automorphisms of the tree $\mathcal{T}_m = \mathcal{T}(Y)$ and let X be a $P(G)$-invariant subset of Y. Then $\mathcal{T}(X)$ is G-invariant and for the resulting representation $\mu : G \to \mathcal{A}(X)$ the group G^μ is state-closed. If G is diagonally or topologically closed then so is G^μ.*

Proof. Let xu be a sequence in $\mathcal{M}(X)$ and let $\alpha \in G$. Then $(xu)^\alpha = x^{\sigma(\alpha)} u^{\alpha_x}$. As $x^{\sigma(\alpha)} \in X$ and $\alpha_x \in G$, it follows that $(xu)^\alpha$ is a sequence in $\mathcal{M}(X)$ and then $\mathcal{T}(X)$ is G-invariant. Also, for any sequence u from X, we have $(\alpha^\mu)_u = (\alpha_u)^\mu$. Thus, G^μ is state-closed.

∎

Virtual Endomorphisms

A *virtual endomorphism* of an abstract group G is a homomorphism $f : H \to G$ from a subgroup H of finite index in G. Let G be a group and consider

$$\mathbf{H} = (H_i \leqslant G \mid [G : H_i] = m_i \ (\ 1 \le i \le s))\,,$$

$$\mathbf{m} = (m_1, \ldots, m_s)\,, \ m = m_1 + \cdots + m_s,$$

$$\mathbf{F} = (f_i : H_i \to G \text{ virtual endomorphisms} \mid 1 \le i \le s)\,;$$

we will call $(\mathbf{m}, \mathbf{H}, \mathbf{F})$ a *G-data* or *data for G*. The $\mathbf{F}$-*core*$(\mathbf{H})$ is the largest subgroup of $\cap_{i=1}^s H_i$ which is normal in G and f_i-invariant for all $i = 1, \ldots, s$. In the case $\mathbf{F}$-core$(\mathbf{H})$ is trivial, we say that $\mathbf{F}$ is *simple*; if $\mathbf{F} = (f)$, the endomorphism f is called simple.

Let $(\mathbf{m}, \mathbf{H}, \mathbf{F})$ be a G-data and for each subgroup H_i in $\mathbf{H}$, choose a right transversal $T_i = \{t_{i1}, t_{i2}, \cdots, t_{im_i}\}$ in G. Define the i-Schreier function $\theta_i : G \times T_i \to H_i$ by

$$\theta_i(g, t_{ij}) = t_{ij} g (t_{ik})^{-1}, \ \text{ where } \ H_i t_{ik} = H_i t_{ij} g,$$

and let σ be the induced permutation representation of G on $T = T_1 \dot{\cup} T_2 \dot{\cup} \cdots \dot{\cup} T_s$, which means that $j^{\sigma_g} = k$ if and only if $H_i t_{ik} = H_i t_{ij} g$ for some $i = 1, \cdots, s$, where $\sigma : g \mapsto \sigma_g$.

The maps in $\mathbf{F}$ extends to a homomorphism

$$\varphi : G \to \mathcal{A}_m$$

defined by

$$g^\varphi = (\theta_i(g, t)^{f_i \varphi} \mid t \in T_i, 1 \le i \le s) \, \sigma_g;$$

this is essentially the contents of the Kaloujnine-Krasner Theorem.

Proposition 1.3.3. *([DSS21, Proposition 2.1]) A group G has a faithful self-similar representation of degree m if, and only if there exists a G-data $(\mathbf{m}, \mathbf{H}, \mathbf{F})$, such that the subgroup*

$$\mathbf{F}\text{-}core(\mathbf{H}) = \langle K \leqslant \cap_{i=1}^s H_i \mid K \lhd G, K^{f_i} \leqslant K, \forall i = 1, \ldots, s \rangle,$$

is trivial.

Example 1.3.4 (Transitive self-similar representations of free abelian groups of finite rank). **a)** Consider $\mathbb{Z}^n = \langle x_1, x_2, \ldots, x_n \rangle$, $H = \langle x_1^2, x_2, \ldots, x_n \rangle$, the endomorphism $f : H \to \mathbb{Z}^n$ that extends the map $x_1^2 \mapsto x_2$, $x_i \mapsto x_{i+1}$, $x_n \mapsto x_1$, $i = 2, \ldots, n-1$, and the transversal $T = \{1, x_1\}$ of H in $\mathbb{Z}^n$. Thus the triple $(2, H, f)$ and T induce the following finite-state transitive self-similar representation

$$(\mathbb{Z}^n)^\varphi = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$$

of $\mathbb{Z}^n$ of degree 2, where

$$\alpha_1 = x_1^\varphi = ((1x_1x_1^{-1})^{f\varphi}, (x_1x_11)^{f\varphi})(1\,2) = (e, x_2^\varphi)(1\,2) = (e, \alpha_2)(1\,2),$$

$$\alpha_2 = x_2^\varphi = ((1x_21^{-1})^{f\varphi}, (x_1x_2x_1^{-1})^\varphi) = (x_3^\varphi, x_3^{f\varphi}) = (\alpha_3, \alpha_3),$$

$$\cdots$$

$$\alpha_{n-1} = x_{n-1}^\varphi = ((1x_{n-1}1^{-1})^{f\varphi}, (x_1x_{n-1}x_1^{-1})^{f\varphi}) = (x_n^\varphi, x_n^\varphi) = (\alpha_n, \alpha_n),$$

$$\alpha_n = x_n^\varphi = ((1x_n1^{-1})^{f\varphi}, (x_1x_nx_1^{-1})^{f\varphi}) = (x_1^\varphi, x_1^\varphi) = (\alpha_1, \alpha_1).$$

b) Let $\eta \in 2\mathbb{Z}_2^\times$ be a transcendental number (over $\mathbb{Q}$). By the proof of Theorem 1.2 in [BS20], the virtual endomorphism $f : 2\mathbb{Z}_2 \to \mathbb{Z}_2$ defined by $x \mapsto x/\eta$, where $x = \sum_{n \in \mathbb{N}} x_n \eta^n$ and $x_n \in \{0, 1\}$, is simple and

$$f : H = \mathbb{Z}[1/\eta] \cap 2\mathbb{Z}_2 \to G = \mathbb{Z}[1/\eta] \cap \mathbb{Z}_2$$

is a well defined endomorphism. Hence $(2, H, f)$ is a simple G-data; note that $G \simeq \mathbb{Z}^{(\omega)}$. Let $T = \{0, 2/\eta\}$ be a transversal for H in G. The data $(2, H, f)$ and the transversal T induce the following self-similar representation $\varphi : G \to \mathcal{A}_2$ of the free abelian group G of countable infinite rank

$$(\langle k/\eta^n \mid k \in \mathbb{Z}, n \in \mathbb{N}\rangle \cap \mathbb{Z}_2)^\varphi.$$

For example,

$$\left(\frac{2}{\eta}\right)^\varphi = \left(e, \left(\frac{4}{\eta^2}\right)^\varphi\right)(1\,2)$$

where if $(x_{2u_1u_2...u_n})^\varphi$ is the state of $(2/\eta)^\varphi$ in the word $u = 2u_1u_2...u_n$, then

$$(x_u)^\varphi = \left(\left(\frac{x_u\eta - 2}{\eta^2}\right)^\varphi, \left(\frac{x_u\eta + 2}{\eta^2}\right)^\varphi\right)(1\,2),$$

if $x_u \in \mathbb{Z}_2^\times$, and

$$(x_u)^\varphi = \left(\left(\frac{x_u}{\eta} \right)^\varphi, \left(\frac{x_u}{\eta} \right)^\varphi \right),$$

otherwise.

An interesting feature in the previous examples is that, if we remove the restriction of transitivity, is possible to achieve finite-state non-transitive self-similar representations, as we will now see.

