Topological methods applied to differential equations models of population dynamics

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Resumo

Este trabalho apresenta alguns resultados sobre métodos topológicos e algumas aplicações na análise da existência e comportamento de soluções perto de singularidades e degenerações de algumas equações diferenciais parciais elípticas não lineares. Em concreto, aplicaremos nossos resultados às equações do tipo Schrödinger e do tipo Carrier, assim como à equação logística com refúgio. Na primeira delas, mostraremos resultados de existência de solução dependendo de dois parâmetros. Para a segunda, estudaremos o comportamento assintótico das soluções quando a difusão da espécie é muito grande em uma zona de seu habitat, e quando existe uma zona de degradação do domínio muito forte.

Resumen

Este trabajo presenta algunos resultados sobre métodos topológicos y algunas aplicaciones en el análisis de la existencia y comportamiento de soluciones cerca de singularidades y degeneraciones de algunas ecuaciones parciales elípticas no lineales. En concreto, aplicaremos nuestros resultados a las ecuaciones del tipo Schrödinger y del tipo Carrier, así como a la ecuación logística con refugio. En la primera de ellas, mostraremos resultados de existencia de solución dependiendo de dos parámetros. Para la segunda, estudiaremos el comportamiento asintótico de las soluciones cuando la difusión de la especie es muy grande en una zona de su hábitat, y cuando existe una zona de degradación del dominio muy fuerte.

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Introduction

The main aim of this thesis is to study some classical and important classes of multi-parameter problems involving models of elliptic partial differential equations arising in population dynamics, using topological tools that have enabled significant advances in understanding the existence, non-existence, and behavior of positive solutions with respect to the parameters.

The book [16] provided a foundational stimulus for authors examining mathematical models of population dynamics, specially reaction-diffusion models. Undoubtedly, problems involving the logistic model (one of the most paradigmatic models in population dynamics) has been widely studied in the last decades in partial differential equations. In this work, we present some results in this line.

When analyzing how variations of parameters in reaction-diffusion models of the logistic equation affect the problem in existence and behavior of positive solutions, some technical difficulties arise. By depending on which term of the equation is the varying parameter associated with, these difficulties may be very significant because it can reflect in singularities or degeneracies in the equation. Our contributions are principally in this direction. To overcome this problem, we provided some abstract results on eigenvalue problems and topological methods in Chapter 1, 2 and 4.

By topological methods, we understand methods that strongly rely on properties that are invariant under constant perturbations. The term "topological methods" in the literature of partial differential (or difference) equations is first [37] (Krasnoselskii, 1956), later [50] (Nirenberg, 1981) then [30] (Du, 2006), [48] (Montreanu, Montreanu and Papageorgiou, 2014) and lastly, in difference equations [10] (Balanov, García-Azpeitia and Krawcewicz, 2018). Nirenberg in [50], quoted [13], [33] and [49] as mathematicians that made important contributions to the field of topological methods.

The first reference quoted in the above paragraph, Krasnoselskii [37], is certainly a pioneer in bifurcation theory, a powerful tool for solving a large class of partial differential equations. In this book, the author mentioned that the concept of bifurcation comes from the stability theory, which dates back the works of Lyapunov and Poincaré. Krasnoselskii was the first researcher to propose a well-known definition of a bifurcation point (in a local sense) and to establish sufficient conditions for a specific value $\lambda = \lambda_0$ to be a bifurcation point for an equation of the form $\Phi(\lambda, u) = 0$, where $\Phi : \mathbb{R} \times E \to E$ is a compact operator satisfying certain structure conditions and E is a Banach space.

In 1971, Crandall and Rabinowitz in [22] established another set of sufficient conditions to obtain local bifurcation for equations involving a regular operator $\Phi(\lambda, \cdot)$, with its derivative being a Fredholm operator satisfying a transversality condition. Also in 1971, Rabinowitz published the celebrated paper [52], which presented a global version of Krasnosel'skii's, now known as Global Bifurcation Alternative of Rabinowitz. Dancer proved in [25] that the statement of the Global Bifurcation Alternative of Rabinowitz is actually stronger than that one done in [52] by bringing up more accurate information about the set of bifurcation points.

Most of the above-mentioned results have a common restriction on the structure of the operator Φ, requiring it to have a part $\mathcal{L}(\lambda)$ that approximates $\Phi(\lambda, \cdot)$, up to a rest $o(\|u\|)$ at $u = 0$. More specifically, most of the above-mentioned results considered the structure

$$\mathcal{L}(\lambda) = I - \lambda L,$$

where L is a compact linear operator. However, López Gómez in [43] complemented Dancer's theorem by allowing $\mathcal{L}(\lambda)$ to be an Fredholm operator with index 0 at $\lambda = \lambda_0$, besides assuming more general assumptions on the regularity of the operator $\lambda \mapsto \mathcal{L}(\lambda)$, while Dai in [24] also proved similar results by weakening the linearity near to $u = 0$, assuming just a homogeneity assumption.

Many different related-bifurcation results have been presented in the literature in recent years allowing for the approach of larger and more sophisticate structure of problems and their associated operators. Among these, we mention, for instance, the structure of nonlinear Sturm-Liouville problems in [51], nonlinear eigenvalue problems in [6], quasilinear problems in [8], $k-$Hessian equation in [24], and others. An important contribution to these efforts is the development of results aimed at solving equations whose operators cannot be defined over the whole space, that is, the space formed by the Cartesian product of $\mathbb{R}$ (where the bifurcation parameter varies) with the underlying Banach space, but it is well-defined just in some open subset of this product space.

In 1971, a version of the Global Alternative Bifurcation Theorem for operators defined just in a bounded open subset was presented in [52] (see Corollary 1.12). There is a number of works exploring this result in different directions. In our purpose, Dai improved it in [23] (2017) to closed subsets that are not necessarily bounded (besides other generalizing features), while Shi and Wang, in [56] (2009), proved a version of this result for regular Fredholm operators with index 0 defined on an open subset that is not necessarily bounded.

In terms of the arguments to prove versions of global alternatives type for operators defined on the whole space versus those ones defined on open subset, the main requirement on the technique is the homotopy property should work for the open subset. In other words, to extend the proof of global alternatives results to more general types of domain, the differences in the structure of the operators in the aforementioned results (Fredholm with index zero as in [56], compact and having a linear part as in [51] or compact and having a homogeneous part as in [23]) are handled to ensure that their structure still satisfies the homotopy property. Besides, whether the operator is defined on a closed or an open set, then the corresponding theorems show that the alternatives are adjusted to each type of domain, as seen, for example, in [23] and [56].

In [51], Rabinowitz dedicated a section to proving some results on existence of continua of solutions for nonlinear eigenvalue problems, where bifurcation does not necessarily need to occur, using the techniques he employed to prove bifurcation theorems. Inspired by [38], Rabinowitz proved Theorem 3.5 in [51] assuming, among others, a priori boundedness of the set of solutions of the operator $\Phi(\lambda,u)=0$ at $\lambda=\lambda_0$, denoted by B, to conclude that a bilateral continua of solutions emanates from some point of B. In Theorem 3.2 of [52], a version of this result was proved by adding a hypothesis on the operator that, in particular, implies that the set B is a singleton. Arcoya, Coster, Jeanjean and Tanaka stated in [9] a consequence of Theorem 3.5 of [51], which explicitly requires that the set B to be a singleton. As mentioned at the end of the previous paragraph, the form of the alternatives depends on the geometry nature of the set B, see, for instance, [9] when B is a singleton, [38] for B being amount finitely, and [51] for B being a bounded set. Finally, we note that all of the results mentioned in this paragraph consider the operator's domain as the whole space.

Although there are global alternative results in the literature for open subsets of the parameter-working space, they consider in general rather regular operators; see, for example, [56]. Here, we are interested in addressing operators with lower regularity, with the aim of restating the main results mentioned in the previous paragraph for open subsets of the parameter-working space. In this regard, we have inspired principally in Theorem 3.5 of [51], which established the existence of a continuum of solutions for the equation

$$0 = \Phi(\lambda, u) = u - K(\lambda, u)$$

with $K : \mathbb{R} \times E \to E$ being a compact operator under the additional assumption that B is a bounded set in E, where E is a Banach space.

For stating our Theorem 0.0.1, let $E = (E, \|\cdot\|)$ be a Banach space and $\mathcal{U} \subset \mathbb{R} \times E$ an open subset. Given $\theta \in \mathbb{R}$, define the θ-partition of $\mathcal{U}$ by

$$\mathcal{U}^\nu := \mathcal{U}^\nu_\theta = \mathbb{R}^\nu_\theta \cap \mathcal{U}, \text{ for } \nu \in \{+, -\},$$

which are relative open subsets, $\overline{\mathcal{U}}^\nu$ the closure of $\mathcal{U}^\nu$, and $\partial\mathcal{U}^\nu$ the boundary of $\mathcal{U}^\nu$ with the relative topology inherited from $\mathbb{R}^\nu_\theta \times E$, where

$$\mathbb{R}^\nu_\theta = \begin{cases} [\theta, +\infty) & \text{if } \nu = +, \\ (-\infty, \theta] & \text{if } \nu = -. \end{cases}$$

As in [2], $K : \mathcal{U} \to E$ will be called a compact operator in the open subset $\mathcal{U}$ if K is compact in every closed and bounded set $C \subset \mathcal{U}$ with $\text{dist}(\partial\mathcal{U}, C) > 0$. Moreover, $\Phi : \mathcal{U} \to E$ will denote the operator

$$\Phi(\lambda, u) := u - K(\lambda, u),$$

and

$$\Phi_\lambda(u) := \Phi(\lambda, u), \; u \in \mathcal{U}_\lambda,$$

where

$$\mathcal{U}_\lambda = \{u \in E; \; (\lambda, u) \in \mathcal{U}\} \text{ for each } \lambda \in \mathbb{R}.$$

Besides, for each $\lambda \in \mathbb{R}$ such that $\mathcal{U}_\lambda \neq \emptyset$, the number $i(\Phi_\lambda, u, 0)$ will denote the Leray-Schauder index of the isolated solution $u \in \mathcal{U}_\lambda$ of $\Phi_\lambda(u) = 0$. For more details about the Leray-Schauder index, we suggest the reference [36].

With these notations, we will look for a connected set $\mathscr{C}$ in

$$\mathscr{S} := \{(\lambda, u) \in \mathcal{U}; \; \Phi(\lambda, u) = 0\},$$

which will be split in two connect subsets of

$$\mathscr{S}^{\nu} := \mathscr{S}^{\nu}_{\theta} := \overline{\{(\lambda,u) \in \mathcal{U}^{\nu};\ \Phi(\lambda,u) = 0,\ \lambda \in \operatorname{int}\mathbb{R}^{\nu}_{\theta}\}}$$

that considers the closure of the set of solutions on the left and right sides of λ.

So, we are ready to state the below theorem.

Theorem 0.0.1 (Continuation Theorem). *Let $K : \mathcal{U} \subset \mathbb{R}\times E \to E$ be a continuous and compact operator in the open subset $\mathcal{U}$. Suppose that $(\lambda_0, u_0) \in \mathcal{U}$ is such that $u_0 \in \mathcal{U}_{\lambda_0}$ is an isolated solution of $\Phi_{\lambda_0}(u) = 0$ with index $i(\Phi_{\lambda_0}, u_0, 0) \neq 0$. Then the set $\mathscr{S}$ contains a pair of connected subsets $\mathscr{C}^{\nu} \subset \mathscr{S}^{\nu} = \mathscr{S}^{\nu}_{\lambda_0}$, for each $\nu \in \{-,+\}$, emanating from (λ_0, u_0) and satisfying one of the following (non-excluding) alternatives:*

i) $\mathscr{C}^{\nu}$ *is unbounded,*

ii) $\operatorname{dist}(\mathscr{C}^{\nu}, \partial\mathcal{U}) = 0$,

iii) $\mathscr{C}^{\nu}$ *meets* $(\lambda_0, u_1) \in \mathcal{U}$ *with* $u_1 \neq u_0$.

The below result highlights that the stronger compactness on K, the more accurate information we obtain on the alternative (ii) above.

Corollary 0.0.1. *Assume the assumptions of Theorem 0.0.1 hold with the compactness of the operator $K : \mathcal{U} \subset \mathbb{R}\times E \to E$ substituted by K is compact on any closed and bounded $C \subset \mathcal{U}$ given. Then the conclusion of Theorem 0.0.1 holds with*

ii)' $\overline{\mathscr{C}^{\nu}}^{E} \cap \partial\mathcal{U} \neq \emptyset$

in the place of the alternative ii).

More precisely, the above result is a corollary of the proof of Theorem 0.0.1 as the reader can see in (2.1).

We highlight at least two important complements of our Theorem 0.0.1 compared to similar ones on the literature, particularly to Theorem 3.5 of [51]. First, we allow the domain of the operator to be an open subset of $\mathbb{R}\times E$ (not necessarily bounded); second, we permit the set B to be unbounded with at least an isolated point, rather than requiring B to be a bounded set as in Theorem 3.5 of [51]. We also note that none of the referenced theorems are directly applicable to our applications (see (3.1.5) and (3.2.1)), since they do not allow singularities on their operators, nor does our operator possess sufficient regularity. For the sake of the clearness, let us summarize below some of the main contributions of the above theorem to the literature:

i) Theorem 0.0.1 (Continuation Theorem) provides a powerful tool that allows us to obtain a connected set of solutions to a wide class of problems that classical theorems cannot handle, due to the restrictions that appear naturally from these problems.

ii) Theorem 0.0.1 complements some previously related results in literature, for instance, by showing the existence of a continuum of solutions for problems that do not necessarily have a priori boundedness of solutions at the emanating-parameter point, as required in Theorem 3.5 of [51]. Also, it complements Theorem 2.2 of [9] by neither requiring that the solution from which emanates a continuum should be unique, nor assuming that the operator need to be well-defined on the whole parameter-working space.

The above Theorem can be useful for solving a large class of partial differential equations that presents some singularity in its structure, preventing the definition of the associated operator in the whole parameter-working space. This occurs when the associated operator must be constrained to a subset to be well-defined. In this direction, let us present new results regarding both the existence of classical positive solutions and qualitative information for the well-studied class of quasilinear Schrödinger equations

$$\begin{cases} -\Delta u - \lambda u \Delta u^2 = \mu u - u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $\lambda, \mu \in \mathbb{R}$, $p > 1$, and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 1$. In particular, we show that the diagram of solutions of the problem ($P_{\lambda,\mu}$) presents some similarity with the one of the problem ($P_{\lambda,\mu}$) with $\lambda = 0$ that is the classical logistic problem

$$\begin{cases} -\Delta u = \mu u - u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$P_{0,\mu}$}$$

Indeed, from the literature (see [32], for example), it is well known that ($P_{0,\mu}$) admits a positive solution u_0, which is unique if it exists, if and only if $\mu > \mu_1$, where $\mu_1 > 0$ stands for the first eigenvalue of $(-\Delta; H_0^1(\Omega))$. Moreover, $\|u_0\|_0 < \mu^{1/(p-1)}$, where $\|u\|_0 := \sup_{\overline{\Omega}} |u(x)|$, for each $u \in C(\overline{\Omega})$. Then for ($P_{\lambda,\mu}$), we have *i*) the result of existence of positive solutions, strictly bounded by $\mu^{1/(p-1)}$, for the problem ($P_{0,\mu}$), for each $\mu > \mu_1$, remains true for the problem ($P_{\lambda,\mu}$) for at least $\lambda > -1/(2\mu^{2/(p-1)})$, ,

ii) the non-existence of positive solutions strictly bounded by $\mu^{1/(p-1)}$ for the problem ($P_{0,\mu}$), for $0 \leq \mu \leq \mu_1$, stays true also for the problem ($P_{\lambda,\mu}$) for at least

$$\lambda \geq \frac{\mu - \mu_1}{2\mu_1 \mu^{2/(p-1)}},$$

In addition, if $1 < p \leq 3$, we proved a more sophisticated estimate for non-existence given by

$$\lambda \geq -\frac{1}{2\mu^{2/(p-1)}},$$

iii) there are no solutions for all $\lambda \in \mathbb{R}$ when $\mu < 0$.

See Proposition 3.1.3 for the above conclusions.

From now on, we will denote by

$$\text{proj}_{\|H\|} A := \{(\lambda, \|u\|_H) \in \mathbb{R}^2; (\lambda, u) \in A\},$$

and

$$\text{proj}_H A := \{(0, \|u\|_H) \in \mathbb{R}^2; (\lambda, u) \in A\},$$

where $A \subset \mathbb{R} \times H$ is a subset, and H is space endowed with the norm $\| \cdot \|_H$, while

$$\text{proj}_\lambda A = \{(\lambda, 0) \in \mathbb{R} \times H; (\lambda, u) \in A\}.$$

To state our next result, let us denote by

$$\mathcal{U} = \left\{(\lambda, u) \in \mathbb{R} \times C_0^1(\overline{\Omega});\ 1 + 2\lambda \|u\|_0^2 > 0\right\} \subset \mathbb{R} \times C_0^1(\overline{\Omega}),$$

and by

$$\partial\mathcal{U} = \left\{(\lambda, u) \in \mathbb{R} \times C_0^1(\overline{\Omega});\ 1 + 2\lambda \|u\|_0^2 = 0\right\} \subset \mathbb{R} \times C_0^1(\overline{\Omega}),$$

where $\|\cdot\|_0$ denotes the norm in $C(\overline{\Omega})$.

In Section 3.1, we apply the above theorem to obtain positive classic solutions of ($P_{\lambda,\mu}$). Precisely, we obtain the following

Theorem 0.0.2 (Quasilinear-Schrödinger-Logistic problem)**.** *Assume that $p > 1$ and $\mu > \mu_1$. Then there exists an unbounded connected set $\mathscr{C} = \mathscr{C}^- \cup \mathscr{C}^+ \subset \mathcal{U}$ of strongly-positive classical solutions (λ, u) of ($P_{\lambda,\mu}$) crossing the axis $\lambda = 0$ at $u = u_0$, where $u_0 \in \operatorname{int} P_{C_0^1(\overline{\Omega})}$ is the unique positive solution of ($P_{0,\mu}$), such that*

$$\Big(-1/(2\mu^{2/(p-1)}), +\infty\Big) \subset \operatorname{proj}_\lambda \mathscr{C} \subset \Big[-1/[2(\mu-\mu_1)^{2/(p-1)}], \infty\Big) \tag{0.0.1}$$

$$\operatorname{proj}_{C(\overline{\Omega})} \mathscr{C}^- \subset \Big[(\mu-\mu_1)^{1/(p-1)}, \mu^{1/(p-1)}\Big], \quad \operatorname{proj}_{C(\overline{\Omega})} \mathscr{C}^+ \subset \Big[0, \mu^{1/(p-1)}\Big], \tag{0.0.2}$$

$$\operatorname{dist}\big(\mathscr{C}^-, \partial\mathcal{U}\big) = \operatorname{dist}(\mathscr{C}, \partial\mathcal{U}) = 0 \tag{0.0.3}$$

and

$$\inf\{1 + 2\lambda \|u\|_0^2;\ (\lambda, u) \in \mathscr{C}^-\} = 0, \tag{0.0.4}$$

where $\mathscr{C}^- = \{(\lambda, u) \in \mathscr{C}; \lambda \leq 0\}$, and $\mathscr{C}^+ = \{(\lambda, u) \in \mathscr{C}; \lambda \geq 0\}$. In addition, the problem ($P_{\lambda,\mu}$) admits:

a) at least one strongly-positive solution $u \in \mathcal{U}$ for each

$$\lambda \in \Big(-1/(2\mu^{2/(p-1)}), +\infty\Big),$$

b) no positive solution $(\lambda, u) \in \mathcal{U}$ for any

$$\lambda \in \Big(-\infty, -1/[2(\mu-\mu_1)^{2/(p-1)}]\Big).$$

Moreover:

i) the problem ($P_{\lambda,\mu}$) admits at most one positive solution in $C_0^1(\overline{\Omega})$ for $\lambda \geq 0$. In particular, the set $\mathscr{C}^+$ is a continuous curve such that $\|u_\lambda\|_0 \to 0$ as $\lambda \to \infty$,

ii) there is no λ-bifurcation point of positive solutions from the trivial solution in the $C(\overline{\Omega})$-norm.