Example 1.3.5 (Non-transitive finite-state self-similar representations of free abelian groups of countable rank). **a)** The $\mathbb{Z}^{(\omega)}$-data $((2,1), (H, \mathbb{Z}^{(\omega)}), (f_1, f_2))$, where

$$H = \{(a_n)_{n \in \mathbb{N}} \mid a_1 \in 2\mathbb{Z}, a_n \in \mathbb{Z}, n > 1\},$$

the endomorphisms f_1 and f_2 given by

$$\begin{aligned} f_1 &: H \longrightarrow \mathbb{Z}^{(\omega)} & f_2 &: \mathbb{Z}^{(\omega)} \longrightarrow \mathbb{Z}^{(\omega)} \\ (a_n)_{n \in \mathbb{N}} &\longmapsto \left(\frac{a_1}{2}, 0, 0, 0, \dots \right), & (a_n)_{n \in \mathbb{N}} &\longmapsto (a_{n+1})_{n \in \mathbb{N}} \end{aligned}$$

and the choice of transversals being $T_1 = \{(0,0,0,\dots), (1,0,0,\dots)\}$ of H in $\mathbb{Z}^{(\omega)}$ and $T_2 = \{(0,0,0,\dots)\}$ of $\mathbb{Z}^{(\omega)}$ in $\mathbb{Z}^{(\omega)}$, induce the following finite-state self-similar representation of degree 3 of the free abelian group of countable rank $\mathbb{Z}^{(\omega)}$,

$$\mathbb{Z}^{(\omega)} \simeq \langle \alpha_1 = (e, \alpha_1, e)(1\,2), \alpha_n = (e, e, \alpha_{n-1}) \mid n = 2, 3, 4, \dots \rangle.$$

b) Note that

$$\mathbb{Z}^n \simeq \langle \alpha_1 = (e, \alpha_1, e)(1\,2), \alpha_n = (e, e, \alpha_{n-1}) \mid i = 2, 3, \dots, n \rangle$$

is a non-transitive finite-state self-similar representation of $\mathbb{Z}^n$.

1.4 Some results on Abelian and Profinite groups

Here, we collect important results that will be referred later in this work for proving our main theorems. Mainly, a result on the structure of finitely generated profinite abelian groups and theorems, proved by E.Specker, on finitely generated summands and on countable subgroups of the Baer-Specker group.

Lemma 1.4.1. *Suppose that a torsion-free abelian group A embeds in $T \oplus B$, where T is torsion and B is torsion-free abelian groups. Then A embeds in B.*

Proof. Notice that $B = \frac{T \oplus B}{T}$. Then

$$\varphi : \frac{T \oplus B}{T} \longrightarrow B$$

is an isomorphism and if

$$(Ta_1)^\varphi = (Ta_2)^\varphi$$
$$(Ta_1 a_2^{-1})^\varphi = T^\varphi$$

then $a_1 a_2^{-1} \in T$ and there exists n such that $(a_1 a_2^{-1})^n = e$. Hence $a_1 a_2^{-1} = e$ and then $a_1 = a_2$, since A is torsion-free. ∎

Theorem 1.4.2. *([RZ00], Theorem 4.3.5) Let G be a finitely generated profinite abelian group with minimal number of generators $d(G) = d$. The G is a direct sum of finitely many procyclic groups; more explicitly,*

$$G \simeq \left[\bigoplus_p \left(\bigoplus_{s(p)} \mathbb{Z}_p \right) \right] \oplus \left[\bigoplus_p \left(\bigoplus_{i \in I_p} L_i(p) \right) \right],$$

where p ranges over all primes, each $L_i(p)$ is a cyclic p-group, each $s(p)$ is a natural number with $s(p) \leq d$ and each I_p is a finite set with $|I_p| \leq d$.

Theorem 1.4.3. *(Specker, [Rob12], 4.4.5) Let G be a cartesian sum of infinite cyclic groups. Then every finite subset of G is contained in a finitely generated summand of G whose direct complement is also a cartesian sum of infinite cyclic groups.*

For the sake of completeness, we state

Theorem 1.4.4. *(Specker, [Rob12], 4.4.6) If G is a cartesian sum of infinite cyclic groups, every countable subgroup of G is free abelian.*

In this text we will often refer to the *product topology* on $P = \mathbb{Z}^\omega$, where is given to $\mathbb{Z}$ the discrete topology. We can describe a basic system of neighborhoods of 0 as $U_n = \{x \in P \mid x_m = 0 \text{ for all } m < n\}$. For any subset A of P, let $\bar{A}$ denote the closure of A in P, that is, $\bar{A}$ equals the set of all $x \in P$ such that for every n, x agrees with some element of A in the first n coordinates.

The following lemma will be useful when we consider the product topology in the last part of our work:

Lemma 1.4.5. *([EM02], Ch. IX, Lemma 2.1) For any subgroup A of $P = \mathbb{Z}^\omega$ of infinite rank, $\bar{A}$ is isomorphic to P.*

Chapter 2

The self-similarity of the free abelian group of uncountable rank

In [DSS23], the authors asked if there is a self-similar free abelian group of uncountable (continuum) rank. In this section we answer positively this question, establishing also the transitivity of this representation. We use arguments of number-theoretic flavour, and in the second section we present a concrete computable representation of the free abelian group of countable rank.

2.1 Transitive self-similar representation of an uncountable free abelian group

We now proceed to prove, using arguments based on [BS20], that the free abelian group of uncountable rank is *transitive* self-similar. First we state some definitions from Transcendental Number Theory, as in [Bak75].

Let $L|K$ be a field extension. We say that $\beta \in L$ is *algebraic over* K if there exists a nonzero $P \in K[x]$ such that $P(\beta) = 0$. Otherwise β is said to be *transcendental* over K. More generally, the elements $\beta_1, \ldots \beta_n \in L$ are said to be *algebraically dependent over* K if there exists a nonzero $P \in K[x_1, \ldots, x_n]$ such that $P(\beta_1, \ldots, \beta_n) = 0$. Otherwise $\beta_1, \ldots, \beta_n$ are said to be *algebraically independent over K*.

Consider a transcendental field extension $L|K$, that is, $L \simeq K(X)$ for some non-empty set X of algebraically independent variables over K. A set $\mathcal{B} \subset L$ is called a *transcendence basis of* $L|K$ if $\mathcal{B}$ is a maximal (with respect to set inclusion) algebraically independent set over K and $L|K(\mathcal{B})$ is an algebraic extension, it means that for every $\alpha \in L$, the element α is algebraic over $K(\mathcal{B})$.

Following the ideas of Bartholdi and Sidki [BS20], we prove:

Theorem A. *The uncountable free abelian group admits a faithful transitive self-similar representation.*

Proof. Let $\eta \in 2\mathbb{Z}_2^\times$ be a transcendental (over $\mathbb{Q}$) dyadic integer, where $\mathbb{Z}_2^\times$ stands for the group of units in $\mathbb{Z}_2$, which is given by $1 + 2\mathbb{Z}_2$. In the following, we represent the elements of $\mathbb{Z}_2$ in η-basis, that is, $z = \sum_{i \geq 0} z_i \eta^i$, $z_i \in \{0,1\}$. Consider the family $(S_\lambda)_{\lambda \in \Lambda}$ of all algebraically independent subsets of $\mathbb{Z}_2 \setminus \overline{\mathbb{Q}}$ which contains η, where $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}_2$. By Zorn's Lemma, $(S_\lambda)_{\lambda \in \Lambda}$ has a maximal element S. Now, we consider the (abelian) group generated by the products of the elements of a subset X of $\mathbb{Q}_2$

$$\mathbb{Z}[X] = \langle x_1 \ldots x_n \mid x_i \in X, n \in \mathbb{N} \rangle.$$

First, write $S_\eta = S \setminus \{\eta\}$. Then η is transcendental over $\mathbb{Z}[S_\eta]$, that is, η does not satisfy a polynomial with coefficients in $\mathbb{Z}[S_\eta]$. This follows directly by the definition of algebraic independence and by the construction of the set S. The second claim is that $\mathbb{Z}[S_\eta]$ is an uncountable free abelian group generated by $\{s^n \mid s \in S_\eta, n \in \mathbb{N}\}$. In fact, in face of the first claim, if any linear combination

$$a_1\xi_1 + a_2\xi_2 + \ldots + a_k\xi_k, (a_i \in \mathbb{Z}, \ \xi_i \in \{s_1 \ldots s_n \mid s_i \in S_\eta, n \in \mathbb{N}\}),$$

were to be null, we would have a polynomial on elements of S_η equaling zero, contradicting the construction of S; hence $\mathbb{Z}[S_\eta]$ is free abelian.