The below pictures show the possible behaviors for the $\operatorname{proj}_{\|C(\overline{\Omega})\|_0} \mathscr{C}$. The grey area represents the non-existence region of positive solutions bounded by $\mu^{1/(p-1)}$ for each $\mu > \mu_1$ given.

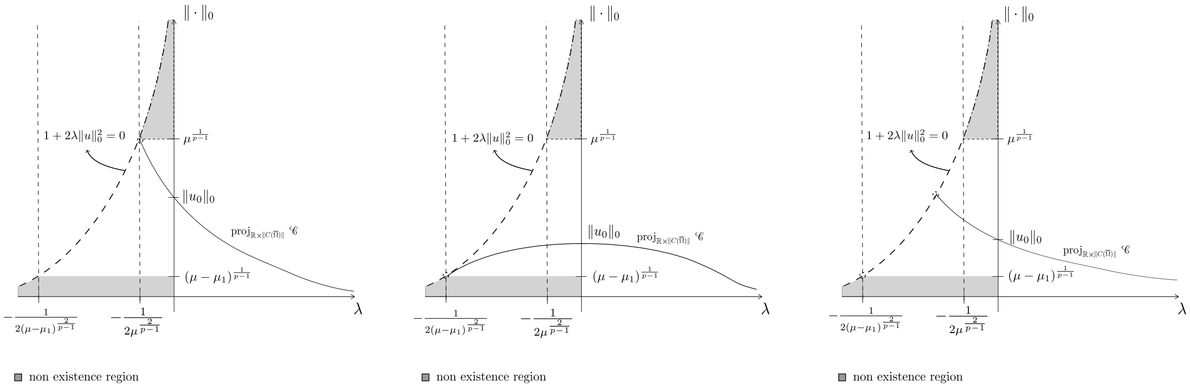

Fig. 1 Possible behaviors of $\mathscr{C}$

For the sake of completeness, we state the above item $i)$ of Theorem 0.0.2, a part of whose statements were already proved in [21]. Our results complement those ones by contributing principally to the existence and qualitative properties of the solutions for the negative λ-range.

Remark 0.0.1. *About Theorem 0.0.2:*

i) despite the fact that (0.0.3) holds true, we are not able to prove that there exists a solution $(\lambda, u) \in \mathscr{C}^- \cap \partial\mathcal{U}$ of the problem ($P_{\lambda,\mu}$) due to the $C^1(\overline{\Omega})$-estimate fails; however, we can infer from (0.0.4) that

$$\overline{\mathrm{proj}_{\|C(\overline{\Omega})\|}\mathscr{C}^-} \cap \left\{(\lambda, s) \in \mathbb{R}^2;\ 1+2\lambda s^2 = 0\right\} \neq \emptyset,$$

ii) $\mathscr{C}$ is connected in the $\mathbb{R}\times C(\overline{\Omega})$-norm as well, and it will be denoted by $\mathrm{proj}_{\|C(\overline{\Omega})\|_0}\mathscr{C}$. In fact, this claim follows from the connectedness of $\mathscr{C}$ in the $\mathbb{R}\times C_0^1(\overline{\Omega})$-norm combined with the continuous embedding $C_0^1(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$.

There are many different types of real-world phenomena that lead to models with nonlinear diffusion terms. One very known is that one presented by Carrier in [17], in the unidimensional case, to model transversal vibrations of elastic membranes. Inspired by this problem, let us introduce a problem that we will be termed by Carrier-type problem, more specifically,

$$\begin{cases} -(1+\lambda|u|_r^q)\Delta u = \mu u - u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \qquad (Q_{\lambda,\mu})$$

where $\lambda \in \mathbb{R}$ is a parameter, $q > 0$, $p > 1$, $r \geq 1$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 1$ and $|u|_r$ is the norm of u in the Lebesgue space $L^r(\Omega)$ of the r-integrable functions.

In Section 3.2, we apply Theorem 0.0.1 to obtain positive classic solutions of ($Q_{\lambda,\mu}$). To state our next Theorem, let us denote by

$$\mathcal{V} := \{(\lambda, u) \in \mathbb{R} \times (H_0^1(\Omega) \cap L^r(\Omega)); 1 + \lambda |u|_r^q > 0\} \subset \mathbb{R} \times H_0^1(\Omega).$$

Precisely, we obtain

Theorem 0.0.3 (Carrier-Type-Logistic problem)**.** *Assume $\mu > \mu_1$ and $p > 1$. Then, there exists an unbounded connected set $\mathscr{C} = \mathscr{C}^- \cup \mathscr{C}^+ \subset \mathcal{V}$ of strongly-positive classical solutions for the problem ($Q_{\lambda,\mu}$) that crosses the axis $\lambda = 0$ at $u = u_0$, where $u_0 \in \operatorname{int} P_{C_0^1(\overline{\Omega})}$ is the unique positive solution of ($Q_{\lambda,\mu}$) with $\lambda = 0$, such that*

$$\left(-1/(\mu^{\frac{q}{p-1}} |\Omega|^{\frac{q}{r}}), +\infty\right) \subset \operatorname{proj}_\lambda \mathscr{C} \subset \left(-1/((\mu - \mu_1)^{\frac{q}{p-1}} |\varphi_1|_r^q), +\infty\right), \tag{0.0.5}$$

$$\operatorname{proj}_{L^r(\Omega)} \mathscr{C}^- \subset \left[(\mu - \mu_1)^{1/(p-1)} |\varphi_1|_r, \mu^{1/(p-1)} |\Omega|^{1/r}\right], \quad \operatorname{proj}_{L^r(\Omega)} \mathscr{C}^+ \subset \left[0, \mu^{1/(p-1)} |\Omega|^{1/r}\right], \tag{0.0.6}$$

and

$$\inf\left\{1 + \lambda |u|_r^q;\ (\lambda, u) \in \mathscr{C}^-\right\} = \inf\left\{1 + \lambda |u|_r^q;\ (\lambda, u) \in \mathscr{C}\right\} = 0, \tag{0.0.7}$$

where $0 < \varphi_1 \in C^2(\Omega) \cap C(\overline{\Omega})$ is the first eigenfunction for the Laplacian operator under homogeneous Dirichlet boundary conditions normalized in $C(\overline{\Omega})$-norm, $\mathscr{C}^- = \{(\lambda, u) \in \mathscr{C}; \lambda \leq 0\}$, and $\mathscr{C}^+ = \{(\lambda, u) \in \mathscr{C}; \lambda \geq 0\}$. In addition:

i) for any $(\lambda, u_\lambda) \in \mathscr{C}$ such that $1 + \lambda |u_\lambda|_r^q \to 0$, one has

$$u_\lambda(x) \to \mu^{1/(p-1)} \text{ uniformly in each compact set } K \subset \Omega, \tag{0.0.8}$$

ii) there is no positive solution $(\lambda, u) \in \mathcal{V}$ to the problem ($Q_{\lambda,\mu}$) for any

$$\lambda \in \left(-\infty, -\frac{1}{(\mu - \mu_1)^{\frac{q}{p-1}} |\varphi_1|_r^q}\right],$$

iii) $\|u_\lambda\|_0 \to 0$ when $\lambda \to +\infty$.

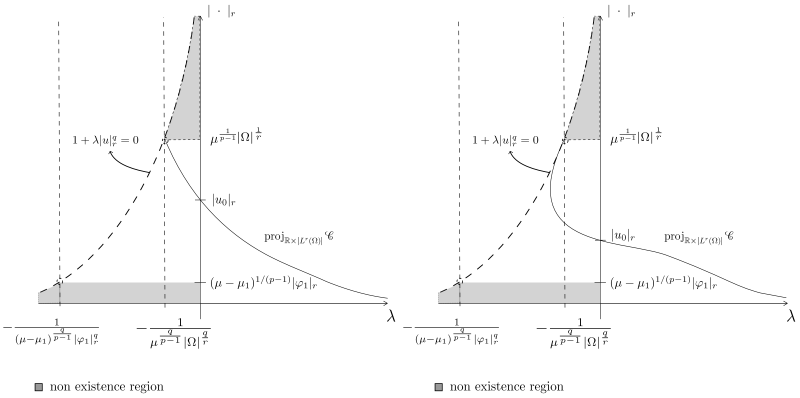

Fig. 2 Possible behaviors of $\mathscr{C}$

Remark 0.0.2. *We claim that $\mathscr{C}$ is connected in the $\mathbb{R} \times L^r(\Omega))$-norm as well, and it will be denoted by* $\text{proj}_{\|L^r(\Omega)\|_0}\mathscr{C}$*. In fact, if $r \leq 2^*$, then the claim follows directly from the connectedness of $\mathscr{C}$ in the $\mathbb{R} \times H_0^1(\Omega))$-norm combined with the continuous embedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$. For $r > 2^*$, let $j : (\mathscr{C}, \mathbb{R} \times L^r(\Omega)) \to \mathbb{Z}$ be any continuous function. Since $L^r(\Omega)$ is dense in $L^{2^*}(\Omega)$, then the extension $\tilde{j} : (\mathscr{C}, \mathbb{R} \times L^{2^*}(\Omega)) \to \mathbb{Z}$ of j is also continuous. So, by using the facts that $H_0^1(\Omega)$ is continuous embedded in $L^{2^*}(\Omega)$ and $\mathscr{C}$ is $\mathbb{R} \times H_0^1(\Omega))$-norm connected, we obtain $(\mathscr{C}, \mathbb{R} \times L^{2^*}(\Omega))$ is also connected and, consequently, $\tilde{j}$ is constant. Then j is constant as well proving the claim.*

Remark 0.0.2 allows us to highlight the possible behavior of $\text{proj}_{\|L^r(\Omega)\|}\mathscr{C}$ in the above pictures.

Let us summarize the main contributions of the last two theorems to the literature.

i) In both Theorems 0.0.2 and 0.0.3 we have shown that the connected set of solutions extends up to the boundary (not on) of the maximal subset of the parameter-working space in which the problem is well-defined. As "maximal", we mean that the problem degenerates on the boundary of such subset.

ii) Both Theorems 0.0.2 and 0.0.3 bring up to literature new and fine estimates on the parameter for existence or non-existence of solutions.

iii) As far as we know, concerning to problems with nonlinear perturbation of the non-local operators as in the problem $(Q_{\lambda,\mu})$, Theorem 0.0.3 is the first result in the literature that provides a connected set of positive solutions, and, in particular, shows the equality

(0.0.7). The aforementioned references about this type of problem assume the hypothesis about the existence of a constant $a_0 > 0$ such that $a(s) > a_0 > 0$ for all s, except for [19] and [18], where the authors consider the case where the function a is not necessarily bounded away from zero. However, they deal with perturbation of the differential operator not depending on the solution.

The positiveness of the solutions lying in the connected set of solutions of ($P_{\lambda,\mu}$) and of ($Q_{\lambda,\mu}$) were obtained as a consequence of an abstract result that we call Positiveness-continuity-principle (see Proposition 2.2.1). Proposition 2.2.1 provides a sufficient condition for positiveness of solutions on a connected of solutions, without requiring that the associated operator to be strongly positive, as it is assumed in Lemma 6.5.4 of [43]. Indeed, Proposition 3.1.2 provides positiveness of fixed points of the operator K introduced 3.1.5, but K is not positive.

Problems ($P_{\lambda,\mu}$) and ($Q_{\lambda,\mu}$) feature nonlinear differential operators, which presented a significant challenge in establishing the existence of positive solutions. This challenge was addressed using Theorem 0.0.1. In Chapter 5, we investigate two further perturbations of the classical diffusive logistic model with refuge (see ($R_{\lambda,\mu}$) and ($S_{\lambda,\mu}$)), where existence was proven via Theorem 0.0.4, a synthesis of techniques from the literature. The primary challenge for problems ($P_{\lambda,\mu}$) and ($Q_{\lambda,\mu}$), however, lies in determining the behavior of positive solutions under parameter variations.

To introduce the results of Chapters 4 and 5, let us present the diffusive logistic problem with refuge

$$\begin{cases} -d\Delta u = \mu u - b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{0.0.9}$$

where $d > 0$, $p > 1$, Ω be a domain of the euclidean space of dimension N, $\mathbb{R}^N$, and $b \in \mathcal{C}(\overline{\Omega})$, b non-negative. We denote

$$B_0 := \operatorname{int}\{x \in \overline{\Omega}; b(x) = 0\}.$$

The scalar $d > 0$ is the diffusion coefficient that measures the diffusion velocity, μ is a real parameter denoting the growth rate of the species, and b measures (inversely) the carrying capacity of the habitat. The refuge B_0 is a concept that derive from the competition between species. That is, it originally represents a region where there is no competition between species (see page 302 of [16], where the author cite this concept as a novelty, in the competition model, introduced by [42], see also [14]). When studying a single species u, B_0 can be interpreted as the limit of large carrying capacity, in this sense, B_0 represents a region of the habitat Ω where the species u has unlimited resources. The Dirichlet condition means that the species is surrounded by a lethal zone.

The existence and uniqueness of positive solution to the problem (0.0.9) was studied by several authors assuming different types of hypotheses. Among them, we quote [44], [1], [39] and [32]. In this work, we state Theorem 0.0.4, which allows the intersection of $\bar{B}_0$ with $\partial\Omega$ to be non empty. For homogeneous Dirichlet boundary condition, this detail was already covered in [32] (see $(A3)$ and Theorem 3.5 in [32]). But, in order to apply the subsupersolution method (among other technical reasons) they assumed that the term that multiplies $b(x)u$ in the (RHS) of (0.0.9) must have a certain degree of regularity (see (A_2) in [32]). In our case, this term corresponds to the function $s \mapsto s^{p-1}$, which not have such regularity for $p < 2$. The regularity hypothesis required by [32] was relaxed in [1], which would cover the case $p < 2$. However, the authors imposed the condition $\bar{B}_0 \subset \Omega$ (see Theorem 4.1 of [1]), which clearly prohibits the set $\bar{B}_0$ to intercepts $\partial\Omega$. Fortunately, Theorem 6.1 of [1], or alternatively [3], allows us apply the subsupersolution method to (0.0.9). By combining these two ideas, that is, Theorem 6.1 of [1] and Theorem 3.5 of [32], we provide a necessary and sufficient condition to the existence of positive solution of a more general version of (0.0.9) (see Theorem 0.0.4). In particular, we have that (0.0.9) possesses a unique positive solution u_μ if and only if

$$d\sigma_1^{\Omega}[-\Delta] < \mu < d\sigma_1^{B_0}[-\Delta], \tag{0.0.10}$$

where $\sigma_1^{\Omega}[-\Delta]$ ($\sigma_1^{B_0}[-\Delta]$) denotes the principal eigenvalue of the operator $-\Delta$ in Ω (B_0, respectively) under homogeneous Dirichlet boundary conditions, formulated in the introduction of Chapter 1.

The introduction of the region B_0 in the classical logistic equation ($P_{0,\mu}$), brought in [32], a new phenomenon in the literature (see (0.0.12)). Precisely, if we restrict the hypotheses about p and b, namely, requiring p large and b regular, then the problem (0.0.9) would fits the conditions in [32], where it was proved that

$$\lim_{d\uparrow\mu/\sigma_1^{\Omega}[-\Delta]} \|u_d\|_{C_0^1(\bar{\Omega})} = 0 \tag{0.0.11}$$

and

$$\lim_{d\downarrow\mu/\sigma_1^{B_0}[-\Delta]} \|u_d\|_0 = +\infty. \tag{0.0.12}$$

Further advancements, providing substantial refinements to these initial findings, were presented in [46] and [39]. For a more fluid presentation, we will integrate the discussion of these refinements with the presentation of our own results, rather than detailing them upfront.

We focus on the behavior of the positive solutions of two problems that extend the formulation of (0.0.9). In Section 5.1, we study the problem

$$\begin{cases} -(1+\lambda a(x))\Delta u = \mu u - b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$R_{\lambda,\mu}$}$$

where $0 \leq a \in C(\overline{\Omega})$ and $0 \lneq b \in C(\overline{\Omega})$. In Section 5.2, we study the problem

$$\begin{cases} -\Delta u + \lambda V(x)u = \mu m(x)u - b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$S_{\lambda,\mu}$}$$

$0 \lneq V(x) \in L^\infty(\Omega)$, $0 \lneq b \in C(\overline{\Omega})$, $0 \neq m \in C(\overline{\Omega})$ possibly changing sign.

Due to the similarity in the proof techniques for the behavior of positive solutions to problems ($R_{\lambda,\mu}$) and ($S_{\lambda,\mu}$), we investigated in Chapter 4 the generalized logistic equation

$$\begin{cases} -\Delta u + \lambda V(x) = \mu m(\lambda,x)u - b(\lambda,x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{0.0.13}$$

that encompasses both as special cases. Indeed, ($R_{\lambda,\mu}$) is a particular case of (0.0.13) with $V \equiv 0$, $m(\lambda,x) = 1/(1+\lambda a(x))$ (for $\lambda > -1/\|a\|_0$) and $b(\lambda,x) = b(x)/(1+\lambda a(x))$. On the other hand, (0.0.13) is reduced to ($S_{\lambda,\mu}$) by making $m(\lambda,x) = m(x)$ and $b(\lambda,x) = b(x)$.

We studied (0.0.13) assuming $0 \leq V(x) \in L^\infty(\Omega)$, $0 \lneq b(\lambda,\cdot) \in C(\overline{\Omega})$, $0 \neq m(\lambda,\cdot) \in C(\overline{\Omega})$ possibly changing sign and $\emptyset \neq M_\lambda^+ := \{x \in \Omega;\ m(\lambda,x) > 0\}$, $B_{0,\lambda} := \text{int}\{x \in \overline{\Omega}; b(\lambda,x) = 0\}$.

Let us denote

$$\mathcal{S} = \{(\lambda,\mu) \in \mathbb{R}^2; \sigma_1^{\Omega}[-\Delta+\lambda V - \mu m(\lambda,x)] < 0 < \sigma_1^{B_{0,\lambda}}[-\Delta+\lambda V(x) - \mu m(\lambda,x)]\}, \tag{0.0.14}$$

where $\sigma_1^{\Omega}[-\Delta+\lambda V - m(\lambda,\cdot)]$ (respectively $\sigma_1^{B_{0,\lambda}}[-\Delta+\lambda V - m(\lambda,\cdot)]$) are the eigenvalues of the operator $-\Delta+\lambda V - m(\lambda,\cdot)$ in Ω (respectively $B_{0,\lambda}$) with zero Dirichlet condition.

For this problem we have the following existence theorem.

Theorem 0.0.4. *Let $\lambda \geq 0$. Then there exists a unique positive solution $u_{\lambda,\mu}$ of (0.0.13) if and only if $(\lambda,\mu) \in \mathcal{S}$.*

Once explained the strategy that we adopted on the text for studying ($R_{\lambda,\mu}$) and ($S_{\lambda,\mu}$), we will present our results about the positive solutions of

$$\begin{cases} -(1+\lambda a(x))\Delta u = \mu u - b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{0.0.15}$$

The main goal is to analyze what happens if the diffusion velocity depends on space, that is, $a \in C^{\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, is a non-negative function with

$$A_0 := \operatorname{int} a^{-1}(\{0\}), \quad \text{and} \quad A_+ := \Omega \setminus \overline{A_0}$$

are smooth subsets, and λ is a real parameter representing the velocity diffusion acting only in A_+. Hence, there exists a subregion, A_0, where the species diffuses in a random way, and another, A_+, where the species is affected by a diffusion and a parameter λ. Since we assume that $1+\lambda a(x) > 0$ for all $x \in \overline{\Omega}$, we suppose through the work that

$$\lambda > -\frac{1}{\|a\|_0}.$$

Let us introduce a motivation of this problem. In [57] (see also [12] and [11]), the authors studied a Fisher-KPP problem in a two-dimensional strip, considered as a field where the species lives. This living space of the species also contains one road, that is assumed to be unidimensional, where potentially fast diffusion occurs. They showed that the survival in large time of the population depends on the rate of diffusion in the road. Later in [54], the problem was studied in an infinite cylindrical domain in $\mathbb{R}^{N+1}$, that when $N = 1$ is reduced to a strip between two straight lines, modelling the effects of two roads with fast diffusion on a strip-shaped field bounded by them. The authors analysed the existence of an asymptotic speed of propagation for solutions, as well as the dependence of this speed on the diffusivity at the boundary and the amplitude of the cylinder. The authors in [20] dealt with the existence in the case of bounded domain and they also made a wide overview including biological context and motivations around this subject, namely, the variation of diffusion rate in some region of the habitat and its influence on the behaviour of the population.

In this context, our diffusion coefficient contains a region where the species diffuses in a random way, A_0, and another region where the species can diffuse very fast, a road, A_+ when λ is large, or even very slow, λ small or even negative.

Hence, our main goal in this paper is to study the influence of this new diffusion coefficient in the logistic equation with refuge.

Precisely, we are interested in the behavior of the positive solution at the extremes of the interval of existence, i. e., $\lambda_*(\mu)$ and $\lambda^*(\mu)$. That is, for each fixed μ, there exists a unique positive solution of ($R_{\lambda,\mu}$) if and only if $\lambda \in (\lambda_*(\mu), \lambda^*(\mu))$, where $\lambda_*(\mu)$ is defined by

$$\mu = \sigma_1^{B_0}\left[-\Delta; \frac{1}{1+\lambda_*(\mu)a(x)}\right], \text{ if } \overline{\mu} := \lim_{\lambda \downarrow -1/\|a\|_0} h(\lambda) < \mu < \sigma_1^{B_0}[-\Delta; \chi_{A_0}]$$

and $\lambda^*(\mu)$ is defined by

$$\mu = \sigma_1^{\Omega}\left[-\Delta; \frac{1}{1+\lambda^*(\mu)a(x)}\right], \text{ if } \underline{\mu} := \lim_{\lambda \downarrow -1/\|a\|_0} \sigma_1^{B_0}\left[-\Delta; \frac{1}{1+\lambda a(x)}\right] < \mu < \sigma_1^{B_0}[-\Delta; \chi_{A_0}]$$

and

$$\lambda^*(\mu) = \infty, \text{ if } \sigma_1^{\Omega}[-\Delta; \chi_{A_0}] \le \mu < \sigma_1^{B_0}[-\Delta; \chi_{A_0}]$$

(see Proposition 5.1.2). For that we will assume that $B_0 \subsetneq \Omega$ and $A_+ \cup B_0$ is constituted by a finite number of connected components. Specifically, we suppose that

$$A_+ \cup B_0 = \bigcup_{i=1}^{d} D_i \bigcup_{i=1}^{m} C_i$$

where $m, d \in \mathbb{N}$, $d, m \ge 0$, $D_i \subset A_+$ and $C_i \not\subset A_+$ are regular connected subsets. That is, we have separated the connected components into those that are fully contained in A_+, and those that are not fully contained in A_+, but intersect to B_0 (see Figure 5.1 where we have described a possible configuration of $A_+ \cup B_0$). Hence,

$$\sigma_1^{A_+\cup B_0}[-\Delta;\chi_{A_0}] = \min_{1\le i\le m}\{\sigma_1^{C_i}[-\Delta;\chi_{A_0}]\},$$

where $\sigma_1^{A_+\cup B_0}[-\Delta;\chi_{A_0}]$ (respectively, $\sigma_1^{C_i}[-\Delta;\chi_{A_0}]\}$) denotes the first eigenvalue of the operator $-\Delta$ with weight χ_{A_0} in $A_+\cup B_0$ (respectively, C_i) with zero Dirichlet boundary condition. We can order the sets C_i such that

$$\sigma_1^{A_+\cup B_0}[-\Delta;\chi_{A_0}] = \sigma_1^{C_1}[-\Delta;\chi_{A_0}] < \ldots < \sigma_1^{C_m}[-\Delta;\chi_{A_0}].$$

On the other hand, we will also write

$$A_+\cup B_0 = \bigcup_{i=1}^{m+d} \mathcal{C}_i,$$

where $\mathcal{C}_i$ are regular connected components, and as above we can order them as

$$\sigma_1^{A_+\cup B_0}[-\Delta] = \sigma_1^{\mathcal{C}_1}[-\Delta] < \sigma_1^{\mathcal{C}_2}[-\Delta] < \cdots < \sigma_1^{\mathcal{C}_{m+d}}[-\Delta].$$

In the following result, we show the behavior of the positive solution of ($R_{\lambda,\mu}$) at the extremes of the existence interval.

Theorem 0.0.5. *Assume that $A_0 \neq \emptyset$, $B_0 \subsetneq \Omega$ and let $\underline{\mu} \le \mu < \sigma_1^{B_0}[-\Delta;\chi_{A_0}]$. Then we have the following behavior of the positive solutions u_λ by depending on the range of μ.*

1) **(about the λ-extreme superior interval of existence).** *Let $\mu > \underline{\mu}$, one has:*

1.1) If $\underline{\mu} < \mu < \sigma_1^{\Omega}[-\Delta;\chi_{A_0}]$, then $\lambda^*(\mu) < \infty$, and

$$\lim_{\lambda\to\lambda^*(\mu)} \|u_\lambda\|_{C^1(\overline{\Omega})} = 0.$$

1.2) If $\sigma_1^{\Omega}[-\Delta;\chi_{A_0}] \le \mu < \sigma_1^{A_+\cup B_0}[-\Delta;\chi_{A_0}]$, then $\lambda^*(\mu) = \infty$, and

$$\lim_{\lambda\to\infty} \|u_\lambda\|_{C^{1,\gamma}(\overline{\Omega})} < +\infty, \text{ for some } 0<\gamma<1$$

and $u_\lambda \to u_\infty$ in $C^1(\overline{\Omega})$, where $u_\infty \equiv 0$ in the case $\mu = \sigma_1^{\Omega}[-\Delta;\chi_{A_0}]$ and u_∞ is the unique positive solution of the problem

$$\begin{cases} -\Delta u = \chi_{A_0}(\mu u - b(x)u^p) & \textit{in } \Omega, \\ \quad u = 0 & \textit{on } \partial\Omega, \\ \quad u > 0 & \textit{in } \Omega, \end{cases} \tag{0.0.16}$$

for $\mu > \sigma_1[-\Delta;\chi_{A_0}]$.

1.3) If $\sigma_1^{\Omega}[-\Delta;\chi_{A_0}] < \sigma_1^{A_+\cup B_0}[-\Delta;\chi_{A_0}] = \mu$, *then*

$$\lim_{\lambda\to\infty} \|u_\lambda\|_2 = +\infty.$$

1.4) If $\mu > \sigma_1^{C_{i_0}}[-\Delta;\chi_{A_0}]$, *for some* $1 \le i_0 \le m$ *and for each* $1 \le i \le i_0$ *we assume that* C_i *is isolated from any other component of* $A_+ \cup B_0$, *and* $a(x)^r \ge M\,\mathrm{dist}(x,\partial C_i)$ *for all* $x \in A_+ \setminus B_0$ *in a neighbourhood of* ∂C_i *for all* $x \in C_i$ *for some* $M > 0$ *and* $0 < r < 1$, *then*

$$\lim_{\lambda\to\infty} u_\lambda(x) = +\infty \quad \textit{for all } x \in C_i \textit{ and } 1 \le i \le i_0.$$

1.5) If $\underline{\mu} < \mu < \sigma_1^{\mathcal{C}_{j_0}}[-\Delta]$, *for some* $1 \le j_0 \le m+d$, *then*

$$\sup_{\Lambda<\lambda<\lambda^*(\mu)} \|u_\lambda\|_{L^\infty(D)} < \infty, \textit{ for all } \ D \subset\subset \Omega \setminus \left(\cup_{i=1}^{j_0-1}\overline{\mathcal{C}_i}\right),$$

where Λ *is any number such that* $\lambda_*(\mu) < \Lambda < \lambda^*(\mu)$. *Moreover, for any* $\mu > \underline{\mu}$, *then*

$$\sup_{\Lambda<\lambda<\lambda^*(\mu)} \|u_\lambda\|_{L^\infty(D)} < \infty, \textit{ for all } \ D \subset\subset \Omega \setminus \left(\cup_{i=1}^{m+d}\overline{\mathcal{C}_i}\right).$$

2) **(about the λ-extreme inferior interval of existence).** *If* $\mu > \overline{\mu}$, *one has*

$$\lim_{\lambda\to\lambda_*(\mu)} u_\lambda(x) = +\infty \quad \textit{for all } x \in B_0,$$

and there exists $M > 0$ *such that*

$$u_\lambda(x) \le M \textit{ in any } D \subset \Omega \setminus \overline{B}_0 \textit{ for all } \lambda_*(\mu) < \lambda < \Lambda,$$

where Λ *is any number such that* $\lambda_*(\mu) < \Lambda < \lambda^*(\mu)$.

(see [53]).

Remark 0.0.3. *We point out that:*

a) *Theorem 0.0.5 can be restated by assuming $A_0 = \emptyset$ with the conclusions understood in according to the conventions that $\sigma_1^{\Omega}[-\Delta;\chi_{A_0}] = \sigma_1^{B_0}[-\Delta;\chi_{A_0}] = \sigma_1^{A_+\cup B_0}[-\Delta;\chi_{A_0}] = +\infty$.*

b) *The conclusion of item 1.4) of Theorem 0.0.5 follows without any additional hypothesis if $A_+ \setminus B_0 = \emptyset$.*

c) *We have not analyzed the behaviour in the inferior extreme in the case $\mu \in (\underline{\mu}, \overline{\mu}]$. Observe that this would lead us to the study of the solution as $\lambda \to -1/\|a\|_0$, and as consequence the study of the logistic equation with unbounded coefficient. This study will be carried out in forthcoming work (see [28] for similar results for the logistic equation without refuge.)*

In Figures 3, 4 and 5, we have represented different possible shapes of the graphics of the $\mu = h(\lambda)$ and $\mu = H(\lambda)$ (see the definitions in (5.1.2)) as well as the existence regions, that is the region defined by (5.1.3).

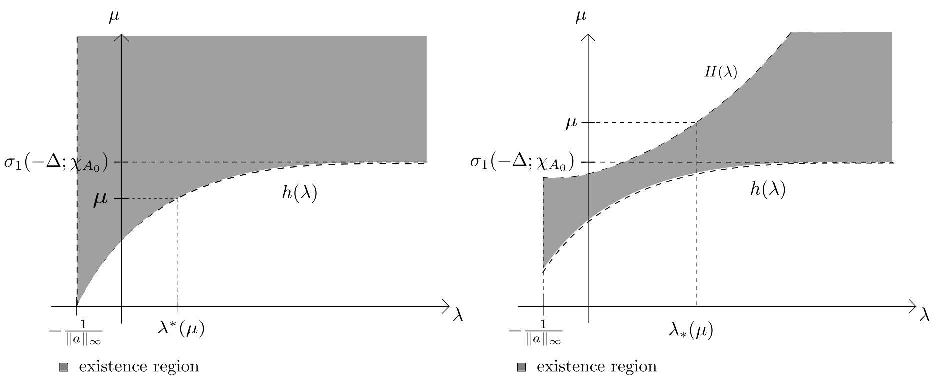

Fig. 3 Existence regions: the case $B_0 = \emptyset$, $A_0 \neq \emptyset$ and $\underline{\mu} = 0$ (left) and the case $A_0, B_0 \neq \emptyset$, and $A_0 \cap B_0 = \emptyset$ and $0 < \underline{\mu} < \overline{\mu}$. (right)

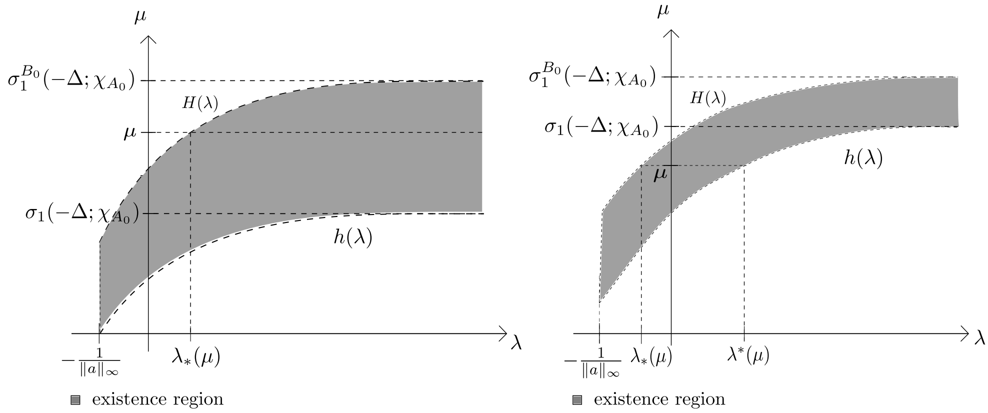

Fig. 4 Existence regions in the case $B_0 \neq \emptyset$: on the left the case $0 = \underline{\mu} < \overline{\mu}$. On the right, the case $0 < \underline{\mu} < \overline{\mu} < +\infty$.

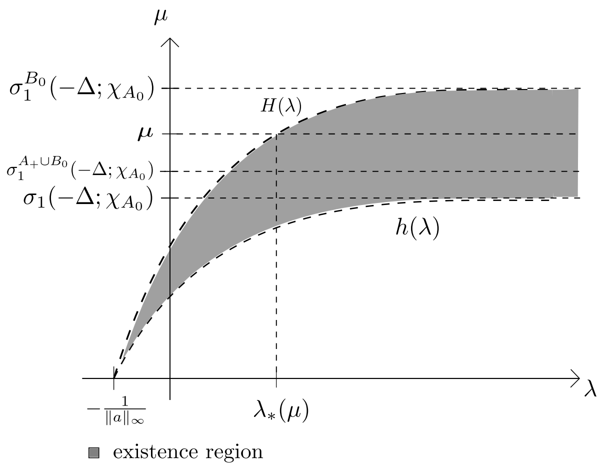

Fig. 5 Existence region in the case $B_0 \neq \emptyset$ and $0 = \underline{\mu} = \overline{\mu}$.

As a consequence of Theorems 0.0.5, we can observe the drastic change that occurs between the problem with homogeneous diffusion coefficient (0.0.9) and the heterogeneous case ($R_{\lambda,\mu}$). Indeed, in the first case there is no solution when the diffusion coefficient is large, however, in the heterogeneous case, when $A_0 \neq \emptyset$ and for birth rates with intermediate values, the population persists for very large values of the diffusion coefficient, and even grows uncontrollably in the refuge and in the fast diffusion zone. Also, we have studied the behavior at the inferior extreme, showing that the solution blows up in the refuge and remains bounded in the rest of the habitat. We will delve deeper into these biological consequences in the last section of the paper.

In Section 5.2, we study the problem

$$\begin{cases} -\Delta u + \lambda V(x)u = \mu m(x)u - b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \qquad (S_{\lambda,\mu})$$

The term $\lambda V(x)$ measures the degradation of the harsh patches of the territory, where $V > 0$, whereas the non-degraded patches of Ω are the regions where $V = 0$.

With regard to the literature on the behavior of the positive solutions of ($S_{\lambda,\mu}$) in the case where μ is fixed in λ varies, we quote [15] and [55]. The authors in [15] study the case which is similar to our analysis with $\lambda \to +\infty$ but with different boundary conditions so that their result can be interpreted as closely related to item 3) of Theorem 0.0.5. The analysis of Theorem 0.0.5 is related to problem (1.3) of [55], by making $m(x) = \chi_G$, $V(x) = \chi_B$, $b(x) = \chi_G$, where G (non degraded area) and B (degraded area) constitute a partition of Ω, $p = 2$ and substituting the Dirichlet boundary condition for Neumann. Our result with respect to the behavior of the positive solution $u_{\lambda,\mu}$ when $\lambda \to +\infty$ compliments their result in the case of the Dirichlet boundary condition. See also [41] and its references.

The next theorem we will present includes the phenomenon of blow-up in the boundary of certain regions of Ω. This is a very fine qualitative information and because of that, Theorems 5.2.2 and 5.2.3 were required. Theorem 5.2.3 was inspired by Theorem 4.8 of [45]. Theorem 5.2.2 is inspired on the pioneer work [46], where the authors proved the following refinement of (0.0.12) for the solution $u_{0,\mu}$ of ($S_{\lambda,\mu}$) with $\lambda = 0$ and $m \equiv 1$.

$$\lim_{\mu \uparrow \sigma_1^{B_0}[-\Delta]} u_{0,\mu} = +\infty \text{ uniformly in compact subsets of } B_0 \qquad (0.0.17)$$

and

$$\lim_{\mu \uparrow \sigma_1^{B_0}[-\Delta]} u_{0,\mu}(x) = +\infty \ \forall\, x \in \partial B_0 \qquad (0.0.18)$$

by assuming $\bar{B}_0 \subset \Omega$.

For the case $m \not\equiv 1$, it was proved in [39] a generalization of (0.0.17) to m possibly changing sign, that is,

$$\lim_{\mu \uparrow \sigma_1^{B_0}[-\Delta;m]} u_{\lambda,\mu} = +\infty \text{ uniformly in compact subsets of } B_0, \qquad (0.0.19)$$

by assuming $\bar{B}_0 \subset \Omega$ and

$$m \geq 0 \text{ in a neighborhood of } B_0. \tag{0.0.20}$$

According to the authors, it was necessary to impose this condition in order to deal with the loss of monotonicity of the positive solutions that occurs if m changes sign. As clearly explained in [39], the analysis of positive solutions of the problem ($S_{\lambda,\mu}$) is harder in the case where m changes sign. Also in [39], the authors generalized (0.0.18) proving

$$\lim_{\mu \uparrow \sigma_1^{B_0}[-\Delta;m]} u_{0,\mu}(x) = +\infty \; \forall \, x \in \partial B_0 \tag{0.0.21}$$

requiring (0.0.20) and $\bar{B}_0 \subset \Omega$.

In [47], the condition (0.0.20) of [39] was relaxed and it was shown that (0.0.19) still holds by imposing $m(x_0) > 0$ for some $x_0 \in B_0$ and more regularity under m. Moreover, we do not require $\bar{B}_0 \subset \Omega$. With techniques similar to [47], we proved Theorem 5.2.2.

In order to enunciate the results about ($S_{\lambda,\mu}$), we will introduce some notations.