In light of these statements, we proceed with the following construction.

Define $G = \mathbb{Z}[S_\eta][1/\eta] \cap \mathbb{Z}_2$ and $H = G \cap 2\mathbb{Z}_2$ subgroups of $\mathbb{Z}_2$; notice that $|G : H| = 2$ and $\mathbb{Z}[S_\eta][1/\eta] = \mathbb{Z}[S_\eta \cup \{1/\eta\}]$ is an uncountable free abelian group. We can see the elements of $G \leq \mathbb{Z}_2$ as written on base η. Also, define $f : H \to G$, $g \mapsto g/\eta$, a virtual endomorphism. The action of f is a "shift" on the η-formal sums; notice that $H^f \leq G$.

We argue that the above action f is self similar. Let g be an element in $H = G \cap 2\mathbb{Z}_2$. As $g \in 2\mathbb{Z}_2$, it has the form $(0, g_1, g_2, \dots)$; also, as $g \in G$, it is written in η-basis. Thus, applying f repeatedly to g, there will exist a natural number n_0 such that $g^{f^{n_0}} \notin H$, as the shift will eventually bring to the first coordinate an entry not divisible by η. Hence f is a simple endomorphism. Furthermore, the associated representation $\varphi : G \to \mathcal{A}_2$ is injective, thus G is self-similar. ∎

2.2 Concrete transitive self-similar representation of the infinite countable free abelian group

The Thue-Morse sequence $t = (t_n)_{n \in \mathbb{N}}$ is defined as follows: $t_n = 0$ if the sum of the binary digits of n is even, and $t_n = 1$ otherwise. Then

$$t = (0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, \dots).$$

Define the element $\tau \in \mathbb{Z}_2$ by $\tau = \sum_{n \in \mathbb{N}} t_n 2^n$. By [AB07, Theorem 6], τ is transcendental. Note that $\tau/2 \in \mathbb{Z}_2^\times$. Consider $\eta = 2(1/(\tau/2)) = 4/\tau$ and $T = \{0, 2/\eta\}$. Thus, by Example 1.3.4-b, we have that $h^f = h/\eta = h\tau/4$ and that $(\mathbb{Z}^\omega)^\varphi$ is generated by $(\langle k/\eta^n \mid k \in \mathbb{Z}, n \in \mathbb{N}\rangle \cap \mathbb{Z}_2)^\varphi$; for example,

$$\left(\frac{2}{\eta}\right)^\varphi = \left(\frac{\tau}{2}\right)^\varphi = \left(e, \left(\frac{\tau^2}{4}\right)^\varphi\right)(1\,2).$$

Then the element $(\tau/2)^\varphi$ is well determined by finding the permutations of each of its states. Note that $(\tau/2)^\varphi = (e, (\tau^2/4)^\varphi)(1\,2)$ and $\tau^2/4 \in \mathbb{Z}_2^\times$, thus

$$\left(\frac{\tau^2}{4}\right)^{\varphi} = \left(\left(\frac{\tau^3 - 2\tau^2}{16}\right)^{\varphi}, \left(\frac{\tau^3 + 2\tau^2}{16}\right)^{\varphi}\right)(1\,2).$$

Now, $(\tau^3 - 2\tau^2)/16 \notin \mathbb{Z}_2^{\times}$ and $(\tau^3 + 2\tau^2)/16 \in \mathbb{Z}_2^{\times}$, hence we have the expressions

$$\left(\frac{\tau^3 - 2\tau^2}{16}\right)^{\varphi} = \left(\left(\frac{\tau^4 - 2\tau^3}{64}\right)^{\varphi}, \left(\frac{\tau^4 - 2\tau^3}{64}\right)^{\varphi}\right)$$

and

$$\left(\frac{\tau^3 + 2\tau}{16}\right)^{\varphi} = \left(\left(\frac{\tau^4 + 2\tau^3 - 8\tau^2}{64}\right)^{\varphi}, \left(\frac{\tau^4 + 2\tau^3 + 8\tau^2}{64}\right)^{\varphi}\right)(1\,2).$$

Therefore,

$$(\tau/2)^{\varphi} = ((e, e), ((x_{211}, x_{212}), (x_{221}, x_{222})(1\,2))(1\,2))(1\,2),$$

where $x_{211} = \left(\frac{\tau^4 - 2\tau^3}{64}\right)^{\varphi}$, $x_{212} = \left(\frac{\tau^4 - 2\tau^3}{64}\right)^{\varphi}$, $x_{221} = \left(\frac{\tau^4 + 2\tau^3 - 8\tau^2}{64}\right)^{\varphi}$, and $x_{222} = \left(\frac{\tau^4 + 2\tau^3 + 8\tau^2}{64}\right)^{\varphi}$.

Also, this concrete representation can be recursively calculated to the subsequent levels of the tree.

Finishing this chapter, we observe that the free abelian group of uncountable rank appears as a subgroup of the Baer-Specker group, where it appears as the group of all bounded sequences with integer coefficients, that is,

$$S = \{a = (a_n)_{n \in \mathbb{N}} \in \mathbb{Z}^{\omega} \mid |a_n| \leq N_a, N_a \in \mathbb{N}, \forall n \in \mathbb{N}\}.$$

The freeness of this group was proven by [Nöb68]; in [Bla02] the author proves that the freeness of S cannot be proved from the usual axioms of set theory minus the axiom of choice (even adding the axiom of dependent choice).

A result about the Baer-Specker group also worth mentioning before the next chapter, which also appears in [Bla02], is the following. Consider the truncation homomorphism

$$T_k : \mathbb{Z}^{\omega} \to \mathbb{Z}^k : (x_n)_{n \in \mathbb{N}} \mapsto (x_n)_{0 \leq n < k},$$

which truncates an infinite sequence after k terms.

Theorem 2.2.1. *[Spe50] Every homomorphism $h : \mathbb{Z}^\omega \to \mathbb{Z}$ factors as $h' \circ T_k$, for some finite k and some $h' : \mathbb{Z}^k \to \mathbb{Z}$.*

This means that, considering "unit vectors" $e_i \in \mathbb{Z}^\omega$ elements with 1 in the i-th coordinate and 0 on every other coordinate, every $h : \mathbb{Z}^\omega \to \mathbb{Z}$ sends all but finitely many e_i's to zero and it is determined by its values on all e_i's.

Chapter 3

The self-similarity of Cartesian Products and of the Baer-Specker group

In this last chapter, following the ideas of the beautiful paper of Dantas, Santos and Sidki [DSS21], we extend the self-similarity of direct sums of self-similar groups to direct products. As a consequence, we obtain the self-similarity of the Baer-Specker group. However, as it will be proved in the last section, this group does not admit any *transitive* self-similar representation.

3.1 Representation of Cartesian Products

The next result has the interesting consequence that, being G self-similar of degree m, the degree of the cartesian product increases just by 1. In consequence of this, we obtain again a concrete self-similar representation of the free abelian group of uncountable rank, this time an *intransitive* one.