Let us denote

$$\mathcal{C}_0 := \{\mu \in C([0,\Lambda]),\ 0 < \Lambda < \infty; (\lambda, \mu(\lambda)) \in \mathcal{S} \; \forall \, \lambda \in [0,\Lambda)\}$$

and

$$\mathcal{C}_\infty := \left\{ \mu \in C([0,+\infty));\ \exists \mu(\infty) := \lim_{\lambda \to +\infty} \mu(\lambda) < \infty \text{ and } (\lambda, \mu(\lambda)) \in \mathcal{S} \; \forall \, \lambda \in [0,+\infty) \right\}.$$

Note that since ($S_{\lambda,\mu}$) is a particular case of (0.0.13) with $m(\lambda,x) = m(x)$ and $b(\lambda,x) = b(x)$, then the above definitions of $\mathcal{C}_0$ and $\mathcal{C}_\infty$ consider as $\mathcal{S}$ the family

$$\mathcal{S} := \{(\lambda,\mu) \in \mathbb{R}^2; \sigma_1^{\Omega}[-\Delta + \lambda V - \mu m] < 0 < \sigma_1^{B_0}[-\Delta + \lambda V - \mu m]\},$$

where $B_0 = \text{int}\{x \in \overline{\Omega}; b(x) = 0\}$.

We will denote $u_\lambda := u_{\lambda,\mu(\lambda)}$ for each $\mu \in \mathcal{C}_0 \cup \mathcal{C}_\infty$ and $D_0 := V_0 \cap B_0$.

Once we have settled these notation, we get the following theorem.

Theorem 0.0.6. *One has:*

1) Let $\mu_0 \in \mathcal{C}_0$ and consider the family u_λ of positive solutions associated to μ_0.

1i) If $\lim_{\lambda\uparrow\Lambda}\mu_0(\lambda)=\sigma_1^{\Omega}[-\Delta+\Lambda V;m]$*, then*

$$\lim_{\lambda\uparrow\Lambda}\|u_\lambda\|_{C_0^1(\overline{\Omega})}=0.$$

1ii) If $\lim_{\lambda\uparrow\Lambda}\mu_0(\lambda)=\sigma_1^{B_0}[-\Delta+\Lambda V;m]$*, then*

$$\lim_{\lambda\uparrow\Lambda}u_\lambda(x)=\infty\;\forall\,x\in B_0. \tag{0.0.22}$$

If $m\in\mathcal{C}^r$ *for some* $r=r(N)\geq 0$*, then the above convergence holds uniformly on compact subsets of* B_0*. Additionally, if*

$$\mu'(\Lambda)\neq\left.\frac{d}{d\lambda}\right|_{\lambda=\Lambda}\sigma_1^{B_0}[-\Delta+\lambda V;m], \tag{0.0.23}$$

$\nabla b=0$ *in* ∂B_0*,* $\bar{B}_0\subset\Omega$*, the functions* m *and* V *are holomorphic in a neighborhood of* $\overline{B_0}$ *and* μ_0 *can be extended to a holomophic function defined in an open interval containing* Λ*, then*

$$\lim_{\lambda\uparrow\Lambda}u_\lambda(x)=\infty\;\forall\,x\in\partial B_0, \tag{0.0.24}$$

by whence

$$\lim_{\lambda\uparrow\Lambda}u_\lambda=\begin{cases}+\infty & x\in\bar{B}_0,\\ L_{\min}(x) & x\in\bar{\Omega}\setminus\bar{B}_0,\end{cases} \tag{0.0.25}$$

where $L_{\min}$ *stands for the minimal large positive solution of the singular problem*

$$\begin{cases}-\Delta u+\lambda V(x)u=\sigma_1^{B_0}[-\Delta+\Lambda V;m]m(x)u-b(x)u^p & \text{in } \Omega\setminus\bar{B}_0,\\ u=+\infty & \text{on } \partial B_0,\\ u=0 & \text{on } \partial\Omega,\\ u>0 & \text{in } \Omega\setminus\bar{B}_0,\end{cases}$$

2) Assume that $V_0\neq\emptyset$*,* $m(x)>0$ *for some* $x\in V_0$ *and* $V_0\not\subseteq B_0$*. Then* $\sigma_1^{V_0}[-\Delta;m]$ *is positive, finite and* $\sigma_1^{V_0}[-\Delta;m]<\sigma_1^{D_0}[-\Delta;m]\leq+\infty$*. Let* $\mu_\infty\in\mathcal{C}_\infty$ *and consider the family* u_λ *of positive solutions associated to* μ_∞*.*

2i) If $\lim_{\lambda\to\infty}\mu_\infty(\lambda)=\sigma_1^{V_0}[-\Delta;m]$*, then*

$$\lim_{\lambda\to\infty}\|u_\lambda\|_{C_0^1(\overline{\Omega})}=0.$$

2ii) If $\sigma_1^{V_0}[-\Delta;m]<\lim_{\lambda\to\infty}\mu_\infty(\lambda)<\sigma_1^{D_0}[-\Delta;m]$*, then* $\lim_{\lambda\to\infty}\|u_\lambda-u_\infty\|_\infty=0$ *where* u_∞ *is the null extension of the unique positive solution of*

$$\begin{cases}-\Delta u=\lim_{\lambda\to\infty}\mu_\infty(\lambda)mu-bu^p & \text{in } V_0\\ u=0 & \text{on } \partial V_0.\end{cases}$$

2iii) If $\emptyset\neq D_0\in\mathcal{C}^2$, $m(x)>0$ *for some* $x\in D_0$ *and* $\lim_{\lambda\to\infty}\mu_\infty(\lambda)=\sigma_1^{D_0}[-\Delta;m]$*, then*

$$\lim_{\lambda\to\infty}u_\lambda(x)=\infty\;\forall\,x\in D_0. \tag{0.0.26}$$

If $m\in\mathcal{C}^r$ *for some* $r=r(N)\geq 0$*, then the above convergence holds uniformly on compact subsets of* D_0*.*

Additionally, assume that there exists a component Γ *of* ∂D_0 *such that* $\Gamma\cap\partial\Omega=\emptyset$, $\Gamma\subset M_+=\{x\in\Omega;m(x)>0\}$ *and* $\Gamma\subset V_0$ *(see Figure 10). Then*

$$\lim_{\lambda\to\infty}u_\lambda(x)=\infty\;\forall\,x\in\Gamma. \tag{0.0.27}$$

In particular, if $\overline{B_0}\subset V_0\subset\overline{V_0}\subset\Omega$ *and* $\partial B_0\subset M_+$*, then*

$$\lim_{\lambda\to+\infty}u_\lambda(x)=\begin{cases}+\infty & x\in\bar{B}_0,\\ L_{\min}(x) & x\in\bar{\Omega}\setminus\bar{B}_0,\end{cases} \tag{0.0.28}$$

where $L_{\min}$ *stands for the minimal large positive solution of the singular problem*

$$\begin{cases}-\Delta u=\sigma_1^{D_0}[-\Delta;m]m(x)u-b(x)u^p & \text{in } V_0\setminus\bar{B}_0,\\ u=+\infty & \text{on } \partial B_0,\\ u>0 & \text{on } V_0\setminus\bar{B}_0\end{cases}$$

3) Assume that $\emptyset \neq V_0 \subset B_0$, $M_+ \cap V_0 \neq \emptyset$ *and let* $0 \leq \lambda_n \to \infty$. *Then there exist sequences* $\overline{\mu}(\lambda_n)$ *and* $\underline{\mu}(\lambda_n)$ *such that* $(\lambda_n, \overline{\mu}(\lambda_n)), (\lambda_n, \underline{\mu}(\lambda_n)) \in \mathcal{S}$ *and* $x_n \in B_0$ *such that*

$$\lim_{n \to +\infty} \underline{\mu}(\lambda_n) = \lim_{n \to +\infty} \overline{\mu}(\lambda_n) = \sigma_1^{D_0}[-\Delta; m],$$

$$\lim_{n \to +\infty} \overline{u}_n(x_n) = \infty \text{ and } \lim_{n \to +\infty} \|\underline{u}_n\|_\infty = 0,$$

where $\overline{u}_n$ *(respectively,* $\underline{u}_n$*) is the sequence of positive solutions associated to* $(\lambda_n, \overline{\mu}(\lambda_n))$ *(respectively,* $(\lambda_n, \underline{\mu}(\lambda_n))$*).*

The convergence (0.0.27) and (0.0.24) are very fine qualitative information about the behavior of the positive solutions of the problems. The proof of them required ingenious techniques presented in Theorems 5.2.2 and 5.2.3, respectively.

For the sake of clarity, we would like to highlight the contributions of Theorem 0.0.6 to the literature.

1) The blow up result given in (0.0.22) complements (0.0.21) proved by [47], showing that the blow up in the boundary of B_0 remains true even if another parameter besides the birth rate, namely the degradation rate, vary simultaneously with the birth rate approximating to a finite value Λ.

2) Item 2*iii*) of Theorem 0.0.6 shows that the well known phenomenon of blow up in the refugee with $\lambda = 0$ when the birth rate μ approximates to its maximum (see (0.0.17)) keeps happening to the positive solutions of ($S_{\lambda,\mu}$) (at least in the subregion V_0 of the refugee) even when the approximation of the birth rate $\mu(\lambda)$ occurs asymptotically as λ grows indefinitely.

3) (0.0.27) extends the result of blow up in the boundary of the refuge given by (0.0.21) to a blow up in a component Γ of ∂D_0 in the case $\lambda = +\infty$ at least for when Γ and m satisfies appropriated conditions.

4) Lemma 2.3.1 extends the a priori bound (4.3) of [32] to the case $\lambda = +\infty$.

In the following figures, we illustrate examples of μ_0, μ_∞ and Γ satisfying the hypotheses of the above theorem.

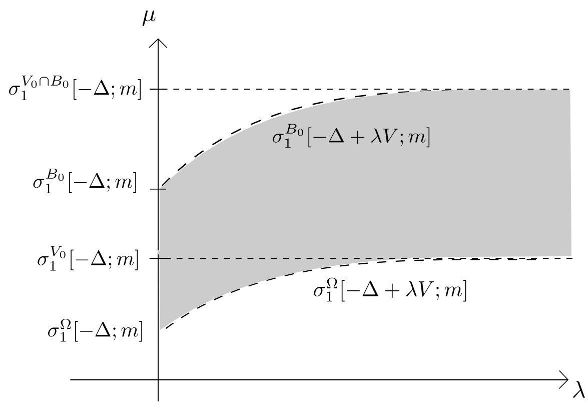

Fig. 6 Existence region

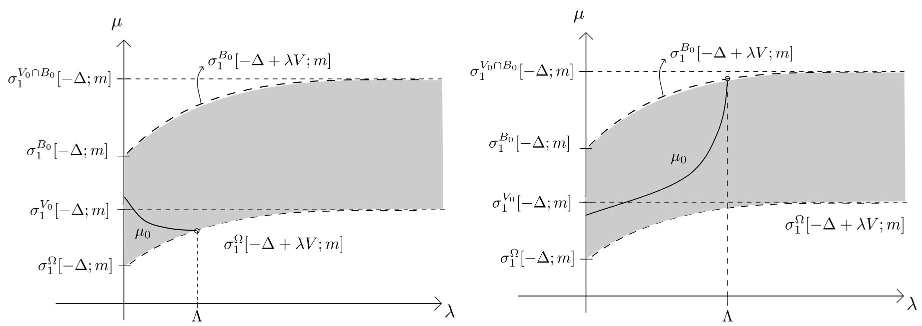

Fig. 7 On the left: an example of μ_0 satisfying item $1i)$ of Theorem 0.0.6. On the right: an example of μ_0 satisfying item $1ii)$ of Theorem 0.0.6.

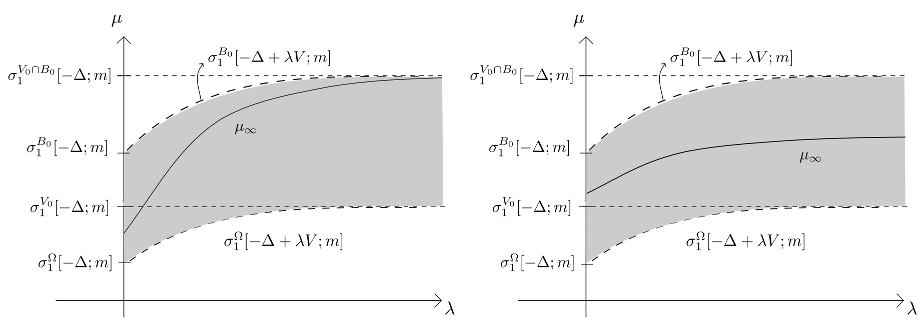

Fig. 8 On the left: an example of μ_∞ satisfying item $2i)$ of Theorem 0.0.6. On the right: an example of μ_∞ of item $2ii)$ of Theorem 0.0.6.

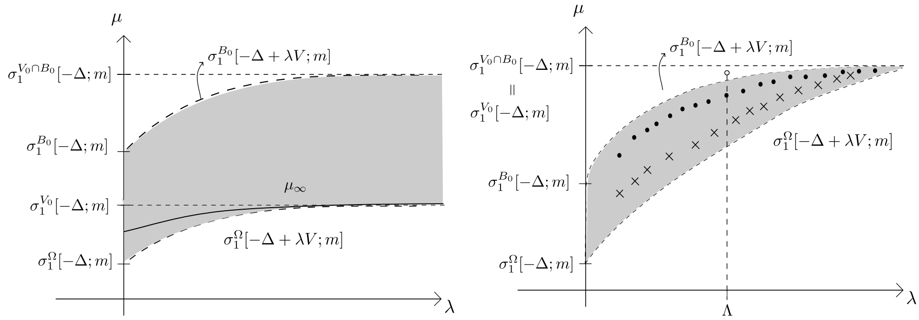

Fig. 9 On the left: an example of μ_∞ satisfying item $2iii)$ of Theorem 0.0.6. On the right: example of sequences $(\lambda_n, \underline{\mu}_n(\lambda_n))$ (represented by dots) and $(\lambda_n, \overline{\mu}_n(\lambda_n))$ (represented by x's) satisfying item $3)$ of Theorem 0.0.6.

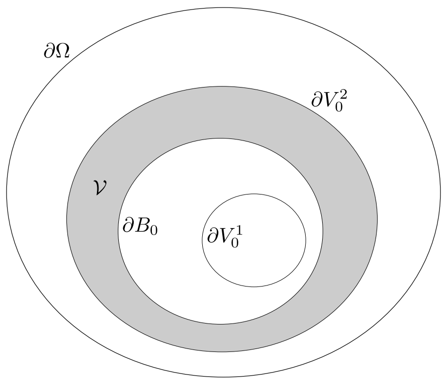

Fig. 10 ∂B_0 is an example of Γ satisfying the hypothesis of item $2iii)$ of Theorem 0.0.6.

Now let us present an overview on the structure of the text.

The content of Chapter 1 is about the behavior of the first eigenvalue and its associated eigenfunction with respect to variations of the potential, domain and weight of the eigenvalue problem operator. The introduction of the chapter provides some basic properties that can be found in [44] and [26]. In Section 1.1, we prove Lemma 1.1.1, that estimates the $H_0^1(\Omega)$-norm of a certain power of the principal eigenfunction from above by the integral of a power of the principal eigenfunction multiplied be the weight. Lemma 1.1.1 also provides an estimate of the L^∞-norm of the principal eigenfunction from above by its L^2-norm. This estimate, will play an important role in the proof of Theorem 4.2.1. In Section 1.2, we prove Theorem 1.2.1 that will play a crucial role in Item 1.4) of Theorem 0.0.5. Section 1.3 is dedicated to provide a fine data of the variation of first eigenvalue with respect to the domain. This information is crucial in the proof of (0.0.24), as the reader can attest in the proof of Theorem 5.2.2. Having explained the role of Sections 1.1, 1.2 and 1.3, which primarily impact Chapters 4 and 5, readers who focus solely on Chapter 2 and are familiar with the basic properties of the principal eigenvalue can safely skip Chapter 1.

Chapter 2 provides the proof of Theorem 0.0.1, an abstract result on topological methods that will be used in the proofs of Theorems 0.0.2 and 0.0.3, which in turn are presented in Sections 3.1 and 3.2, respectively.

Chapters 1 and 4 provide together the abstract results that are used in the proofs of Theorems 0.0.5 and 0.0.6, which in turn are presented in Sections 5.1 and 5.2, respectively.

In conclusion, this work presents some results on topological methods and some applications in analyzing the existence and behavior of solutions near singularities and degeneracies of some nonlinear elliptic partial equations.

Chapter 1

Refined results on weighted eigenvalue problems

Theorems 0.0.5 and 0.0.6 provide qualitative information of the positive solutions of ($S_{\lambda,\mu}$) and (0.0.13), respectively. By depending on the variation of the parameters λ and μ in these problems, the associated family of positive solutions $u_{\lambda,\mu}$ may be uniformly bounded or blows-up. Both phenomena strongly rely on the behavior of the positive eigenfunctions and the associated eigenvalues with respect to the potential, the weight and the domain. This chapter focuses on establishing and proving results concerning these behaviors.

Let us give a brief overview on the content of this chapter. Before Section 1.1, we present some basic properties of the first eigenvalue that can be implied or deduced from [44] and it will be used frequently in this text. In Section 1.1, we proof a $L^\infty(\Omega)$-estimate of Moser's type that will play a crucial rule in the proof of Theorem 4.2.1 which, in turn, pavements the case of uniformly boundedness mentioned in the previous paragraph. In Section 1.2, we prove Theorem 1.2.1 that explicit the limit of the first eigenvalue with respect to exploding potentials. In a particular case, we can also determine the limit of the associated family of positive eigenfunctions (see Remark 1.2.1). By relaxing hypothesis (2.7) of Theorem 2.4 in [32], Theorem 1.2.1 provides a complementary result. This weakened condition is crucial for Corollary 5.1.1 (as noted in Remark 5.1.1). Finally, in Section 2.3, we prove a result about fine qualitative information about the first eigenvalue with respect to the perturbation of the domain. This result will play a key role in proving (0.0.24).

Before enunciating some properties of the first eigenvalue, we will establish the following notations and conventions. Given open subset U of Ω and $m \in L^\infty(\Omega)$, we will denote by $\sigma_1^U[-\Delta; m]$ the principal eigenvalue, in the sense of [26] or [44], of the problem

$$\begin{cases} -\Delta u = \mu m(x) u & \text{in } U, \\ \quad u = 0 & \text{on } \partial U, \\ \quad u > 0 & \text{on } U \end{cases} \tag{1.0.1}$$

whenever the set $\{m > 0\} \cap U$ has positive Lebesgue's measure. On the contrary, we will adopt the convention that $\sigma_1^U[-\Delta; m] = +\infty$ as U is the empty set and $\sigma_1[-\Delta] = \sigma_1[-\Delta; 1]$.

We recall some well-known properties of the first eigenvalue $\sigma_1^U[-\Delta + c]$, $c \in L^\infty(\Omega)$, of the problem

$$\begin{cases} -\Delta u + c(x) u = \mu u & \text{in } U, \\ \qquad\qquad u = 0 & \text{on } \partial U, \\ \qquad\qquad u > 0 & \text{on } U, \end{cases}$$

see for instance [44].

Proposition 1.0.1. *One has:*

1. *The map $c \in L^\infty(\Omega) \mapsto \sigma_1^U[-\Delta + c]$ is continuous and increasing.*

2. *Assume $c \geq 0$ in U, $c \neq 0$ in U, then*

$$\lim_{r \to -\infty} \sigma_1^U[-\Delta + rc] = -\infty, \qquad \lim_{r \to +\infty} \sigma_1^U[-\Delta + rc] = \sigma_1^{U \cap C_0}[-\Delta],$$

where

$$C_0 := \operatorname{int} c^{-1}(\{0\}).$$

Take now $m \in L^\infty(\Omega)$, $m \geq 0$, $m \neq 0$ in U and consider the eigenvalue problem

$$\begin{cases} -\Delta u + c(x) u = \mu m(x) u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{1.0.2}$$

Observe that the study of principal eigenvalue of (1.0.2) is equivalent to study of the zeros of the map

$$r(\mu) := \sigma_1^U[-\Delta + c - \mu m]. \tag{1.0.3}$$

Thanks to Proposition 1.0.1, the map $\mu \mapsto r(\mu)$ is continuous, decreasing, and

$$\lim_{\mu\to-\infty} r(\mu) = \sigma_1^{U\cap M_0}[-\Delta+c], \qquad \lim_{\mu\to+\infty} r(\mu) = -\infty,$$

where $M_0 := \operatorname{int} m^{-1}(\{0\})$, and we are using the convention $\sigma_1^{U\cap M_0}[-\Delta+c] = +\infty$ if $U\cap M_0 = \emptyset$.