Theorem B. *If G is a self-similar group of degree m, then the unrestricted direct product $G^\omega = \prod_{n \in \mathbb{N}} G$ admits a faithful self-similar representation of degree $m + 1$. For $G = \mathbb{Z}$, there is a faithful self-similar representation of the Baer-Specker group $\mathbb{Z}^\omega$ of degree 3.*

Proof. Let G be a state-closed group. By Proposition 1.3.3, there is G-data $(\mathbf{m}, \mathbf{H}, \mathbf{F})$ such that $\mathbf{H}$ is $\mathbf{F}$-core-free.

(1) The subgroup

$$L_i = \{(h, g_2, g_3, ...) \in G^\omega \mid h \in H_i, \ g_j \in G \text{ for } j \geq 2\}$$

has index m_i in G^ω. For each $i = 1, ..., s$ define the homomorphism $\bar{f}_i : L_i \to G^\omega$ by

$$(h, g_2, g_3, ...)^{\bar{f}_i} = (h^{f_i}, e, e, e, ...)$$

and the homomorphism $\bar{f}_{s+1} : L_{s+1} = G^\omega \to G^\omega$ by

$$(g_1, g_2, ...)^{\bar{f}_{s+1}} = (g_2, g_3, ...).$$

Consider the $\mathbf{F}$-core

$$\langle L \leqslant \cap_{i=1}^{s+1} L_i \mid L \lhd G^\omega, L^{\bar{f}_i} \leqslant L, \forall i = 1, ..., s+1 \rangle.$$

Taking a subgroup L normal in G^ω and $\bar{f}_i$-invariant, observe that successive aplications of the endomorphisms gives us that L must be trivial, so as the $\mathbf{F}$-core. By Proposition 1.3.3, the group G^ω is self-similar with respect to the data $(\overline{\mathbf{m}}, \overline{\mathbf{H}}, \overline{\mathbf{F}})$, where $\overline{\mathbf{m}} = (m_1, ..., m_s, 1)$, $\overline{\mathbf{H}} = (L_1, ..., L_s, L_{s+1})$ and $\overline{\mathbf{F}} = (\bar{f}_1, ..., \bar{f}_s, \bar{f}_{s+1})$.

(2) Now consider $G = \mathbb{Z}$ and the endomorphism $f : 2\mathbb{Z} \to \mathbb{Z}$ defined by $2k \mapsto k$. Note that f is simple. By (1), the Baer-Specker group $\mathbb{Z}^\omega$ is self-similar with respect to the data $((2, 1), (H, \mathbb{Z}^\omega), (f_1, f_2))$ where

$$H = \{(a_n)_{n \in \mathbb{N}} \mid a_0 \in 2\mathbb{Z}, a_n \in \mathbb{Z}, n > 0\}$$

and the endomorphisms f_1 and f_2 are given by

$$f_1 : H \longrightarrow \mathbb{Z}^\omega \qquad\qquad f_2 : \mathbb{Z}^\omega \longrightarrow \mathbb{Z}^\omega$$

$$(a_n)_{n \in \mathbb{N}} \longmapsto (a_0^f, 0, 0, 0, \dots), \qquad (a_n)_{n \in \mathbb{N}} \longmapsto (a_{n+1})_{n \in \mathbb{N}}.$$

On defining the transversals $T_1 = \{(0,0,0,\dots),(1,0,0,\dots)\}$ and $T_2 = \{(0,0,0,\dots)\}$ of H and $\mathbb{Z}^\omega$ in $\mathbb{Z}^\omega$ we obtain the following concrete representation of $\mathbb{Z}^\omega$ in $\mathcal{A}_3$

$$\mathbb{Z}^\omega \simeq \{\alpha_0^{a_0} \alpha_1^{a_1} \alpha_2^{a_2} \cdots \alpha_n^{a_n} \cdots \mid (a_n)_{n \in \mathbb{N}} \in \mathbb{Z}^\omega\},$$

where

$$\alpha_0 = (e, \alpha_0, e)(1\,2), \alpha_n = (e, e, \alpha_{n-1}), \text{ for } n = 1, 2, 3, \dots.$$

■

Notice that the above infinite product $\alpha_0^{a_0} \alpha_1^{a_1} \alpha_2^{a_2} \cdots \alpha_n^{a_n} \cdots$ is well-defined for any sequence $(a_n)_{n \in \mathbb{N}} \in \mathbb{Z}^\omega$. Since α_n acts nontrivially only from the $(n+1)$-th level onwards, the considered infinite product has, for each level of the tree, only a finite number of automorphisms acting on it, resulting in a well-defined action on each level. Also, this is true for the topological (and the diagonal-topological) closure both in transitive and intransitive cases, as proven in [[DSS23], Theorem A] that A^* and $\overline{\Delta(A)}$ are well defined.

It is natural then to state the following corollary:

Corollary 3.1.1. *If K is a subgroup of G^ω such that $(K \cap L_i)^{\bar{f}_i} < K, i = 1, \dots, s$ and $K^{\bar{f}_{s+1}} < K$, then K is self-similar. In particular, the free abelian group of uncountable rank S is self-similar.*

Proof. Let K be a subgroup as stated and the G^ω-data $(\overline{\mathbf{m}}, \overline{\mathbf{H}}, \overline{\mathbf{F}})$. Observe that the subgroups $(K \cap L_i)$ have finite index in K. As in the previous theorem, we apply successively the endomorphisms $\bar{f}_1, \dots, \bar{f}_s$ to elements of the form $(k', k_2, k_3, \dots)$, where $k' \in (K \cap L_i)$ and $k_i \in K$. Then

$$(k', k_2, k_3, \dots)^{\bar{f}_i} = ((k')^{f_i}, e, e, e, \dots),$$

and also, applying $\bar{f}_{s+1}$, that is the shift operator, we see that the resulting $\bar{\mathbf{F}}$-core

$$\langle L \leqslant \cap_{i=1}^{s+1}(H \cap L_i) \mid L \lhd K, \ L^{\bar{f}_i} \leqslant L, \forall i = 1, ..., s+1 \rangle$$

is trivial, and thus K is self-similar. Now, consider the uncountable free abelian group

$$S = \{a = (a_n)_{n \in \mathbb{N}} \in \mathbb{Z}^\omega \mid |a_n| \leq N_a, N_a \in \mathbb{N}, \forall n \in \mathbb{N}\}$$

(refer to [Fuc70, Section 97] for a proof that S is an uncountable free abelian group, also to end of section 2.2 of the last chapter for more information). Considering a subgroup H of index 2 in S (such as the subgroup of sequences beginning with 0) such that $(H \cap S)^{f_1} \leq H$, $H^{f_2} \leq H$, we have that S is self-similar with respect to the S-data $((2,1), (S \cap H, S), (f_1, f_2))$.

Furthermore, applying the Kaloujnine-Krasner theorem we obtain the following concrete representation of S in $\mathcal{A}_3$

$$S \simeq \langle \alpha_0^{a_0} \alpha_1^{a_1} \alpha_2^{a_2} \cdots \alpha_n^{a_n} \cdots \mid a = (a_n)_{n \in \mathbb{N}} \in S \rangle,$$

where

$$\alpha_0 = (e, \alpha_0, e)(1\,2), \alpha_n = (e, e, \alpha_{n-1}), \ \text{for } n = 1, 2, 3, \ldots.$$

∎

As an important example that motivates our work, we present in detail, as in [DS18], $\mathbb{Z}^{(\omega)}$ embedded in a also transitive self-similar group, namely $\mathbb{Z}^{(\omega)} \wr C_2$. It will follow this example a corollary of our Theorem B that generalizes this result.