Hence, we get:

Proposition 1.0.2. *Assume that $m \geq 0$, $m \neq 0$ in U. Then, there exists the principal eigenvalue of (1.0.2), denoted by $\sigma_1^U[-\Delta+c;m]$, if and only if $\sigma_1^{U\cap M_0}[-\Delta+c] > 0$. Moreover, the map (1.0.3) verifies that*

$$r(\mu) > 0 \text{ if } \mu < \sigma_1^U[-\Delta+c;m] \text{ and } r(\mu) < 0 \text{ if } \mu > \sigma_1^U[-\Delta+c;m].$$

Finally,

$$\sigma_1^U[-\Delta+c;m] > 0 \text{ if and only if } \sigma_1^U[-\Delta+c] > 0.$$

Let us note that, according to our notation,

$$\sigma_1^U[-\Delta+c] = \sigma_1^U[-\Delta+c;1].$$

The following properties of $\sigma_1^U[-\Delta+c;m]$ will be used along the paper (see [44] and [26]).

Proposition 1.0.3. *Assume that $m \geq 0$, $m \neq 0$ in Ω and $\sigma_1^{U\cap M_0}[-\Delta+c] > 0$.*

1. *The map $c \in L^\infty(\Omega) \mapsto \sigma_1^U[-\Delta+c;m]$ is continuous and increasing.*
2. *The map $m \in L^\infty_+(\Omega) \mapsto \sigma_1^U[-\Delta+c;m]$ is continuous and decreasing.*
3. *Assume that $U_1 \subseteq U_2 \subset \Omega$, then $\sigma_1^{U_2}[-\Delta+c;m] \leq \sigma_1^{U_1}[-\Delta+c;m]$. If moreover, $U_1 \subsetneq U_2$, then $\sigma_1^{U_2}[-\Delta+c;m] < \sigma_1^{U_1}[-\Delta+c;m]$.*

1.1 $L^\infty(\Omega)$ -estimates of Moser's type

In this section we will prove a $L^\infty(\Omega)$- estimate of Moser's type that will be very useful in the proofs of the Section 4.2.

Lemma 1.1.1. *Let $\{m_\lambda\}_{\lambda\in\Lambda}$ and $\{h_\lambda\}_{\lambda\in\Lambda}$ be families of functions $L^\infty(\Omega)$ such that $h_\lambda, m_\lambda \geq 0$ for all $\lambda \in \Lambda$. Suppose that φ_λ is the positive eigenfunction in $H_0^1(\Omega)$ associated to $\sigma_1^\Omega[-\Delta + h_\lambda; m_\lambda] > 0$. If*

$$\sup_{\lambda\in\Lambda} \sigma_1^\Omega[-\Delta + h_\lambda; m_\lambda] < +\infty \text{ and } \varphi_\lambda \in L^{q+1}(\Omega) \text{ for some } q \geq 0,$$

then there is a constant $D > 0$ such that

$$\|\psi_\lambda^r\|^2_{H_0^1(\Omega)} \leq D \int_\Omega m_\lambda \varphi_\lambda^{q+1} \ \forall\ \lambda \in \Lambda,$$

where $\varphi_\lambda = k\psi_\lambda$, $k = r^{1/r}/\left(q^{1/(2r)}\right)$ and $r = (q+1)/2$.
Additionally, if $\sup_{\lambda\in\Lambda} \|m_\lambda\|_{L^\infty(\Omega)} < +\infty$, then there is a $C > 0$ such that

$$\|\varphi_\lambda\|_{L^\infty(\Omega)} \leq C(1 + \|\varphi_\lambda\|_{L^2(\Omega)}). \tag{1.1.1}$$

Proof. By the definition of φ_λ, we have that

$$\int_\Omega \nabla\varphi_\lambda \nabla\phi + \int_\Omega h_\lambda \varphi_\lambda \phi = \sigma_1^\Omega[-\Delta + h_\lambda; m_\lambda] \int_\Omega m_\lambda \varphi_\lambda \phi \ \forall\ \phi \in H_0^1(\Omega).$$

By making $\phi = \varphi_\lambda^q$, we deduce that

$$\begin{aligned} \frac{q}{r^2} \int_\Omega |\nabla(\psi_\lambda^r)|^2 &= \int_\Omega q\varphi_\lambda^{q-1} |\nabla\varphi_\lambda|^2 \\ &= \int_\Omega \nabla\varphi_\lambda \nabla(\varphi_\lambda^q) \\ &= \sigma_1^\Omega[-\Delta + h_\lambda; m_\lambda] \int_\Omega m_\lambda \varphi_\lambda^{q+1} - \int_\Omega h_\lambda \varphi_\lambda^{q+1} \\ &\leq C \sup_{\lambda\in\Lambda} \sigma_1^\Omega[-\Delta + h_\lambda; m_\lambda] \int_\Omega m_\lambda \varphi_\lambda^{q+1}, \end{aligned}$$

where $u_\lambda = kv_\lambda$, $k = r_0^{1/r}/\left(q_0^{1/(2r)}\right)$, $r = (q+1)/2$.
Now let us prove the second statement of the lemma. Let $\beta \geq 1$ and define $z = \varphi_\lambda + 1$, $\varphi = z^\beta - 1 \geq 0$. Note that $\nabla\varphi = \beta z^{\beta-1}\nabla z$ and $\nabla z = \nabla\varphi_\lambda$. So by taking φ as a test function in

the definition of φ_λ, we obtain

$$\begin{aligned}
\beta\int_\Omega z^{\beta-1}|\nabla z|^2 &\leq \beta\int_\Omega z^{\beta-1}|\nabla z|^2+\int_\Omega h_\lambda\varphi \\
&= \sigma_1^\Omega[-\Delta+h_\lambda;m_\lambda]\int_\Omega m_\lambda\varphi_\lambda\varphi \\
&= \sigma_1^\Omega[-\Delta+h_\lambda;m_\lambda]\int_\Omega m_\lambda(z-1)(z^\beta-1) \\
&= \sigma_1^\Omega[-\Delta+h_\lambda;m_\lambda]\int_\Omega m_\lambda(z^{\beta+1}-z-z^\beta+1) \\
&= \sigma_1^\Omega[-\Delta+h_\lambda;m_\lambda]\int_\Omega m_\lambda(z^{\beta+1}-z^\beta-\varphi_\lambda) \\
&\leq \sigma_1^\Omega[-\Delta+h_\lambda;m_\lambda]\int_\Omega m_\lambda z^{\beta+1} \\
&\leq \sup_{\lambda\in\Lambda}\sigma_1^\Omega[-\Delta+h_\lambda;m_\lambda]\sup_{\lambda\in\Lambda}\|m_\lambda\|_{L^\infty(\Omega)}\int_\Omega z^{\beta+1},
\end{aligned}$$

so

$$\beta\int_\Omega z^{\beta-1}|\nabla z|^2\leq \sup_{\lambda\in\Lambda}\sigma_1^\Omega[-\Delta+h_\lambda;m_\lambda]\sup_{\lambda\in\Lambda}\|m_\lambda\|_{L^\infty(\Omega)}\int_\Omega z^{\beta+1}.$$

Therefore, by argueeing as in Lemma 6.7 of [27], we deduce that

$$\|\varphi_{\lambda_n}\|_\infty\leq C(\|\varphi_{\lambda_n}\|_2+1).$$

□

1.2 On eigenfunction and eigenvalue limits of unbounded potentials

This section complements Theorem 2.4 in [32]. Specifically, we relax hypothesis (2.7) of that theorem, a modification crucial for Corollary 5.1.1.

Theorem 1.2.1. *Let* $\{m_\lambda\}_{\lambda\geq 0}$ *and* $\{q_\lambda\}_{\lambda\geq 0}$ *be families of funtions such that* $0\lneq m_\lambda\in L^\infty(\Omega)$, $m_\lambda\to m_\infty$ *in* $L^\infty(\Omega)$, $0\leq q_\lambda\in L^\infty(\Omega)$ *and* Ω_V *a connected open subset of* Ω. *Suppose that there is some* $\delta>0$ *such that*

$$|\{x\in U_\delta;\ m_\infty(x)>0\}|>0, \tag{1.2.1}$$

where $U_\delta = \{x \in \Omega_V;\ \mathrm{dist}(x, \partial\Omega_V) \geq \delta\}$,

$$q_\lambda(x) \to +\infty\ \forall\, x \in K,$$

where K *is a compact subset of* $\overline{\Omega} \setminus \Omega_V$. *Moreover, assume that there exists some constant* M *and some* $V \in L^\infty(\Omega_V)$ *such that*

$$q_\lambda(x) \to V(x) \text{ for a.e. } x \in \Omega_V$$

and

$$q_\lambda(x) \leq \frac{M}{d(x)} \text{ for a.e. } x \in \Omega_V, \tag{1.2.2}$$

where $d(x) := \mathrm{dist}(x, \partial\Omega_V)$

Let φ_λ *be the positive eigenfunction associated to*

$$\sigma_1^\Omega[-\Delta + q_\lambda;\ m_\lambda]$$

with $\|\varphi_\lambda\|_{L^2(\Omega)} = 1$. *Also, let* φ_∞ *be the positive eigenfunction associated to* $\sigma_1^{\Omega_V}[-\Delta, m_\infty]$. *Then the following convergences hold.*

$$\sigma_1^\Omega[-\Delta + q_\lambda; m_\lambda] \to \sigma_1^{\Omega_V}[-\Delta + V, m_\infty]$$

and

$$\lim_{\lambda \to +\infty} \left(\|\varphi_\lambda - \varphi_\infty\|^2_{H_0^1(\Omega)} + \int_\Omega q_\lambda \varphi_\lambda^2 \right) = \int_\Omega V(x) \varphi_\infty^2$$

Proof. Note that by using (1.2.2) and the convergence $m_\lambda \to m_\infty$ in $L^\infty(\Omega)$, we get that

$$\begin{aligned} 0 &< \sigma_1^\Omega[-\Delta + q_\lambda; m_\lambda] \\ &\leq \sigma_1^{U_\delta}[-\Delta + q_\lambda; m_\lambda] \\ &\leq \sigma_1^{U_\delta}\left[-\Delta + \frac{M}{\inf_{U_\delta} d(x)}; m_\lambda\right] \\ &\leq \sigma_1^{U_\delta}\left[-\Delta + \frac{M}{\inf_{U_\delta} d(x)}; m_\infty\right] + 1 \\ &< +\infty \end{aligned} \tag{1.2.3}$$

for all $\lambda \geq \lambda_0$, for some large λ_0. So

$$0 \leq \sup_{\lambda \geq 0} \sigma_1^{\Omega}[-\Delta + q_\lambda; m_\lambda] < +\infty. \tag{1.2.4}$$

By the definition of φ_λ, we have

$$\begin{cases} -\Delta\varphi_\lambda + q_\lambda \varphi_\lambda = \sigma_1^{\Omega}[-\Delta + q_\lambda; m_\lambda] m_\lambda \varphi_\lambda & \text{in } \Omega, \\ \varphi_\lambda = 0 & \text{on } \partial\Omega, \\ \varphi_\lambda > 0 & \text{on } \Omega \end{cases} \tag{1.2.5}$$

Take $\lambda_n \to +\infty$. Note that by testing (1.2.5) against φ_{λ_n}, we deduce that φ_{λ_n} is bounded in $H_0^1(\Omega)$. So there is a $\varphi_\infty \in H_0^1(\Omega)$ such that (up to a subsequence) $\varphi_{\lambda_n} \rightharpoonup \varphi_\infty$ in $H_0^1(\Omega)$ and $\varphi_{\lambda_n} \to \varphi_\infty$ in $L^2(\Omega)$. Note that $\varphi_\infty \in H_0^1(\Omega)$ is non-negative and satisfies $\|\varphi_\infty\|_{L^2(\Omega)} = 1$ and so $\varphi_\infty \neq 0$.

We claim that $\varphi_\infty = 0$ in $\Omega \setminus \Omega_V$. Indeed, let $D \subset\subset \Omega \setminus \Omega_V$, consider a positive $\varphi \in C_0^\infty(D)$ and let us abuse the notation by denoting φ as the null extension of φ to Ω. So

$$\begin{aligned} \int_\Omega \nabla\varphi_\infty \nabla\varphi + \liminf\left(\int_{\Omega_V^c} q_{\lambda_n}\varphi_{\lambda_n}\varphi\right) &\leq \liminf\left(\int_\Omega \nabla\varphi_{\lambda_n}\nabla\varphi + \int_{\Omega_V^c} q_{\lambda_n}\varphi_{\lambda_n}\varphi\right) \\ &\leq \liminf\left(\int_\Omega \nabla\varphi_{\lambda_n}\nabla\varphi + \int_\Omega q_{\lambda_n}\varphi_{\lambda_n}\varphi\right) \\ &= \liminf\left(\sigma_1^{\Omega}[-\Delta + q_\lambda; m_\lambda]\int_\Omega m_\lambda \varphi_{\lambda_n}\varphi\right) \\ &< +\infty, \end{aligned}$$

that is,

$$\liminf\left(\int_{\Omega_V^c} q_{\lambda_n}\varphi_{\lambda_n}\varphi\right) < +\infty.$$

By the arbitrariness of the positive φ, we deduce that

$$\varphi_\infty(x) = 0 \;\forall\, x \in \Omega \setminus \Omega_V.$$

Now, let us prove that φ_∞ is the positive eigenfunction associated to $\sigma_1^{\Omega_V}[-\Delta+V,m_\infty]$. Take $\varphi \in H_0^1(\Omega_V)$. Observe that $\varphi \in H_0^1(\Omega_V) \subset H_0^1(\Omega)$, so we can test (1.2.5) against φ as follows.

$$\begin{aligned} \int_{\Omega_V} \nabla\varphi_{\lambda_n}\nabla\varphi + \int_{\Omega_V} q_{\lambda_n}\varphi_{\lambda_n}\varphi &= \int_{\Omega} \nabla\varphi_{\lambda_n}\nabla\varphi + \int_{\Omega} q_{\lambda_n}\varphi_{\lambda_n}\varphi \\ &= \sigma_1^{\Omega}\left[-\Delta+q_{\lambda_n};m_{\lambda_n}\right]\int_{\Omega} m_\lambda\varphi_{\lambda_n}\varphi \\ &= \sigma_1^{\Omega}\left[-\Delta+q_{\lambda_n};m_{\lambda_n}\right]\int_{\Omega_V} m_\lambda\varphi_{\lambda_n}\varphi. \end{aligned} \tag{1.2.6}$$

Note that

$$\lim_{n\to+\infty} q_{\lambda_n}\varphi = V\varphi \ \forall\, x\in\Omega_V.$$

Moreover,

$$q_{\lambda_n}(x)\varphi \le M\frac{\varphi}{d(x)} \ \forall\, x\in\Omega_V.$$

But $|\varphi|/d \in L^2(\Omega_V)$ by the Hardy Inequallity. Consequently, by passing to the limit in (1.2.6), we deduce that φ_∞ is a nontrivial non negative solution of

$$\begin{cases} -\Delta v + V(x)v = \lim_n \sigma_1{}^{\Omega}\left[-\Delta+q_{\lambda_n};m_{\lambda_n}\right] m_\infty v & \text{in } \Omega_V, \\ \varphi_\lambda = 0 & \text{on } \partial\Omega_V, \\ \varphi_\lambda > 0 & \text{on } \Omega_V \end{cases}$$

By the Strong Maximum Principle, it follows that φ_∞ is the positive eigenfunction associated to $\sigma_1^{\Omega_V}[-\Delta,m_\infty]$ and

$$\lim_n \sigma_1^{\Omega}\left[-\Delta+q_{\lambda_n};m_{\lambda_n}\right] = \sigma_1^{\Omega_V}\left[-\Delta+V;m_\infty\right] \tag{1.2.7}$$

up to a subsequence.

Now, by testing (1.2.5) against $\varphi_{\lambda_n} - \varphi_\infty$, we deduce that

$$\begin{aligned}
\int_\Omega |\nabla(\varphi_{\lambda_n} - \varphi_\infty)|^2 &= \int_\Omega \nabla\varphi_{\lambda_n}\nabla(\varphi_{\lambda_n} - \varphi_\infty) - \int_\Omega \nabla\varphi_\infty\nabla(\varphi_{\lambda_n} - \varphi_\infty) \\
&= \int_\Omega \nabla\varphi_{\lambda_n}\nabla(\varphi_{\lambda_n} - \varphi_\infty) - o_n(1) \\
&= \sigma_1^\Omega\left[-\Delta + q_{\lambda_n}; m_{\lambda_n}\right]\int_\Omega m_{\lambda_n}\varphi_{\lambda_n}(\varphi_{\lambda_n} - \varphi_\infty) - \\
&\quad - \int_\Omega q_{\lambda_n}(\varphi_{\lambda_n} - \varphi_\infty) - o_n(1) \\
&= \tilde{o}_n(1) - \int_\Omega q_{\lambda_n}\varphi_{\lambda_n}(\varphi_{\lambda_n} - \varphi_\infty) - o_n(1).
\end{aligned}$$

That is,

$$\int_\Omega |\nabla(\varphi_{\lambda_n} - \varphi_\infty)|^2 + \int_\Omega q_{\lambda_n}\varphi_{\lambda_n}^2 = \tilde{o}_n(1) - o_n(1) + \int_\Omega q_{\lambda_n}\varphi_{\lambda_n}\varphi_\infty. \tag{1.2.8}$$

Now, observe that

$$q_{\lambda_n}(x)\varphi_\infty(x) \le M\frac{\varphi_\infty(x)}{d(x)} \;\forall\, x \in \Omega.$$

But

$$q_{\lambda_n}(x)\varphi_\infty(x) \to V(x)\varphi_\infty(x) \text{ for almost every } x \in \Omega.$$

So

$$\int_\Omega q_{\lambda_n}\varphi_{\lambda_n}\varphi_\infty \to \int_\Omega V(x)\varphi_\infty^2.$$

Finally, by passing to the limit in (1.2.8), we deduce that

$$\int_\Omega |\nabla(\varphi_{\lambda_n} - \varphi_\infty)|^2 + \int_\Omega q_{\lambda_n}\varphi_{\lambda_n}^2 \to \int_\Omega V(x)\varphi_\infty^2.$$

□

Remark 1.2.1. *Observe that in the particular case $V \equiv 0$, we have that $\varphi_\lambda \to \varphi_\infty$ in $H_0^1(\Omega)$.*

1.3 First variations of the first eigenvalues due to domain perturbations

The results of this section sharpens [47] by refining the condition of the function R (see condition iii) of Theorem 1.3.1). This refinement is necessary to prove (5.2.22).