Example 3.1.2. Let $G = \mathbb{Z}^{(\omega)} \wr C_2$, with $C_2 = \langle \sigma \rangle$, $B = \mathbb{Z}^{(\omega)} \times \mathbb{Z}^{(\omega)}$ and consider $H \leq G$ given by

$$H = \langle \ ((2n_1, n_2, n_3, \ldots), e), \ (e, (m_1, m_2, m_3 \ldots)) \ \rangle.$$

Denote the two generators of H by x and y, respectively. Notice that $|B : H| = 2$ and

$|G : H| = 4$. Then, define the endomorphism f as

$$f : H \longrightarrow G$$

$$x \longmapsto ((n_1, n_2, n_3, \ldots), e)$$

$$y \longmapsto (e, (m_2, m_3, m_4, \ldots))$$

Now, consider the following group G:

$$G = \langle\ ((\underbrace{0, 0, \ldots, 0}_{i-1}, 1, 0, \ldots), e)\ ,\ \sigma \mid i = 1, 2, 3, \ldots\ \rangle,$$

where we define $x_i = ((\underbrace{0, 0, \ldots, 0}_{i-1}, 1, 0, \ldots), e)$ and $y_i = (e, (\underbrace{0, 0, \ldots, 0}_{i-1}, 1, 0, \ldots))$, both with the only nonzero entry on the i-th position.

Choose a transversal $T = \{e, \sigma, x_1, x_1\sigma\}$ of H in G; observe that $H\sigma x_1 = Hy_1\sigma = H\sigma$ and $H\sigma y_1 = Hx_1\sigma$. Now, calculating $\varphi : G \to \mathcal{A}_4$:

for $i = 1$:

$$x_1^\varphi = ((ex_1x_1^{-1})^{f\varphi}, (\sigma x_1 \sigma)^{f\varphi}, (x_1x_1e)^{f\varphi}, (x_1\sigma x_1\sigma x_1^{-1})^{f\varphi})(13)$$

$$= (e, e, x_1^\varphi, e)(13)$$

$$y_1^\varphi = ((ey_1e)^{f\varphi}, (\sigma y_1\sigma x_1^{-1})^{f\varphi}, (x_1y_1x_1^{-1})^{f\varphi}, (x_1\sigma y_1\sigma)^{f\varphi})(24)$$

$$= (e, e, e, x_1^\varphi)(24);$$

for $i \geq 2$:

$$x_i^\varphi = ((ex_ie)^{f\varphi}, (\sigma x_i\sigma)^{f\varphi}, (x_1x_ix_1^{-1})^{f\varphi}, (x_1\sigma x_i\sigma x_1^{-1})^{f\varphi})$$

$$= (x_i^\varphi, y_{i-1}^\varphi, x_i^\varphi, y_{i-1}^\varphi)$$

$$y_i^\varphi = ((ey_ie)^{f\varphi}, (\sigma y_i\sigma)^{f\varphi}, (x_1y_ix_1^{-1})^{f\varphi}, (x_1\sigma y_i\sigma x_1^{-1})^{f\varphi})$$

$$= (y_{i-1}^\varphi, x_i^\varphi, y_{i-1}^\varphi, x_i^\varphi).$$

Defining $x_i^\varphi = \alpha_i$ and $y_i^\varphi = \beta_i$ we have

$$\alpha_1 = (e, e, \alpha_1, e)(13), \ \beta_1 = (e, e, e, \alpha_1)(24) \text{ and}$$

$$\alpha_i = (\alpha_i, \beta_{i-1}, \alpha_i, \beta_{i-1}), \ \beta_i = (\beta_{i-1}, \alpha_i, \beta_{i-1}, \alpha_i), \text{ for } i \geq 2.$$

Therefore we have that $\langle \alpha_i, \ \beta_j \mid i, j = 1, 2, \ldots \rangle \simeq \mathbb{Z}^{(\omega)} \times \mathbb{Z}^{(\omega)} \simeq \mathbb{Z}^{(\omega)}$ is a state-closed and finite-state representation of $\mathbb{Z}^{(\omega)}$. Notice then, if we consider $C_2 = \langle \sigma = (12)(34) \rangle$, we obtain a transitive representation for $\mathbb{Z}^{(\omega)} \wr C_2$.

The following theorem deals with embeddings of self-similar groups, possibly intransitive, into transitive ones. A permutation group $K \leq Sym(\{1, \ldots, s\})$ is called *regular* if it is transitive and has order s.

Theorem 3.1.3. *[DSS21]   Let G be a self-similar group of degree m and orbit-type $(m_1, \ldots, m_s)$, also let K be a regular subgroup of $Sym(\{1, \ldots, s\})$. Then $G \wr K$ admits a faithful transitive self-similar representation of degree $m_1.m_2.\ldots.m_s.s$.*

Now, we apply this embedding to the groups considered in this work.

Corollary 3.1.4. *If G is a self-similar group, then $G \wr C_r$ is transitive self-similar group. In particular, $\mathbb{Z}^\omega \wr C_2$ and $S \wr C_2$ are transitive self-similar of degree 6.*

Proof. The corollary follows directly from Theorem B and the above theorem. ∎

3.2   The non-transitivity of the Baer-Specker Group

In this final section we establish that, despite being self-similar, the Baer-Specker group does not afford a *transitive* self-similar representation. Therefore, this is the first example of a torsion-free abelian self-similar group that is not transitive. As a result with its own interest, we prove that the Baer-Specker group does not embed in a direct product of $\mathbb{Z}_{p_i}$, with p_i prime numbers.

In the discussion of the results from now on, since for any integer m we can consider the prime decomposition of $m = \prod_{1 \leq i \leq s} p_i^{k_i}$, we have the decomposition $\mathbb{Z}_m = \bigoplus_{1 \leq i \leq s} \varepsilon_i \mathbb{Z}_{p_i^{k_i}}$, where ε_i are the orthogonal idempotents such that $1 = \sum_{1 \leq i \leq s} \varepsilon_i$. Thus, it will suffice to

establish our results to direct sums of the group of the p-adic integers for some primes. For more, see [AF92].

To achieve our results, first we are interested in the homomorphisms $f : \mathbb{Z}^\omega \to \mathbb{Z}_p$. Let I be a an indexing set, $e_i \in \mathbb{Z}^\omega$ elements with 1 in the i-th coordinate and 0 on every other coordinate. For a countable sequence of indices $i_1, i_2, \ldots$ and $b_1, b_2, \cdots \in \mathbb{Z}$, the expected condition

$$f \left(\sum_{j=1}^\infty b_j e_{i_j} \right) = \sum_{j=1}^\infty b_j f(e_{i_j}) \tag{$*$}$$

may fail even if the right side converges. There is, however, a specific circumstance where it holds.

Lemma 3.2.1. *If $b_j \to 0$ in $\mathbb{Z}_p$, then ($*$) holds.*

Proof. It is clear that if $b_j \to 0$ then the series $\sum b_j f(e_{i_j})$ converges for any group homomorphism f and any choice of indices i_j. Moreover, for any $k \geq 0$, there exists $N > 0$ such that if $j \geq N$ then $b_j = p^k c_j$ with $c_j \in \mathbb{Z}$.

Hence, we have:

$$
\begin{aligned}
f \left(\sum_{j=1}^\infty b_j e_{i_j} \right) - \sum_{j=1}^\infty b_j f(e_{i_j}) &= f \left(\sum_{j=N}^\infty b_j e_{i_j} \right) - \sum_{j=N}^\infty b_j f(e_{i_j}) \\
&= f \left(p^k \sum_{j=N}^\infty c_j e_{i_j} \right) - p^k \sum_{j=N}^\infty c_j f(e_{i_j}) \\
&= p^k \left(f \left(\sum_{j=N}^\infty c_j e_{i_j} \right) - \sum_{j=N}^\infty c_j f(e_{i_j}) \right) \\
&= 0 \mod p^k.
\end{aligned}
$$

Since the difference between the left and right sides of ($*$) is congruent to $0 \mod p^k$ for every $k \geq 0$, it must be equal to zero in $\mathbb{Z}_p$. ∎

Theorem C. *The Baer-Specker group $\mathbb{Z}^\omega$ does not embed in $\mathbb{Z}_p$, for any prime p.*

Proof. We prove that for any homomorphism $f : \mathbb{Z}^\omega \to \mathbb{Z}_p$, the kernel of f is nontrivial. If $f(e_i) = 0$ for some index i, we are done. Otherwise, assume that $f(e_i) \neq 0$ for all $i \in I$

and write $f(e_i) = p^{a_i} u_i$ where $a_i \geq 0$ and $u_i \in \mathbb{Z}_p^\times$. There are two possibilities to consider.

Case 1: There are infinitely many indices $i_1, i_2, \dots$ with $a_{i_j} = C$ constant. In this case, observe that the closed subgroup generated by any one of the $f(e_{i_j})$ is precisely $p^C \mathbb{Z}_p$. In particular, all the $f(e_{i_j})$ generate the same subgroup modulo p^k for every $k \geq 0$. We inductively choose elements $b_2, b_3, \dots \in \mathbb{Z}_{\geq 0}$ such that

$$f(e_{i_1}) \equiv b_2 f(e_{i_2}) \qquad\qquad (\mathrm{mod}\ p^{C+1})$$

$$f(e_{i_1}) - b_2 f(e_{i_2}) \equiv b_3 f(e_{i_3}) \qquad\qquad (\mathrm{mod}\ p^{C+2})$$

$$f(e_{i_1}) - b_2 f(e_{i_2}) - b_3 f(e_{i_3}) \equiv b_4 f(e_{i_4}) \qquad\qquad (\mathrm{mod}\ p^{C+3})$$

$$\vdots$$

$$f(e_{i_1}) - \sum_{j=2}^n b_j f(e_{i_j}) \equiv b_{n+1} f(e_{i_{n+1}}) \qquad\qquad (\mathrm{mod}\ p^{C+n})$$

From our choice of b_j, we have:

$$f(e_{i_1}) \equiv b_2 f(e_{i_2}) + \cdots + b_n f(e_{i_n}) \quad (mod\, p^{C+n-1}), \tag{3.1}$$

from which follows that

$$b_{n+1} \in p^{n-1}\mathbb{Z} \tag{3.2}$$

for every n.

We now consider the element $e = \sum_{j=2}^\infty b_j e_{i_j}$ of $\mathbb{Z}^\omega$. By 3.2 and Lemma 3.2.1, we have

$$f(e) = \sum_{j=2}^\infty b_j f(e_{i_j}).$$

Moreover, it is clear that the element $e_{i_1} - e$ is nonzero, since it has a nontrivial projection

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onto the i_1-th factor. However:

$$f(e_{i_1}) - f(e) = f(e_{i_1}) - \sum_{j=2}^{n} b_j f(e_{i_j})$$

$$\equiv f(e_{i_1}) - b_2 f(e_{i_2}) - \cdots - b_n f(e_{i_n}) \pmod{p^{n-1}}, \text{by } (3.2)$$

$$\equiv 0 \pmod{p^{n-1}}, \text{by } (3.1).$$

Hence, since $f(e_{i_1} - e)$ is zero mod p^{n-1} for all n, it must be identically zero in $\mathbb{Z}_p$. We have therefore built a nonzero element $e_{i_1} - e$ in $\ker f$.

Case 2: For each $a \geq 0$, there are only finitely many indices i with $a_i = a$. We can then find a sequence of indices $i_1, i_2, \ldots$ for which the sequence $a_{i_1} < a_{i_2} < a_{i_3} < \ldots$ is strictly increasing. Once again, choose $b_2, b_3, \cdots \in \mathbb{Z}_{\geq 0}$ inductively such that

$$p^{a_{i_2}} f(e_{i_1}) \equiv b_2 f(e_{i_2}) \qquad \pmod{p^{a_{i_2} + a_{i_3}}}$$

$$p^{a_{i_2}} f(e_{i_1}) - b_2 f(e_{i_2}) \equiv b_3 f(e_{i_3}) \qquad \pmod{p^{a_{i_3} + a_{i_4}}}$$

$$\vdots$$

$$p^{a_{i_2}} f(e_{i_1}) - \sum_{j=2}^{n} b_j f(e_{i_j}) \equiv b_{n+1} f(e_{i_{n+1}}) \qquad \pmod{p^{a_{i_{n+1}} + a_{i_{n+2}}}}$$

From our choice of b_j, we have

$$p^{a_{i_2}} f(e_{i_1}) \equiv \sum_{j=2}^{n} b_j f(e_{i_j}) \pmod{p^{a_{i_n} + a_{i_{n+1}}}}, \tag{3.3}$$

from which it follows that

$$b_{n+1} \in p^{a_{i_n}} \mathbb{Z} \tag{3.4}$$

for all n.

Let $e = \sum_{j=2}^{\infty} b_j e_{i_j}$. By 3.4 and Lemma 3.2.1, we have

$$f(e) = \sum_{j=2}^{\infty} b_j f(e_{i_j}).$$

Once again the element $p^{a_{i_2}} e_{i_1} - e$ is nonzero, and we find

$$f(p^{a_{i_2}} e_{i_1}) - f(e) = p^{a_{i_2}} f(e_{i_1}) - \sum_{j=2}^{n} b_j f(e_{i_j})$$

$$\equiv p^{a_{i_2}} f(e_{i_1}) - b_2 f(e_{i_2}) - \cdots - b_n f(e_{i_n}) \ (mod \ p^{a_{i_n}}), \text{by } (3.4)$$

$$\equiv 0 \ (mod \ p^{a_{i_n}}), \text{by } (3.3).$$

Hence, since $\mathbb{Z}_p \simeq \varprojlim \mathbb{Z}_p / p^{a_{i_n}} \mathbb{Z}_p$, $p^{a_{i_2}} e_{i_1} - e$ is our desired element in $ker f$. ∎

In possession of this theorem, we generalize it to a result that holds its own interest.

Theorem D. *There is no embedding of the Baer-Specker group in a finite direct product* $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$, *where* $p_i, 1 \leq i \leq n$ *are prime numbers.*

Proof. Suppose there is an embedding of $\varphi : P \hookrightarrow \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$, and we will denote $P \simeq P^{\varphi}$. First we prove that, if H is a finite index subgroup of $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$, then there is a product topology in P^{φ} such that $H \cap P^{\varphi}$ is a union of open sets of P^{φ} in this topology. In fact, the subgroup $K = H \cap P^{\varphi}$ has finite index in P^{φ}. Let $x_1, \ldots, x_s \in P^{\varphi}$ such that $P^{\varphi} = \dot{\cup}_{i=1}^{s}(x_i + K)$. By 1.4.3, there exists a minimal r such that $\langle x_1, \ldots, x_s \rangle$ is a finite index subgroup in $\mathbb{Z}^r$ and $P^{\varphi} = \mathbb{Z}^r \oplus P_1$, where $P_1 \simeq P$ and $P_1 \leq K$. Furthermore, if $(y_1, y_2, \ldots, y_r, \ldots) \in K$ is in the same coordinate system of $\mathbb{Z}^r \oplus P_1$, then $(y_1, y_2, \ldots, y_r, \ldots) \in \{y_1\} \times \cdots \times \{y_r\} \times P_1$ and K is a union of open sets in the product topology of $P^{\varphi} = \mathbb{Z}^r \oplus P_1$, in this coordinate system.

Now, consider the projection $\pi_1 : \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n} \to \mathbb{Z}_{p_1}$. We proceed to prove that $ker (\pi_1|_{P^{\varphi}})$ is closed in some product topology in P^{φ}. Notice that $ker \ \pi_1$ is closed in $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$; thus $(ker \ \pi_1)^{\complement}$ is open. Also,

$$(ker \ (\pi_1|_{P^{\varphi}}))^{\complement} = P^{\varphi} \cap (ker \ \pi_1)^{\complement}.$$

As $(ker \ \pi_1)^{\complement}$ is open in $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$, there is a finite index subgroup H in $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$ such that

$$(ker \ \pi_1)^{\complement} = \bigcup_{i=1}^{t}(z_i + H).$$

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Since

$$P^\varphi \cap \bigcup_{i=1}^{t}(z_i + H) = \bigcup_{i=1}^{t}(P^\varphi \cap (z_i + H)),$$

the intersections $P^\varphi \cap (z_i + H)$, $i = 1, \ldots, t$, can be treated as in the first half of the proof and all will be open in the same product topology in the same coordinate system.