Let us set for sufficiently small $\delta \sim 0$,

$$B_\delta := \begin{cases} \{x \in \Omega \ : \ \text{dist}\,(x, B_0) < \delta\}, \text{ if } \delta > 0 \\ \{x \in B_0 \ : \ \text{dist}\,(x, \partial B_0) > -\delta\}, \text{ if } \delta < 0. \end{cases}$$

Theorem 1.3.1. *There exists a bijection $T_\delta \in \mathcal{C}^2(\bar{B}_0; \mathbb{R}^N)$, $T_\delta : \bar{B}_0 \to \bar{B}_\delta$ and $\varepsilon_0 > 0$ satisfying the following properties:*

i) The family T_δ is real holomorphic in δ for $\delta \sim 0$, i.e., every T_δ is a $\mathcal{C}^2$-diffeomorphism that can be expressed in the form

$$T_\delta(x) = x + \delta R(x), \quad x \in \bar{B}_0, \tag{1.3.1}$$

with $R \in \mathcal{C}^2(\bar{B}_0; \mathbb{R}^N)$ and

$$\|D_x^k R\|_{\infty, B_0} := \sup_{x \in \bar{B}_0} \|D_x^k R(x)\|, \quad 0 \le k \le 2.$$

ii) For each $x \in \bar{B}_0$ such that $\text{dist}(x, \partial B_0) < \varepsilon_0$, it is well defined the normal projection of x onto ∂B_0, $\pi(x) \in \partial B_0$,

iii) $R(x) = n(\pi(x))$ for all $x \in \bar{B}_0$ such that $\text{dist}(x, \partial B_0) < \varepsilon_0/4$, where n is the outward unit normal vector in ∂B_0.

Proof. Since $\partial B_0 \in \mathcal{C}^2$, then there exists $\varepsilon_0 > 0$ such that the set

$$A = \{x \in \mathbb{R}^N; \text{dist}(x, \partial B_0) < \varepsilon_0\}$$

is a tubular neighborhood of ∂B_0 due to [29], in the sense that, for each $x \in A$, there exist unique $z \in \partial B_0$ and $-\varepsilon_0 < \tau < \varepsilon_0$ such that

$$x = z - \tau n(z), \tag{1.3.2}$$

where $n(z)$ is the outward normal vector in z. By reducing $\varepsilon_0 > 0$ if necessary, we have functions $\tau \in C^2(A,\mathbb{R})$ and $\pi \in C^2(A,\partial B_0)$ such that

$$x = \pi(x) - \tau(x)n(\pi(x)),\ \forall\, x \in A. \tag{1.3.3}$$

By the definition of τ, we have

$$\tau(x) = \mathrm{dist}(x,\partial B_0)\ \forall\, x \in A. \tag{1.3.4}$$

Let $\hat{\tau} \in C^2(\bar{B}_0;\mathbb{R})$ be the extension of τ to $\bar{B}_0$ defined by $\hat{\tau}(x) = \varepsilon_0$ if $\mathrm{dist}(x,\partial B_0) \geq \varepsilon_0$. Let $\hat{n} \in C^2(\bar{B}_0;\mathbb{R}^N)$ be any smooth extension of the field n to $\bar{B}_0$ and consider any function $\zeta \in C^3([0,\infty);[0,\infty))$ satisfying

$$\zeta(\tau) = 1,\ \tau \in [0,\varepsilon_0/4),\ \zeta(\tau)\zeta'(\tau) < 0,\ \tau \in (\varepsilon_0/4,\varepsilon_0/2),\ \zeta(\tau) = 0,\ \tau \geq \varepsilon_0/2.$$

Define the mapping

$$R(x) := \zeta(\hat{\tau}(x))\hat{n}(x)\ \forall\, x \in \overline{\Omega}.$$

By using (1.3.4), it is easy to check that

$$R(x) = \begin{cases} 0 \text{ if } \mathrm{dist}(x,\partial B_0) \geq \varepsilon_0/2, & (1.3.5)\\ \zeta(\tau(x))n(\pi(x)) \text{ if } \mathrm{dist}(x,\partial B_0) < \varepsilon_0/2, & (1.3.6)\\ n(\pi(x)) \text{ if } \mathrm{dist}(x,\partial B_0) < \varepsilon_0/4. & (1.3.7)\end{cases}$$

Let us define

$$T_\delta(x) = x + \delta R(x),\ \forall\, x \in \bar{B}_0.$$

Let us prove that T_δ is a bijection. Observe that by (1.3.5), the restriction of T_δ to $V := \{x \in B_0; \mathrm{dist}(x,\partial B_0) \geq \varepsilon_0/2\}$ is the identity map. On the other hand, if $x \in U := \{x \in \bar{B}_0; \mathrm{dist}(x,\partial B_0) \leq \varepsilon_0/2\}$, then $T_\delta(x) = x$ or

$$T_\delta(x) = x + \delta\zeta(\tau(x))n(\pi(x)).$$

Then using (1.3.3), it can be easily verified that T_δ is a bijection from U to $U \cup \{x \in \mathbb{R}^N \setminus B_0;\ \mathrm{dist}(x,\partial B_0) \leq \delta\}$ ($U \cup \{x \in B_0;\ \mathrm{dist}(x,\partial B_0) \leq -\delta\}$, respectively), if $\delta > 0$ ($\delta < 0$, respectively). The function T_δ satisfies all the requirements of the theorem. □

Consider the principal eigenvalue, $\sigma_1^{B_\delta}[-\Delta-\mu(\lambda)m(x)+\lambda V(x)]$, of the linear eigenvalue problem

$$\begin{cases} -\Delta\varphi-\mu(\lambda)m(x)\varphi+\lambda V(x)\varphi=\sigma\varphi & \text{in } B_\delta, \\ \varphi=0 & \text{on } \partial B_\delta, \\ \varphi>0 & \text{in } B_\delta \end{cases}$$

is well defined, as well as its associated principal eigenfunction, φ_λ, normalized so that $\|\varphi_\lambda\|_2=1$. Moreover, as a direct consequence of the strong maximum principle, $\varphi_\lambda \gg 0$ for every $\lambda \geq 0$ and sufficiently small $\delta \geq 0$.

Take now a real function μ analytic in $(\Lambda-\eta,\Lambda+\eta)$, $\eta>0$ and such that

$$\mu(\Lambda)=\sigma_1^{B_0}[-\Delta+\Lambda V(x);m(x)].$$

On the other hand, consider the principal eigenvalue, $\sigma_1^{B_\delta}[-\Delta-\mu(\lambda)m(x)+\lambda V(x)]$ and its principal eigenfunction, φ_λ, normalized so that $\|\varphi_\lambda\|_2=1$.

The next result is the main theorem of this section. It is a substantial extension of Theorem 2.1 of López-Gómez and Sabina de Lis [46].

Theorem 1.3.2. *Let $\eta>0$ and assume that μ is an analytic function defined in $(\Lambda-\eta,\Lambda+\eta)$. Also assume that m and V are analytic in a neighborhood of $\overline{B_0}$ and*

$$\mu(\Lambda)=\sigma_1^{B_0}[-\Delta+\Lambda V(x);m(x)]. \tag{1.3.8}$$

Then, so is the family $\left(\sigma_1^{B_\delta}[-\Delta-\mu(\lambda)m(x)+\lambda V(x)],\psi_{\lambda,\delta}\right)$, where $\psi_{\lambda,\delta}(x)=\varphi_{\lambda,\delta}(T_\delta(x))$ for all $x\in\bar{B}_0$. In particular,

$$\begin{cases} \psi_{\lambda,\delta}=\psi_{\Lambda,0}+\delta\psi_1^{(0,1)}+(\lambda-\Lambda)\psi_1^{(1,0)}+r(\lambda,\delta), \\ \sigma_1^{B_\delta}[-\Delta+\lambda V(x)-\mu(\lambda)m(x)]=\delta\lambda_1^{(0,1)}+(\lambda-\Lambda)\lambda_1^{(1,0)}+g(\lambda,\delta), \end{cases} \tag{1.3.9}$$

where r and g are functions that satisfy

$$\lim_{\delta\to 0}\frac{r(\Lambda,\delta)}{\delta}=\lim_{\delta\to 0}\frac{g(\Lambda,\delta)}{\delta}=0.$$

Moreover,

$$\lambda_1^{(0,1)} = -\int_{\partial B_0} \left(\frac{\partial \psi_{\Lambda,0}}{\partial n}\right)^2 dS < 0, \qquad \lambda_1^{(1,0)} = -\int_{B_0} (\mu'(\Lambda)m(x) - V(x))\varphi_{\Lambda,0}^2. \quad (1.3.10)$$

Proof. To simplify the notation, we denote by

$$f(\lambda,\delta) := \sigma_1^{B_\delta}[-\Delta + \lambda V(x) - \mu(\lambda)m(x)]$$

To find out $\lambda_1^{(1,0)}$, we can proceed as follows. By setting and differentiating with respect to λ the problem

$$\begin{cases} -\Delta\varphi_{\lambda,0} + \lambda V(x)\varphi_{\lambda,0} - \mu(\lambda)m(x)\varphi_{\lambda,0} = f(\lambda,0)\varphi_{\lambda,0} & \text{in } B_0, \\ \varphi_{\lambda,0} = 0 & \text{on } \partial B_0, \\ \varphi_{\lambda,0} > 0 & \text{in } B_0 \end{cases} \quad (1.3.11)$$

we obtain that

$$\begin{cases} (-\Delta + \lambda V(x) - \mu(\lambda)m(x) - f(\lambda,0))\varphi'_{\lambda,0} = (\mu'(\lambda)m(x) - V(x) + f'(\lambda,0))\varphi_{\lambda,0} & \text{in } B_0, \\ \varphi'_{\lambda,0} = 0 & \text{on } \partial B_0, \end{cases}$$

Thus, multiplying by $\varphi_{\lambda,0}$ and integrating by parts in B_0, we find that

$$f'(\lambda,0) = -\frac{\int_{B_0}(\mu'(\lambda)m(x) - V(x))\varphi_{\lambda,0}^2}{\int_{B_0}\varphi_{\lambda,0}^2}. \quad (1.3.12)$$

Consequently, particularizing at $\lambda = \Lambda$ yields

$$\lambda_1^{(1,0)} = -\int_{B_0} (\mu'(\Lambda)m(x) - V(x))\varphi_{\Lambda,0}^2.$$

The calculation of $\lambda_1^{(0,1)}$ is much more complicated and involved. Let us set

$$y := T_\delta(x), \qquad T_\delta^{-1}(y) := (h_1(y,\delta),\ldots,h_N(y,\delta)).$$

A calculation leads us to

$$\mathscr{L}(x,D_x,\lambda,\delta)\psi_{\lambda,\delta} = f(\lambda,\delta)\psi_{\lambda,\delta} \quad \text{in } B_0, \psi_{\lambda,\delta} = 0 \quad \text{on } \partial B_0, \tag{1.3.13}$$

where we have denoted

$$\mathscr{L}(x,D_x,\lambda,\delta) := -\sum_{k,\ell=1}^{N} \langle D_y h_k, D_y h_\ell \rangle \frac{\partial^2}{\partial x_k \partial x_\ell} - \sum_{\ell=1}^{N} \Delta_y h_\ell \frac{\partial}{\partial x_\ell} - \mu(\lambda) m(T_\delta(\,\cdot\,)) + \lambda V(T_\delta(\,\cdot\,)).$$

Now, arguing as in [46] it is easily seen that the coefficients $\langle D_y h_k, D_y h_\ell \rangle$ and $\Delta_y h_\ell$ are real analytic in δ for $\delta \sim 0$, and that they are given by

$$\langle D_y h_k, D_y h_\ell \rangle = \delta_{k\ell} - \delta \left(\frac{\partial R_k}{\partial x_\ell}(x) + \frac{\partial R_\ell}{\partial x_k}(x) \right) + o(\delta^2), \tag{1.3.14}$$

$$\Delta_y h_\ell(T_\delta(x)) = -\delta \Delta_x R_\ell(x) + o(\delta^2). \tag{1.3.15}$$

Moreover, for $x \in B_0$ and δ sufficiently close to zero, we have

$$m(T_\delta(x)) \;=\; m(x + \delta R(x)) = m(x) + \sum_{k=1}^{\infty} \delta^k M^{(k)}(x)$$

and

$$V(T_\delta(x)) = V(x) + \sum_{k=1}^{\infty} \delta^k G^{(k)}(x). \tag{1.3.16}$$

By combining (1.3.14), (1.3.15), (1.3.16) and (1.3.16), we deduce that for each given $\varphi \in H_0^1(\Omega)$ and $\psi \in L^2(\Omega)$, there exists $a_{ij} \in \mathbb{R}$ such that

$$\int_{B_0} \varphi \mathscr{L}(x,D_x,\lambda,\delta)\psi = \sum_{i,j=0}^{\infty} a_{ij} \delta^i (\lambda - \Lambda)^j \text{ for } (\lambda,\delta) \sim (\Lambda, 0).$$

According to Kato [35] (see [46]), the above equation implies that the family

$$\big(f(\lambda,\delta), \varphi_{\lambda,\delta} \big)$$

is real analytic at $(\Lambda, 0)$, for $t \sim \Lambda$ and $\delta \geq 0$.

By substituting (1.3.14) and (1.3.15) in (1.3.13) with $\lambda = \Lambda$, it follows that

$$-\sum_{k,\ell=1}^{N}\left[\delta_{k\ell}-\delta\left(\frac{\partial R_k}{\partial x_\ell}(x)+\frac{\partial R_\ell}{\partial x_k}(x)\right)\right]\frac{\partial^2 \psi_{\Lambda,\delta}}{\partial x_k \partial x_\ell}(x)+\delta\sum_{\ell=1}^{N}\Delta_x R_\ell \frac{\partial \psi_{\Lambda,\delta}}{\partial x_\ell}(x) \tag{1.3.17}$$

$$-\mu(\Lambda)m(T_\delta(x))\psi_{\Lambda,\delta}(x)+\Lambda V(T_\delta(x)) = \tag{1.3.18}$$

$$f(\lambda,\delta)\psi_{\Lambda,\delta}(x)+o(\delta^2). \tag{1.3.19}$$

On the other hand,

$$\mu(\Lambda)m(T_\delta(x))-\Lambda V(T_\delta(x)) = w(x)+\delta w^{(1)}(x)+o(\delta^2), \tag{1.3.20}$$

where, owing to (1.3.1),

$$w^{(1)}(x) = \left.\frac{d}{d\delta}\right|_{\delta=0}[\mu(\Lambda)m(T_\delta(x))-\Lambda V(T_\delta(x))] =$$

$$\langle \mu(\Lambda)\nabla m(x)-\Lambda\nabla V(x), \left.\frac{d}{d\delta}\right|_{\delta=0} T_\delta\rangle =$$

$$\langle \mu(\Lambda)\nabla m(x)-\Lambda\nabla V(x), R\rangle. \tag{1.3.21}$$

Thus, by substituting (1.3.20) into (1.3.17), we get

$$-\sum_{k,\ell=1}^{N}\left[\delta_{k\ell}-\delta\left(\frac{\partial R_k}{\partial x_\ell}(x)+\frac{\partial R_\ell}{\partial x_k}(x)\right)\right]\frac{\partial^2 \psi_{\Lambda,\delta}}{\partial x_k \partial x_\ell}(x)+\delta\sum_{\ell=1}^{N}\Delta_x R_\ell \frac{\partial \psi_{\Lambda,\delta}}{\partial x_\ell}(x) \tag{1.3.22}$$

$$-\mu(\Lambda)m(x)\psi_{\Lambda,\delta}(x)+\Lambda V(x)\psi_{\Lambda,\delta}(x) = \delta w^{(1)}(x)\psi_{\Lambda,\delta}(x) \tag{1.3.23}$$

$$+\sigma_1^{B_\delta}[-\Delta-\mu(\Lambda)m(x)+\Lambda V(x)]\psi_{\Lambda,\delta}(x)+o(\delta^2). \tag{1.3.24}$$

Moreover, since (1.3.9) holds and substituting it in (1.3.22) with $\lambda = \Lambda$ yields to

$$\begin{aligned}
&-\sum_{k,\ell=1}^{N}\left[\delta_{k\ell}-\delta\left(\frac{\partial R_k}{\partial x_\ell}(x)+\frac{\partial R_\ell}{\partial x_k}(x)\right)\right]\frac{\partial^2\psi_{\Lambda,0}}{\partial x_k\partial x_\ell}\\
&-\delta\sum_{k,\ell=1}^{N}\left[\delta_{k\ell}-\delta\left(\frac{\partial R_k}{\partial x_\ell}(x)+\frac{\partial R_\ell}{\partial x_k}(x)\right)\right]\frac{\partial^2\psi_1^{(0,1)}}{\partial x_k\partial x_\ell}\\
&+\delta\sum_{\ell=1}^{N}\Delta_x R_\ell\frac{\partial\psi_{\Lambda,0}}{\partial x_\ell}+\delta^2\sum_{\ell=1}^{N}\Delta_x R_\ell\frac{\partial\psi_1^{(0,1)}}{\partial x_\ell}-\mu(\Lambda)m(x)\psi_{\Lambda,0}(x)\\
&+\Lambda V(x)\psi_{\Lambda,0}(x)-\delta(\mu(\Lambda)m(x)-\Lambda V(x))\psi_1^{(0,1)}\\
&=\delta w^{(1)}(x)\psi_{\Lambda,0}(x)+\delta^2 w^{(1)}(x)\psi_1^{(0,1)}(x)+\delta\lambda_1^{(0,1)}\psi_{\Lambda,0}+o(\delta^2).
\end{aligned}$$

But, since

$$-\sum_{k,\ell=1}^{N}\delta_{k\ell}\frac{\partial^2\psi_{\Lambda,0}}{\partial x_k\partial x_\ell}=-\Delta_x\psi_{\Lambda,0}=\mu(\Lambda)m(x)\psi_{\Lambda,0}-\Lambda V(x)\psi_{\Lambda,0},$$

substituting this identity into the previous one, we find that

$$\begin{aligned}
&\sum_{k,\ell=1}^{N}\delta\left(\frac{\partial R_k}{\partial x_\ell}(x)+\frac{\partial R_\ell}{\partial x_k}(x)\right)\frac{\partial^2\psi_{\Lambda,0}}{\partial x_k\partial x_\ell}-\delta\sum_{k,\ell=1}^{N}\left[\delta_{k\ell}-\delta\left(\frac{\partial R_k}{\partial x_\ell}(x)+\frac{\partial R_\ell}{\partial x_k}(x)\right)\right]\frac{\partial^2\psi_1^{(0,1)}}{\partial x_k\partial x_\ell}\\
&+\delta\sum_{\ell=1}^{N}\Delta_x R_\ell\frac{\partial\psi_{\Lambda,0}}{\partial x_\ell}+\delta^2\sum_{\ell=1}^{N}\Delta_x R_\ell\frac{\partial\psi_1^{(0,1)}}{\partial x_\ell}-\delta(\mu(\Lambda)m(x)-\Lambda V(x))\psi_1^{(0,1)}\\
&=\delta w^{(1)}(x)\psi_{\Lambda,0}(x)+\delta^2 w^{(1)}(x)\psi_1^{(0,1)}(x)+\delta\lambda_1^{(0,1)}\psi_{\Lambda,0}+O(\delta^2).
\end{aligned}$$

Consequently, dividing by δ and letting $\delta\to 0$, it becomes apparent that

$$\begin{aligned}
&\sum_{k,\ell=1}^{N}\left(\frac{\partial R_k}{\partial x_\ell}(x)+\frac{\partial R_\ell}{\partial x_k}(x)\right)\frac{\partial^2\psi_{\Lambda,0}}{\partial x_k\partial x_\ell}-\Delta_x\psi_1^{(0,1)}+\sum_{\ell=1}^{N}\Delta_x R_\ell\frac{\partial\psi_{\Lambda,0}}{\partial x_\ell}-\mu(\Lambda)m(x)\psi_1^{(0,1)}+\\
&\Lambda V(x)\psi_1^{(0,1)}=w^{(1)}(x)\psi_{\Lambda,0}(x)+\lambda_1^{(0,1)}\psi_{\Lambda,0}.
\end{aligned}$$