Hence,

$$P^\varphi \cap (ker\ \pi_1)^{\complement} = (ker\ \pi_1|_{P^\varphi})^{\complement}$$

is open in some product topology and then $ker\ \pi_1|_{P^\varphi}$ is closed in some product topology.

For the next step, consider $ker\ \pi_1 = \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$. If $ker\ \pi_1|_{P^\varphi}$ has infinite rank in P^φ then, by Lemma 1.4.5, there is a copy of P in $ker\ \pi_1|_{P^\varphi} \leq \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$. Then, we proceed by induction and conclude that $P \leq ker\ \pi_1 \ldots \pi_n|_{P^\varphi} \leq \mathbb{Z}_{p_n}$ which, by Theorem C, can not occur. If $ker\ \pi_1|_{P^\varphi}$ has finite rank, then

$$P \lesssim \frac{P^\varphi}{ker\ \pi_1|_{P^\varphi}} \lesssim \mathbb{Z}_{p_1},$$

which again can not occur by Theorem C. In either case we have a contradiction and the result is proven. ∎

Proposition 3.2.2. *Let A be a torsion-free self-similar abelian group of degree $m \geq 2$. Then A is isomorphic to a subgroup of a torsion-free self-similar abelian group B of degree s, $2 \leq s \leq m$, where B is diagonally-topologically closed, that is, $B = B^*$ in $\mathcal{A}_s$.*

Proof. Given that A^* is torsion-free, then take $B = A^*$. If not, we have that $exp(Tor(A^*)) \mid m < \infty$ and $Tor(A^*)$ is a direct summand of A^*. Notice that $A_2 = \frac{A^*}{Tor(A^*)}$ is self-similar of degree $r_2 \mid r_1 = m$. Also, by Lemma 1.4.1 A is isomorphic to a subgroup of A_2. If A_2^* is torsion-free, take $B = A_2^*$, if not, repeat the process with A_2^*. The process must end in a finite number of steps, since if r_n is prime, then A_n^* is torsion-free. ∎

Theorem E. *The Baer-Specker group $\mathbb{Z}^\omega$ does not afford a transitive self-similar representation in $\mathcal{A}_m$, for any positive integer m.*

Proof. Let $A = \mathbb{Z}^\omega$. As shown in Theorem B, A is self-similar. Suppose by absurd that

A is also transitive. Consider that A^* is torsion-free; for if not, then by Proposition 3.2.2 we can take a group $B = B^*$ containing A and apply the arguments to follow.

As A^* is torsion-free, then by Theorem 1.2.8 A^* can be seen as a finitely generated $\mathbb{Z}_m$-module and as a pro-m group. Then, by Theorem 1.4.2, we can consider

$$A \leq A^* \simeq \bigoplus_{p|m} \left(\bigoplus_{s(p)} \mathbb{Z}_p \right),$$

for and $s(p) \leq d$ and, as we have a finite direct sum, $s(p) = 0$ for infinitely many primes. But, by Theorem D, an embedding of $A = \mathbb{Z}^\omega$ in such a sum can not occur. Thus, we reach a contradiction, and therefore, the Baer-Specker group does not have a transitive self-similar representation. ∎

3.3   A torsion-free transitive self-similar overgroup of the Baer-Specker group

In the light of the results of this last chapter, a natural question arises: *is it possible to embed the Baer-Specker group in a torsion-free transitive self-similar group?*

We proceed to provide such an embedding, with the additional property of being metabelian.

Theorem F. *Let G^ω be a self-similar group of orbit-type $(m_1, \ldots, m_s)$. Then the group $G^\omega \wr_Y \mathbb{Z}$, where $Y = \{1, \ldots, s\}$, is transitive self-similar of degree $m_1 \ldots \ldots m_s . s^2$. In particular, $\mathbb{Z}^\omega \wr_{\{1,2\}} \mathbb{Z}$ is a torsion-free transitive self-similar metabelian group of degree 8.*

Proof. Consider the G-data $(\mathbf{m}, \mathbf{L}, \mathbf{F})$ and $s^2\mathbb{Z} \leq \mathbb{Z}$; also, let $f_{s+1} : s^2\mathbb{Z} \to \mathbb{Z}$, $s^2 n \mapsto sn$ a virtual endomorphism. Now, considering the subgroup

$$H = L_1 \times \cdots \times L_s \times s^2\mathbb{Z},$$

we define $f : H \to (G^\omega)^s \times s\mathbb{Z} \leq (G^\omega) \wr_Y \mathbb{Z}$ as

$$h^f = (l_1^{f_1}, \ldots, l_s^{f_s}, (s^2 n)^{f_{s+1}}) = (l_1^{f_1}, \ldots, l_s^{f_s}, sn).$$

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Clearly the action of f_{s+1} does not leave any nontrivial element of $s^2\mathbb{Z}$ invariant; thus any f-invariant element of H must be in $L_1 \times \cdots \times L_s \times \{0\}$. Also, the action of $\mathbb{Z}$ is permutational (addition modulo s) on the coordinates $l_1, \ldots, l_s$; but by hypotheses we have that f_i, $1 \leq i \leq s$ are simple virtual endomorphisms in $\mathbf{F}$, and hence f that do not leave any subgroup $K \leq H$, $K \lhd G$ invariant. Thus $(G^\omega) \wr_Y \mathbb{Z}$ is transitive self-similar.

Now, for $\mathbb{Z}^\omega \wr_{\{1,2\}} \mathbb{Z}$, by Theorem B we have that $\mathbb{Z}^\omega$ is self-similar with respect to the $\mathbb{Z}^\omega$-data $((2,1), (H, \mathbb{Z}^\omega), (f_1, f_2))$ as defined there. Choosing the subgroup $2^2\mathbb{Z} \leq \mathbb{Z}$, we obtain the desired representation of degree 8.

Consider $G = (\prod_{i \in \mathbb{N}} \langle \alpha_i \rangle) \wr_{\{1,2\}} \langle z \rangle$, $H = \langle \alpha_1^2, \alpha_i \alpha_j^z, z^4 \mid i \geq 2, j \geq 1 \rangle$ and the transversal $T = \{e, \alpha_1, z, \alpha_1 z, z^2, \alpha_1 z^2, z^3, \alpha_1 z^3\}$ of H in G. Then, we obtain the following transitive self-similar representation of G:

$$\alpha_1 = (e, \alpha_1, e, e, e, \alpha_1, e, e)(12)(56),$$

$$\alpha_i = (\alpha_{i-1}, \alpha_{i-1}, \alpha_{i-1}^z, \alpha_{i-1}^z, \alpha_{i-1}, \alpha_{i-1}, \alpha_{i-1}^z, \alpha_{i-1}^z), i \geq 2,$$

$$\alpha_1^z = (e, e, e, \alpha_1, e, e, e, \alpha_1)(34)(78),$$

$$\alpha_i^z = (\alpha_{i-1}^z, \alpha_{i-1}^z, \alpha_{i-1}, \alpha_{i-1}, \alpha_{i-1}^z, \alpha_{i-1}^z, \alpha_{i-1}, \alpha_{i-1}), i \geq 2, \text{ and}$$

$$z = (e, e, e, e, e, e, z^2, z^2)(1357)(2468).$$

$\blacksquare$

Notice that, considering $A = \langle \alpha_1, \alpha_i, \alpha_1^z, \alpha_i^z, \ i \geq 2 \rangle$, we see that $\overline{\Delta(A)}$ is isomorphic to $\mathbb{Z}^\omega$ and thus the Baer-Specker group appears as a state-closed subgroup of the overgroup obtained above.