On the other hand, by applying the Fredholm Alternative and using that

$$\psi_{\Lambda,0}(x)=\varphi_{\Lambda,0}(x),\qquad \|\psi_{\Lambda,0}\|_{L^2(B_0)}=\|\varphi_{\Lambda,0}\|_{L^2(B_0)}=1,$$

we deduce that

$$\lambda_1^{(0,1)} = \int_{B_0} \sum_{k,\ell=1}^{N} \left(\frac{\partial R_k}{\partial x_\ell}(x) + \frac{\partial R_\ell}{\partial x_k}(x) \right) \frac{\partial^2 \psi_{\Lambda,0}}{\partial x_k \partial x_\ell} \psi_{\Lambda,0} + \sum_{\ell=1}^{N} \int_{B_0} \Delta_x R_\ell \frac{\partial \psi_{\Lambda,0}}{\partial x_\ell} \psi_{\Lambda,0} - \int_{B_0} w^{(1)} \psi_{\Lambda,0}^2. \tag{1.3.25}$$

By arguing as in [47], that is, by integrating by parts, divergence theorem and using the definition of $\psi_{\Lambda,0}$, we have that

$$\lambda_1^{(0,1)} = -2 \int_{\partial B_0} \langle R, \nabla \psi_{\Lambda,0} \rangle \langle \nabla \psi_{\Lambda,0}, n \rangle dS + \int_{\partial B_0} |\nabla \psi_{\Lambda,0}|^2 \langle R, n \rangle dS + \int_{B_0} \langle \nabla(\mu(\Lambda) m(x) - \Lambda V(x)), R \rangle \psi_{\Lambda,0}^2 - \int_{B_0} w^{(1)} \psi_{\Lambda,0}^2.$$

Therefore, owing to (1.3.21), we find that

$$\lambda_1^{(0,1)} = -2 \int_{\partial B_0} \langle R, \nabla \psi_{\Lambda,0} \rangle \langle \nabla \psi_{\Lambda,0}, n \rangle dS + \int_{\partial B_0} |\nabla \psi_{\Lambda,0}|^2 \langle R, n \rangle dS. \tag{1.3.26}$$

Finally, taking into account that

$$\nabla \psi_{\Lambda,0}(x) = \frac{\partial \psi_{\Lambda,0}}{\partial n}(x) n(x) \quad \text{for all } x \in \partial B_0,$$

it becomes apparent that (1.3.26) provides us with the identity

$$\lambda_1^{(0,1)} = -\int_{\partial B_0} \langle R, n \rangle \left(\frac{\partial \psi_{\Lambda,0}}{\partial n} \right)^2 dS.$$

On the other hand, one can assume that $R|_{\partial B_0} = n$, where n stands for the unit outward normal field of ∂B_0 (see Section 3 of [46]). Therefore.

$$\lambda_1^{(0,1)} = -\int_{\partial B_0} \left(\frac{\partial \psi_{\Lambda,0}}{\partial n} \right)^2 dS < 0. \tag{1.3.27}$$

This ends the proof. □

Conclusion of Chapter 1

The $H_0^1(\Omega)$-estimate in Lemma 1.1.1 complements Lemma 6.7 of [27] by incorporating the weight m_λ. Notably, this estimate does not require uniform boundedness of m_λ with respect to λ. While this aspect was not explored here due to the uniform boundedness of m_λ in our applications, it presents an avenue for future research. Another potential direction for future work is to investigate the case where the limit V of the potentials q_λ in Theorem 1.2.1 is non-zero. Theorem 1.3.2 demonstrates that the foundational ideas of [46], later refined in [47], remain effective even with the inclusion of an additional parameter. We emphasize that its application in Theorem 5.2.2 to prove blow-up on ∂B_0 necessitates the transversality condition (0.0.23). This technical necessity naturally raises the question of whether blow-up on ∂B_0 still occurs without this additional condition.

Chapter 2

Connected set of solutions from a continuation theorem on open sets

This chapter is focused in providing the abstract results on topological methods that will be applied in the following chapter in order to obtain existence and qualitative information about positive solutions of the problems ($P_{\lambda,\mu}$) and ($Q_{\lambda,\mu}$), noted in the Introduction.

Section 2.1 is dedicated to a continuation theorem. In Section 2.2 we prove a positiveness-continuity-principle based on connectedness properties. This principle we guarantee the strong positiveness of the solutions lying in the connected sets of solutions of ($P_{\lambda,\mu}$) and ($Q_{\lambda,\mu}$). Finally, in Section 2.3, we prove an equivalence between two concepts from distinct contexts: the abstract formulation of compact operators K and the realm of PDEs. Specifically, we connect the proximity of a set of positive solutions for the equation $u = K(\lambda, u)$ (in the sense defined in this chapter) with a region in the parameter space of a PDE that corresponds to singularities.

2.1 A Continuation Theorem for operators defined on open subsets

The main goal of this section is to prove Theorem 0.0.1.

After this motivation, let us fix additional notations as those done in the Introduction. For each open subset A (A^{ν}) of $\mathcal{U}$ ($\mathcal{U}^{\nu}$) and $\lambda \in \mathbb{R}$, the λ-slice of A (A^{ν}) is defined by

$$A_{\lambda} = \{u \in E;\ (\lambda, u) \in A\}\ (A^{\nu}_{\lambda} = \{u \in E;\ (\lambda, u) \in A^{\nu}\})$$

and the λ-projection of A is given by

$$\text{proj}_\lambda A = \{\lambda \in \mathbb{R};\ (\lambda,u) \in A \text{ for some } u \in E\}.$$

Let us denote by $\mathscr{B}_r(\lambda,u)$ ($\mathscr{B}_r^\nu(\lambda,u)$) the open ball of $\mathcal{U}$ ($\mathcal{U}^\nu$) centered at (λ,u) with radius $r>0$, $B_r(u)$ is the open ball of E centered at u with radius $r>0$, and remind that a subset S of $\overline{\mathcal{U}}$ is considered a maximal connected subset if it is connected and not properly contained in any other connected subset of S.

Before starting, let us give a briefing on our strategy for the proof of our theorem. First, we show that $\mathscr{S}^+ \cap \mathscr{S}^- \neq \emptyset$. So, we can take, for each side, the component of solutions with $\lambda \neq \lambda_0$. In the sequel, we assume by absurd that $\mathscr{S}^\nu$ (for $\nu \in \{-,+\}$) does not satisfy neither of the alternatives $i), ii)$ nor $iii)$. As a consequence of this assumption, we can use a topological result due to Whyburn (see Lemma 2.1.3) in order to construct an admissible open neighbourhood of this component (see Lemma 2.1.4). Finally, by using a generalized homotopy property (see Lemma 2.1.1), we get to a contradiction with the fact that the index of Φ at $\lambda = \lambda_0$ relative to 0 be different from zero.

Now we will enunciate some auxiliary lemmas. The first one guarantees that the λ-degree function $\lambda \mapsto \deg(\Phi_\lambda, \mathcal{O}_\lambda, 0)$ is locally constant.

Lemma 2.1.1. *Let $\nu \in \{-,+\}$ and $\mathcal{O}^\nu$ be a bounded and open subset of $\mathcal{U}^\nu$ such that $0 \notin \Phi(\partial\mathcal{O}^\nu)$ and $\text{dist}(\mathcal{O}^\nu, \partial\mathcal{U}^\nu) > 0$. Then*

$$\deg(\Phi_\lambda, \mathcal{O}_\lambda, 0) \textit{ is constant for every } \lambda \in \text{proj}_\lambda(\mathcal{U}^\nu).$$

Proof. This lemma is a slightly generalization of Theorem 4.1 of [7], where the set $[a,b]\times U$ is substituted by the open subset $\mathcal{U}^\nu$ of $\mathbb{R}^\nu_{\lambda_0}$. The proof is essentially the same. □

The next result shows that $\mathscr{S}^- \cap \mathscr{S}^+$ contains certain isolated solutions.

Lemma 2.1.2. *Let $(\lambda_0,u_0) \in \mathcal{U}$. Suppose that $u_0 \in E$ is an isolated solution of the equation $\Phi_{\lambda_0}(u) = 0$ and that the index $i(\Phi_{\lambda_0}, u_0, 0) \neq 0$. Then $(\lambda_0,u_0) \in \mathscr{S}^- \cap \mathscr{S}^+$.*

Proof. Take a $\nu \in \{-,+\}$ and assume that $(\lambda_0,u_0) \notin \mathscr{S}^\nu$. Therefore, there is $r>0$ such that (λ_0,u_0) is the only solution of $\Phi(\lambda,u) = 0$ in $\overline{\mathscr{B}_r}^\nu(\lambda_0,u_0)$ so that

$$\deg(\Phi_{\lambda_0}, (\mathscr{B}_r^\nu(\lambda_0,u_0))_{\lambda_0}, 0) = \text{i}(\Phi_{\lambda_0}, u_0, 0) \neq 0.$$

Now, by applying Lemma 2.1.1, we deduce that

$$\begin{aligned}\deg(\Phi_{\lambda_0},(\mathscr{B}_r^\nu(\lambda_0,u_0))_{\lambda_0},0) &= \deg(\Phi_r,(\mathscr{B}_r^\nu(\lambda_0,u_0))_r,0)\\ &= \deg(\Phi_\lambda,(\mathscr{B}_r^\nu(\lambda_0,u_0))_\lambda,0)\\ &= \deg(\Phi_r,(\mathscr{B}_r^\nu(\lambda_0,u_0))_r,0)\\ &= \deg(\Phi_r,\emptyset,0)=0.\end{aligned}$$

for $|\lambda-\lambda_0|\geq r$. This contradiction ends the proof. □

Also we will use the following classical result by [58].

Lemma 2.1.3 (Ch. I, Item (9.3), [58])**.** *Let Z be a compact metric space and A and B be disjoint closed subsets of Z. Then either there exists a continuum of Z intersecting both A and B or*

$$Z=Z_A\cup Z_B,$$

where Z_A and Z_B are two disjoint compact subsets of Z containing A and B, respectively.

Finally, we are in a position to establish the following crucial result:

Lemma 2.1.4. *Let $(\lambda_0,u_0)\in\mathcal{U}$, $u_0\in E$ be an isolated solution of $\Phi_{\lambda_0}(u)=0$, and $\mathscr{C}^\nu$ be the component of $\mathscr{S}^\nu$ containing (λ_0,u_0) for any $\nu\in\{-,+\}$ given. If $\mathscr{C}^\nu$ does not satisfy neither of the alternatives i), ii) nor iii) of Theorem 0.0.1, then there exists a bounded open subset $\mathcal{O}^\nu$ of $\mathcal{U}^\nu$ containing $\mathscr{C}^\nu$ such that the set*

$$\partial(\mathcal{O}^\nu)\cap\mathscr{S}=\emptyset \tag{2.1.1}$$

and

$$u=u_0 \text{ if } (\lambda,u)\in\mathscr{S} \text{ and } u\in\mathcal{O}^\nu_{\lambda_0}. \tag{2.1.2}$$

Proof. By the assumption that the alternative $i)$ does not occur, we have that $\overline{\mathscr{C}^\nu}$ is a bounded and closed subset of E. Moreover, since the alternative $ii)$ does not occur, that is,

$$\operatorname{dist}(\overline{\mathscr{C}^\nu},\partial\mathcal{U})>0, \tag{2.1.3}$$

we have from the compactness of the operator K in the open subset $\mathcal{U}$ that K is compact in $\overline{\mathscr{C}^\nu}$.

Since

$$\operatorname{dist}(\overline{\mathscr{C}^\nu},\partial\mathcal{U}^\nu)\geq\operatorname{dist}(\overline{\mathscr{C}^\nu},\partial\mathcal{U})>0,$$

because $\partial\mathcal{U}^\nu \subset \partial\mathcal{U}$, we have that

$$\mathcal{U}_\delta^\nu := \{(\lambda,u)\in\mathcal{U}^\nu;\ \mathrm{dist}((\lambda,u),\overline{\mathscr{C}^\nu}) < \delta\}$$

is an open bounded subset of $\mathcal{U}^\nu$ satisfying

$$\partial\mathcal{U}_\delta^\nu \cap \overline{\mathscr{C}^\nu} = \emptyset \tag{2.1.4}$$

for each $0 < \delta < \mathrm{dist}(\overline{\mathscr{C}^\nu}, \partial\mathcal{U}^\nu)$.

By using (2.1.3) again, we can shorten $\delta > 0$, if necessary, to still obtain $\mathrm{dist}(\overline{\mathcal{U}_\delta^\nu}, \partial\mathcal{U}) > 0$. So, it follows from the compactness of the operator K in the open subset $\mathcal{U}$ that K is compact in $\overline{\mathcal{U}_\delta^\nu}$ as well. Combining the compactness of K in $\overline{\mathcal{U}_\delta^\nu}$ with the fact that $u_0 \in E$ is an isolated solution of $\Phi_{\lambda_0}(u) = 0$, and the assumption that $\mathscr{C}^\nu$ does not satisfy *iii*), we are able to take $\delta > 0$ small enough such that

$$u = u_0 \ \text{ if } (\lambda_0,u)\in\mathscr{S} \text{ and } u \in \left(\mathcal{U}_\delta^\nu\right)_{\lambda_0}. \tag{2.1.5}$$

Since $\mathscr{S}$ is a relative-closed set in $\mathcal{U}$, and $\mathrm{dist}(\overline{\mathcal{U}_\delta^\nu}\cap\mathscr{S}, \partial\mathcal{U}) \geq \mathrm{dist}(\overline{\mathcal{U}_\delta^\nu}, \partial\mathcal{U}) > 0$, we are in conditions to apply Lemma 2.1.3 to the sets

$$Z := \overline{\mathcal{U}_\delta^\nu}\cap\mathscr{S},\ A := \overline{\mathscr{C}^\nu},\ \text{and } B := \partial(\mathcal{U}_\delta^\nu)\cap\mathscr{S}.$$

Let us consider the two alternatives given by Lemma 2.1.3. If there were a continuum $\mathscr{F}^\nu$ of $\mathscr{S}$ connecting A and B, then there would exists a $p \in \mathscr{F}^\nu \cap B$ so that $p \in \partial\mathcal{U}_\delta^\nu$ whence follows together with (2.1.4) that $p \notin A \supseteq \mathscr{C}^\nu$ and so $\mathscr{C}^\nu$ would be a proper subset of the continuum $\mathscr{F}^\nu \cup \mathscr{C}^\nu$ of $\mathscr{S}$, that leads to the contradiction of the maximality of $\mathscr{C}^\nu$. So, the second possibility of Lemma 2.1.3 must occur, that is, there exist two compact sets Z_A and Z_B containing A and B, respectively, such that

$$Z = Z_A \cup Z_B \ \text{ and } \ Z_A \cap Z_B = \emptyset. \tag{2.1.6}$$

Since $Z_A \subset Z = \overline{\mathcal{U}_\delta^\nu}\cap\mathscr{S}$, we have from (2.1.6), that

$$Z_A \cap \partial(\mathcal{U}_\delta^\nu) = \emptyset$$

whence follows together with $Z_A \subset \overline{\mathcal{U}^\nu_\delta} \cap \mathcal{S}$ again that

$$Z_A \subset \mathcal{U}^\nu_\delta. \tag{2.1.7}$$

As a consequence of (2.1.7), (2.1.6), the compactness of Z_A, and Z_A and Z_B to be disjoint compact sets, we are able to take an open bounded neighbourhood $\mathcal{O}^\nu \subset \mathcal{U}^\nu$ of Z_A such that

a) $\overline{\mathcal{O}^\nu} \cap Z_B = \emptyset$,

b) $\overline{\mathcal{O}^\nu} \subset \mathcal{U}^\nu_\delta$,

that leads to

$$\partial(\mathcal{O}^\nu) \cap \mathcal{S} = \emptyset.$$

On the contrary, there would exist

$$q \in \partial(\mathcal{O}^\nu) \cap \mathcal{S}, \tag{2.1.8}$$

leading to

$$q \in \partial(\mathcal{O}^\nu) \subset \overline{\mathcal{O}^\nu} \subset \mathcal{U}^\nu_\delta \subset \overline{\mathcal{U}^\nu_\delta},$$

that would imply by (2.1.8) again that

$$q \in \overline{\mathcal{U}^\nu_\delta} \cap \mathcal{S} = Z. \tag{2.1.9}$$

On the other hand, one follows from $a)$ and $q \in \partial(\mathcal{O}^\nu)$ that $q \notin Z_B$. Besides this, $q \notin Z_A$ because $Z_A \subset \mathcal{O}^\nu$ and $q \in \partial(\mathcal{O}^\nu)$ so that $q \notin Z = Z_A \cup Z_B$, which contradicts (2.1.9). This proves (2.1.1). To complete the proof of the Lemma, we just note that (2.1.2) is a consequence of (2.1.5) and $b)$. This finishes the proof. □

Proof of Theorem 0.0.1-Completed. Take $\nu \in \{-,+\}$. Since $i(\Phi_{\lambda_0}, u_0, 0) \neq 0$, we have from Lemma 2.1.2 that $(\lambda_0, u_0) \in \mathcal{S}^\nu$ showing that the component $\mathcal{C}^\nu$ of $\mathcal{S}^\nu$ containing (λ_0, u_0) is not empty. At least one of the alternatives $i)$, $ii)$ or $iii)$ must be true. On the contrary, we would obtain from Lemma 2.1.4 the existence of an admissible bounded open subset $\mathcal{O}^\nu$ of $\mathcal{U}^\nu$ such that the only solution of $\Phi(\lambda_0, u) = 0$ would be u_0 in $\mathcal{O}^\nu_{\lambda_0}$ (see (2.1.2)).

From this conclusion, and the fact that $\mathcal{O}^\nu_\lambda$ is an open subset of E for each $\lambda \in \mathbb{R}^\nu_{\lambda_0}$, because $\mathcal{O}^\nu$ is an open subset of $\mathcal{U}^\nu$, we can apply Lemma 2.1.1 and excision properties, to infer that

$$0 = \deg(\Phi_{\overline{\lambda}}, \emptyset, 0) = \deg(\Phi_{\overline{\lambda}}, \mathcal{O}^\nu_{\overline{\lambda}}, 0) = \deg(\Phi_{\lambda_0}, \mathcal{O}^\nu_{\lambda_0}, 0) = \mathrm{i}(\Phi_{\lambda_0}, u_0, 0) \neq 0,$$

for any fixed $\overline{\lambda} > 0$ larger then the diameter of $\mathcal{O}^{\nu}$. This is impossible. This ends the proof. □

Proof of Corollary 0.0.1. The proof follows by the fact that the inequality (2.1.3), in the proof of Lemma 2.1.4, remains true under the compactness assumption required on K in the Corollary 0.0.1. □

2.2 Positiveness-continuity-principle

The below result, inspired in Lemma 6.5.4 of [43], is a positiveness-continuity-principle because it brings up necessary and sufficient conditions for positiveness of all elements belonging to a connected subset when we know that at least one element of this connected subset is positive. In particular, it can be used to give positiveness of solutions belonging to the connected set given by Theorem 0.0.1.

Proposition 2.2.1 (Positiveness-continuity-principle)**.** *Assume that (E,P) is an ordered Banach space with $\text{int}P \neq \emptyset$, and let $\mathscr{C}$ be any connected subset of $\mathbb{R}\times E$ such that $\mathscr{C}\cap[\mathbb{R}\times\text{int}P]\neq\emptyset$ and*

$$\mathscr{C}\cap[\mathbb{R}\times(P\setminus\{0\})]\subset\mathbb{R}\times\text{int}P. \tag{2.2.1}$$

Then

$$\mathscr{C}\cap(\mathbb{R}\times\{0\})=\emptyset \text{ if and only if } \mathscr{C}\subset\mathbb{R}\times\text{int}P.$$

In particular,

$$\text{if there is no } \lambda\in\mathbb{R} \text{ such that } (\lambda,0)\in\overline{\mathscr{C}\cap\mathbb{R}\times(P\setminus\{0\})}, \tag{2.2.2}$$

then

$$\mathscr{C}\cap(\mathbb{R}\times\{0\})=\emptyset, \tag{2.2.3}$$

and consequently, $\mathscr{C}\subset\mathbb{R}\times\text{int}P$.