Final Comments

The goal of this work was to contribute with the understanding of self-similar abelian groups. As expected, new questions arise in the process of research and writing, also related problems existing in the literature become research goals in the short-term after the thesis.

The first problem to be cited, inspired by the prodigious theorem by G. Higman, that asserts that every finitely generated recursively presented group can be embedded as a subgroup of some finitely presented group; also [Hou07] proved that any countable abelian group is in the center of some finitely presented group. Then, we ask a variation related to our theme: *"Let G be a countable abelian group of automorphisms of the tree. Is it possible to embed G in some finitely presented self-similar group?"*

The second one arises naturally after this work and asks about the self-similarity of the group

$$\mathbb{Z} \, Wr \, \mathbb{Z} \simeq \left(\prod_{\mathbb{Z}} \mathbb{Z} \right) \rtimes \mathbb{Z}.$$

The third one, where the authors in [DSS23] ask *if there is a torsion-free abelian group of automorphisms of the tree that is not self-similar.* This problem is connected and relevant to the present work since, after some investigation, it is possible that such group is a subgroup of the Baer-Specker group $\mathbb{Z}^{\omega}$. Some papers lead us to search for some subgroup with erratic, or "pathological" behaviour. In the [DG85] the authors prove that $\mathbb{Z}^{\omega}$ has subgroups that $G \simeq G \oplus G \oplus G$, but $G \not\simeq G \oplus G$. Also, the techniques in the paper [Ric20], with the suggestive title of "Pathological Abelian Groups: a friendly example", hints us a direction for the solution.

The construction that follows below aims on the previous problem, but has its own

relevance, since it extends Theorem 1.2.10 to representations any degree. As a possible generalization of the definition of the subgroups $D_m(j)$ is the following. Let $P \leq Sym(m)$ be an abelian permutation group without fixed points generated by $P_1, \ldots, P_s$, where $P_i \leq Sym(O_{m_i})$, $O_{m_i} = \{o_{i,1}, \ldots, o_{i,m_i}\}$ being the orbit of size m_i. Given the decomposition of P_i in cyclic subgroups of order $p_{it}^{r_{it}}$, $1 \leq t \leq k_i$, we have a corresponding decomposition of each $m_i = p_{i1}^{r_{i1}} \ldots p_{ik_i}^{r_{ik_i}}$. Then we can write $P_i = \langle \sigma_{p_{i1}} \rangle \oplus \cdots \oplus \langle \sigma_{p_{ik_i}} \rangle$. Also, choosing a prime p_{it_i}, we write $m_i = q_i p_{it_i}^{r_{it_i}}$. We can rearrange the terms to write each σ_{it_i} as a product of cycles

$$
\sigma_{it_i} = \overbrace{\underbrace{(o_{i,1} \ldots o_{i,p_{it_i}^{r_{it_i}}})}_{p_{it_i}^{r_{it_i}}} \ldots \underbrace{(o_{i,m_i - p_{it_i}^{r_{it_i}} + 1} \ldots o_{i,m_i})}_{p_{it_i}^{r_{it_i}}}}^{q_i}.
$$

Now we define the automorphisms $\alpha_{\sigma_{it_i}} = (\alpha_1, \alpha_2, \ldots, \alpha_m)\sigma_{it_i} = (\alpha_l)_{l \in \{1,\ldots,m\}}\sigma_{it_i}$, where

$$
\alpha_l = \begin{cases} \alpha_{\sigma_{it_i}}^{x_i^{j_{i,t_i}-1}}, & \text{if } l \in O_{m_i} \text{ and } l = o_{i,sp_{i,t_i}^{r_i t_i}}, \ s \in \mathbb{N} \\ e, & \text{otherwise.} \end{cases}
$$

Then, we define the groups

$$
D(\sigma_{it_i}, j_{it_i}) = \langle \alpha_{\sigma_{it_i}}, \alpha_{\sigma_{it_i}}^{x_i}, \ldots, \alpha_{\sigma_{it_i}}^{x_i^{j_{i,t_i}-1}} \rangle.
$$

Also we define

$$
D(P_i, \mathbf{j}_i) = \langle D(\sigma_{it_i}, j_{it_i}) \mid 1 \leq t_i \leq k_i \rangle,
$$

where $\mathbf{j}_i = (j_{1t_1}, \ldots, j_{ik_i})$.

Finally, we construct

$$
D(P, \mathbf{J}) = \langle D(P_i, \mathbf{j}_i) \mid 1 \leq i \leq m_i \rangle,
$$

where $\mathbf{J} = (\mathbf{j}_1, \ldots, \mathbf{j}_{m_i})$.

Theorem 3.3.1. *The following affirmations hold:*

i) $D(\sigma_{it_i}, j_{it_i})$, $D(P_i, \mathbf{j}_i)$ and $D(P, \mathbf{J})$ are self-similar torsion-free abelian groups;

ii) $D(P_i, \mathbf{j}_i) = \bigoplus D(\sigma_{it_i}, j_{it_i})$ *and* $D(P, \mathbf{J}) = \bigoplus D(P_i, \mathbf{j}_i)$;

iii) $\overline{\Delta(D(P_i, \mathbf{j}_i))} = \bigoplus \overline{\Delta(D(\sigma_{it_i}, j_{it_i}))}$ *and* $\overline{\Delta(D(P, \mathbf{J}))} = \bigoplus \overline{\Delta(D(P_i, \mathbf{j}_i))}$;

iv) $\overline{\Delta(D(P, \mathbf{J}))} = \prod_{p_{it_i} \in \pi(m_i), 1 \le i \le s} \prod_\omega \mathbb{Z}_{p_{it_i}}^{(j_{it_i})}$.

Illustrating this construction, we present now a concrete example.

Example 3.3.2. Let P be the intransitive permutation group generated by 3 elements as below:

$$P = \langle \sigma_1 = (1\ 4)(2\ 5)(3\ 6), \sigma_2 = (1\ 2\ 3)(4\ 5\ 6), \sigma_3 = (7\ 8\ 9) \rangle.$$

Then we compute the groups $D_p(\sigma_i, j_i)$ for chosen $j_i \in \mathbb{N}$ fixed:

$$D(\sigma_1, j_1) = \langle \alpha = (e, e, e, \alpha^{x_1^{j_1-1}}, \alpha^{x_1^{j_1-1}}, \alpha^{x_1^{j_1-1}}, e, e, e)\sigma_1, \alpha^{x_1}, \alpha^{x_1^2}, \dots, \alpha^{x_1^{j_1-1}} \rangle$$
$$D(\sigma_2, j_2) = \langle \beta = (e, e, \beta^{x_1^{j_2-1}}, e, e, \beta^{x_1^{j_2-1}}, e, e, e)\sigma_2, \beta^{x_1}, \beta^{x_1^2}, \dots, \beta^{x_1^{j_2-1}} \rangle,$$
$$D(\sigma_3, j_3) = \langle \gamma = (e, e, e, e, e, e, e, e, \gamma^{x_2^{j_3-1}})\sigma_3, \gamma^{x_2}, \gamma^{x_2^2}, \dots, \gamma^{x_2^{j_3-1}} \rangle$$

and by Theorem 3.3.1, item *iv)*, we conclude

$$\overline{\Delta(D(P, ((j_1, j_2), (j_3))))} \simeq \prod_\omega (\mathbb{Z}_2^{(j_1)} \times \mathbb{Z}_3^{(j_2)} \times \mathbb{Z}_3^{(j_3)})$$

Then, the idea is to prove that, if A is a self-similar torsion-free abelian group in $\mathcal{A}_m$, where in A^* these conditions also hold, then A must embed in $\overline{\Delta(D(P, \mathbf{J}))}$, where P is the induced permutation group of A. Hence if we find such a group that do not satisfy this embedding, our example is found. However, given the size of $\overline{\Delta(D(P, \mathbf{J}))}$, this task be may hard to tackle.

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