Proof. Suppose that $\mathscr{C}\cap(\mathbb{R}\times\{0\})=\emptyset$. By (2.2.1), it is sufficient to prove that

$$\mathscr{C}\subset[\mathbb{R}\times(P\setminus\{0\})].$$

Suppose by absurd that the above inclusion does not hold, that is, there would exist a $q \in \mathscr{C}$ such that $q \notin [\mathbb{R} \times (P \setminus \{0\})]$. Since $\mathscr{C} \cap (\mathbb{R} \times \{0\}) = \emptyset$, one has that $q \notin \mathbb{R} \times P$, that is,

$$\mathscr{C} \cap [(\mathbb{R} \times E) \setminus (\mathbb{R} \times P)] \neq \emptyset.$$

On the other hand, we have by assumption that $\mathscr{C} \cap [\mathbb{R} \times \operatorname{int} P] \neq \emptyset$, which implies that

$$\mathscr{C} \cap [\mathbb{R} \times P] \supseteq \mathscr{C} \cap [\mathbb{R} \times \operatorname{int} P] \neq \emptyset$$

so that there exists a

$$q_0 \in \mathscr{C} \cap [\mathbb{R} \times \partial P], \tag{2.2.4}$$

after applying the Customs Theorem.

Using again $\mathscr{C} \cap (\mathbb{R} \times \{0\}) = \emptyset$, we have that $q_0 \notin \mathbb{R} \times \{0\}$, which follows that $q_0 \in \mathscr{C} \cap [\mathbb{R} \times (P \setminus \{0\})]$ so that we have that $q_0 \in \mathbb{R} \times \operatorname{int} P$ after using (2.2.1), but this contradicts (2.2.4). The reverse inclusion is immediate once that $0 \notin \operatorname{int} P$.

Now let us prove (2.2.3). Suppose by contradiction that (2.2.2) holds true, but (2.2.3) not. Since $0 \in E \setminus \operatorname{int} P$, then

$$\mathscr{C} \cap [(\mathbb{R} \times E) \setminus (\mathbb{R} \times \operatorname{int} P)] \neq \emptyset.$$

On the other hand, by using the assumption $\mathscr{C} \cap [\mathbb{R} \times \operatorname{int} P] \neq \emptyset$, we obtain from the Customs Theorem that there exists

$$(\lambda, u) \quad \in \quad \mathscr{C} \cap \partial^{\mathscr{C}}(\mathbb{R} \times \operatorname{int} P),$$

where $\partial^{\mathscr{C}}$ denotes the boundary of $(\mathbb{R} \times \operatorname{int} P)$ relative to the topology in $\mathscr{C}$. Thus

$$(\lambda, u) \in \overline{\mathbb{R} \times \operatorname{int} P}^{\mathscr{C}}, \tag{2.2.5}$$

and so $u \in P$. Moreover,

$$(\lambda, u) \in \overline{\mathbb{R} \times (\operatorname{int} P)^c}^{\mathscr{C}} \tag{2.2.6}$$

By combining (2.2.5) and (2.2.6), we deduce that $u \in \partial(\operatorname{int} P)$, which implies that $u = 0$, because if the contrary were true, we would have $u \in \operatorname{int} P$ (due to the hypothesis (2.2.1)), but this is impossible due to $u \in \partial(\operatorname{int} P)$. After this, we have from $\operatorname{int} P \subset P \setminus \{0\}$ and (2.2.5) that

$$(\lambda, u) = (\lambda, 0) \in \mathscr{C} \cap \overline{\mathbb{R} \times (P \setminus \{0\})},$$

which contradicts (2.2.2). This ends the proof. □

Remark 2.2.1. *We point out that the condition (2.2.1) can be showed by classical Strong Maximum Principles in the context of partial differential equations.*

2.3 Connecting singularities in PDEs with $\text{dist}(\mathscr{C}, \partial\mathcal{U}) = 0$

Let us state a lemma that will be essential in many proofs of Sections 3.1 and 3.2.

Let $\alpha, s > 0$, F be a normed vector space with norm $\|\cdot\|_F$, E be a subspace of F and $Y : E \to F$ be a continuous function. Set

$$\mathcal{U}_F = \{(\lambda, u) \in \mathbb{R}_0^- \times E; 1 + \lambda\alpha\|Y(u)\|_F^s > 0\}.$$

Lemma 2.3.1. *Let C be a subset of $\mathcal{U}_F$ such that $\text{proj}_\lambda C$ is bounded and $I_C > 0$. Then there exists a positive constant $L > 0$, depending only on the size of $\text{proj}_\lambda C$, such that*

$$\min\left\{\frac{1}{2}, L\,\text{dist}(C, \partial\mathcal{U}_F)\right\} \leq I_C, \tag{2.3.1}$$

where

$$I_C = \inf\{1 + \alpha\lambda\|Y(u)\|_F^s; (\lambda, u) \in C\}.$$

In addition, if $Y : E \to F$ is a uniformly continuous function such that $Y(C)$ is bounded, then

$$I_C = 0 \Leftrightarrow \text{dist}(C, \partial\mathcal{U}_F) = 0.$$

Proof. First let us prove (2.3.1). Since Y is continuous, then

$$\partial\mathcal{U}_F = \{(\lambda, u) \in \mathbb{R}_0^- \times E; 1 + \lambda\alpha\|Y(u)\|_F^s = 0\}.$$

Let $(\lambda, u) \in C$. If $Y(u) = 0$, then

$$1 + \lambda\alpha\|Y(u)\|_F^s = 1,$$

and the claim follows trivially. If $Y(u) \neq 0$, then

$$\left(-\frac{1}{\alpha\|Y(u)\|_r^s}, u\right) \in \partial\mathcal{U}_F$$

and so

$$\text{dist}(C, \partial\mathcal{U}_F) \leq \left\| (\lambda, u) - \left(-\frac{1}{\alpha \|Y(u)\|_F^s}, u \right) \right\|_{\mathbb{R}\times E} = \frac{1 + \lambda\alpha \|Y(u)\|_F^s}{\alpha \|Y(u)\|_F^s}. \tag{2.3.2}$$

Let $\Lambda < \inf \text{proj}_\lambda C$. Note that since $C \subset \mathcal{U}_F \subset \mathbb{R}_0^- \times E$, then $\inf \text{proj}_\lambda C \leq 0$ and consequently $\Lambda < 0$. So it is well defined the set

$$C_\Lambda = \left\{ (\lambda, u) \in C; \lambda \in \mathbb{R} \text{ and } \ \alpha \|Y(u)\|_F^s < -\frac{1}{2\Lambda} \right\}$$

which is possible empty.

So, we obtain from (2.3.2), that

$$1 + \lambda\alpha \|Y(u)\|_F^s \geq -\frac{\text{dist}(C, \partial\mathcal{U}_F)}{2\Lambda} \ \forall\, (\lambda, u) \in C \setminus C_\Lambda$$

whence follows

$$1 + \lambda \|Y(u)\|_F^s \geq -\frac{\text{dist}(C, \partial\mathcal{U}_F)}{2\Lambda} \ \forall\, (\lambda, u) \in C \tag{2.3.3}$$

if $C_\Lambda = \emptyset$ so that

On the other hand, if $C_\Lambda \neq \emptyset$, then

$$1 + \lambda\alpha \|Y(u)\|_F^s \geq 1 + \Lambda\alpha \|Y(u)\|_F^s > 1 - \Lambda \left(\frac{1}{2\Lambda} \right) = \frac{1}{2}, \ \forall\, (\lambda, u) \in C_\Lambda.$$

Since $C = (C \setminus C_\Lambda) \cup C_\Lambda$, then the inequality that includes both cases ($C_\Lambda = \emptyset$ or $C_\Lambda \neq \emptyset$) is

$$I_C \geq \min \left\{ \frac{1}{2}, L\,\text{dist}(C, \partial\mathcal{U}_F) \right\},$$

where $L = -1/(2\Lambda) > 0$ and the first part of Lemma 2.3.1 is proved.

Now let us prove the second part of Lemma 2.3.1. Assume that $\text{dist}(C, \partial\mathcal{U}_F) = 0$. So there exist sequences $(\lambda_n, u_n) \in C$ and $(\ell_n, v_n) \in \partial\mathcal{U}_F$ such that

$$\|(\lambda_n, u_n) - (\ell_n, v_n)\|_{\mathbb{R}\times E} \to 0 \tag{2.3.4}$$

Since $Y(C)$ is bounded, then $(\lambda_n, \|Y(u_n)\|_F^s)$ is bounded and so

$$(\lambda_n, \|Y(u_n)\|_F^s) \to (\lambda, \xi).$$

Moreover, $\|Y(u_n) - Y(v_n)\|_F \to 0$ because P is uniformly continuous $\|u_n - v_n\|_E \to 0$. So

$$|\|Y(u_n)\|_F - \|Y(v_n)\|_F| \leq \|Y(u_n) - Y(v_n)\|_F \to 0.$$

Consequently,

$$(\ell_n, \|Y(v_n)\|_F^s) \to (\lambda, \xi)$$

due to (2.3.4). So

$$\lim_n (1 + \lambda_n \|Y(u_n)\|_F^s) = 1 + \lambda\xi = \lim_n (1 + \ell_n \|Y(v_n)\|_F^s) = \lim_n 0 = 0.$$

Which implies that $I_C = 0$.

□

Conclusion of Chapter 2

Theorem 0.0.1 presents a continuation theorem for perturbations of the identity that satisfy a compactness condition within an open subset. Furthermore, its formulation aligns with the definition of a global alternative theorem, as introduced by [52].

It should be noted that the assertions of Theorem 0.0.1 are confined on a unique real Banach space E. When dealing with PDEs, the space where we search for solutions, however, may stratify in subspaces. For example, if we are searching for solutions in $C(\overline{\Omega})$, then $C^1(\overline{\Omega})$ is a subspace of $C(\overline{\Omega})$. In this case, Theorem 0.0.1 may alienate some information about the norm of the solutions with respect to subspaces of E, as can be observed in the conclusion of Chapter 3. This technical issue naturally leads to the open problem of finding an alternative formulation of Theorem 0.0.1 that incorporates layers of the space E. Note that Lemma 2.3.1 already addresses aspects of this question.

Chapter 3

Two parameters quasilinear Schrodinger and Carrier logistic problem

Theorem 0.0.1 can be useful for solving a large class of partial differential equations that presents some singularity in its structure, preventing the definition of the associated operator in the whole parameter-working space. This occurs when the associated operator must be constrained to a subset to be well-defined. In this direction, we will present new results in this chapter regarding both the existence of classical positive solutions and qualitative information for the well-studied class of quasilinear Schrödinger equations ($P_{\lambda,\mu}$) and a Carrier type problem ($Q_{\lambda,\mu}$)

3.1 Quasilinear Schrödinger Operator with logistic perturbation

This section is devoted to study existence and qualitative information about the positive solutions of

$$\begin{cases} -\Delta u - \lambda u \Delta u^2 = \mu u - u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $\lambda, \mu \in \mathbb{R}$, $p > 1$, and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 1$.

To do so, we will present some notations, auxiliary results that and finally to prove Theorem 0.0.2.

3.1.1 Connected set of positive solutions

Let us begin by noting that we are interested in classical solutions for the problem ($P_{\lambda,\mu}$) so that is enough to find classical solutions to the problem

$$\begin{cases} -(1+2\lambda u^2)\Delta u = \mu u - u^p + 2\lambda u|\nabla u|^2 & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

because $\Delta(u^2) = 2u\Delta u + 2|\nabla u|^2$ for all $u \in C^2(\Omega)$.

It is well known that the problem (3.1.1) with $\lambda = 0$ admits a unique positive solution $u = u_0 \in C^2(\Omega) \cap C(\overline{\Omega})$ if, and only if, $\mu > \mu_1$, where $\mu_1 > 0$ is the first eigenvalue of the Laplacian Dirichlet problem. In addition, this unique solution satisfies $0 < u_0(x) < \mu^{1/(p-1)}$ in Ω. As the problem (3.1.1) is a perturbation of the pure logistic problem, we are concerned in understanding how the diagram of solutions of the problem (3.1.1) is affected by the term $\lambda u \Delta u^2$ for $\lambda \in \mathbb{R}$.

Now we will show that the positivity and the boundedness from above by $\mu^{1/(p-1)}$ for the solutions of the problem (3.1.1) are linked with the sign of $1 + 2\lambda u^2$ in $\overline{\Omega}$ that is the same as $1 + 2\lambda \|u\|_0^2$ due to the regularity of u. To do so, let us define the positive cone of $C_0^1(\overline{\Omega})$ by $P_{C_0^1(\overline{\Omega})} := \{u \in C_0^1(\overline{\Omega}); u \geq 0\}$ whose interior is

$$\operatorname{int} P_{C_0^1(\overline{\Omega})} = \left\{ u \in C_0^1(\overline{\Omega}); u(x) > 0 \; \forall \, x \in \Omega, \; \frac{\partial u}{\partial \eta}(x) < 0 \; \forall \, x \in \partial\Omega \right\},$$

where η is the unit outward normal vector in Ω.

Lemma 3.1.1 (Boundeness-positivity continuation). *For any $\mu > 0$ given, let $u \in C^2(\Omega) \times C(\overline{\Omega})$ be a non-negative and non-trivial solution of the problem (3.1.1):*

i) if $\|u\|_0 < \mu^{1/(p-1)}$, then $1 + 2\lambda \|u\|_0^2 > 0$,

ii) if $1 + 2\lambda \|u\|_0^2 \geq 0$, then $\|u\|_0 \leq \mu^{1/(p-1)}$,

iii) if $1 + 2\lambda \|u\|_0^2 > 0$, then $u \in \operatorname{int} P_{C_0^1(\overline{\Omega})}$,

iv) if $p \geq 2$, $\lambda \leq 0$ and $1 + 2\lambda \|u\|_0^2 > 0$, then $\|u\|_0 < \mu^{1/(p-1)}$.

Proof. of *i*). Let $(\lambda, u) \in \mathbb{R} \times [C^2(\Omega) \cap C(\overline{\Omega})]$ be a solution of (3.1.1), and $x_0 \in \Omega$ be a maximum point of u so that $u(x_0) = \|u\|_0$, and $\Delta u(x_0) \leq 0$. So, by the non-negativity of u, we have

$$\begin{aligned}\left(1+2\lambda\|u\|_0^2\right)(-\Delta u(x_0)) &= \left(1+2\lambda u(x_0)^2\right)(-\Delta u(x_0)) \\ &= u(x_0)(\mu - u(x_0)^{p-1}) \\ &= \|u\|_0(\mu - \|u\|_0^{p-1}), \end{aligned} \tag{3.1.2}$$

which implies by the assumption $\|u\|_0 < \mu^{1/(p-1)}$ that in fact $-\Delta u(x_0) > 0$. So, $1+2\lambda\|u\|_0^2 > 0$.
Proof of *ii*). It follows from the non-negativity of u and (3.1.2), that $1+2\lambda\|u\|_0^2 \geq 0$ implies $\|u\|_0 \leq \mu^{1/(p-1)}$.
Proof of *iii*). We will consider two cases. First, we assume that $\lambda \geq 0$. Then (λ, u) satisfies

$$\begin{cases} -\Delta u + \dfrac{u^{p-1}}{1+2\lambda u^2} u = \dfrac{u}{1+2\lambda u^2}(\mu + 2\lambda|\nabla u|^2) \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \end{cases}$$

whence follows that u is a supersolution (in the sense of [44]) of

$$\mathcal{L}_1 := -\Delta + \frac{u^{p-1}}{1+2\lambda u^2}$$

in Ω under homogeneous Dirichlet boundary conditions on $\partial\Omega$. Since $u \geq 0$ and $1+2\lambda\|u\|_0^2 > 0$, we conclude that $u^{p-1}/(1+2\lambda u^2) \in L^\infty(\Omega)$ and $u^{p-1}/(1+2\lambda u^2) \geq 0$ in Ω, which imply, by Theorem 7.5.2 of [44], that $\mathcal{L}_1$ satisfies the Strong Maximum Principle so that $u \in \operatorname{int} P_{C_0^1(\overline{\Omega})}$.

Now, if $\lambda < 0$. Then (λ, u) satisfies

$$\begin{cases} \left(-\Delta - \dfrac{2\lambda|\nabla u|^2}{1+2\lambda u^2} + \dfrac{u^{p-1}}{1+2\lambda u^2}\right) u = u\left(\dfrac{\mu}{1+2\lambda u^2}\right) \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, u is a supersolution of

$$\mathcal{L}_2 := -\Delta - \frac{2\lambda|\nabla u|^2}{1+2\lambda u^2} + \frac{u^{p-1}}{1+2\lambda u^2}$$

in Ω under the boundary condition $u=0$ on $\partial\Omega$. Since u is a non-negative $C^2(\Omega)$-function and $1+2\lambda\|u\|_0^2>0$, we get $(-2\lambda|\nabla u|^2+u^{p-1})/(1+2\lambda u^2)\in L^\infty(\Omega)$. Moreover, in view of the assumption $\lambda<0$, we have $(-2\lambda|\nabla u|^2+u^{p-1})/(1+2\lambda u^2)\geq 0$. Again by Theorem 7.5.2 of [44], the operator $\mathcal{L}_2$ satisfies the Strong Maximum Principle and, hence, $u\in \operatorname{int} P_{C_0^1(\overline{\Omega})}$.
Proof of *iv*). Define $w=\mu-u^{p-1}$. Since $p\geq 2$, we obtain

$$\begin{aligned}\Delta w &= -\operatorname{div}((p-1)u^{p-2}\nabla u)\leq (p-1)u^{p-2}(-\Delta u)\\ &= (p-1)u^{p-1}\left(\frac{(\mu-u^{p-1})}{1+2\lambda u^2}+\frac{2\lambda|\nabla u|^2}{1+2\lambda u^2}\right),\end{aligned}$$

which implies, together with $\lambda\leq 0$, that w satisfies

$$\begin{cases}-\Delta w+\dfrac{(p-1)u^{p-1}}{1+2\lambda u^2}w=-\dfrac{2\lambda|\nabla u|^2}{1+2\lambda u^2}(p-1)u^{p-1}\geq 0 & \text{in } \Omega,\\ \qquad\qquad w>0 & \text{on } \partial\Omega,\end{cases}$$

whence follows that $\mu>u(x)^{p-1}$ for all $x\in\overline{\Omega}$ by the Strong Maximum Principle. Consequently, $\|u\|_0<\mu^{1/(p-1)}$. This ends the proof. □

After Lemma 3.1.1, it is natural to look for positive solutions to the problem (3.1.1) in the open set

$$\mathcal{U}=\left\{(\lambda,u)\in\mathbb{R}\times C_0^1(\overline{\Omega});\ 1+2\lambda\|u\|_0^2>0\right\}, \tag{3.1.3}$$

where the searching of classical solutions for the problem (3.1.1) is equivalent to do the same for the problem

$$\begin{cases}-\Delta u=\dfrac{\mu u-u^p+2\lambda u|\nabla u|^2}{1+2\lambda u^2} & \text{in } \Omega,\\ \quad u=0 & \text{on } \partial\Omega.\end{cases} \tag{3.1.4}$$

Clearly, if $(\lambda,u)\in\mathcal{U}$, then

$$\frac{\mu u-u^p+2\lambda u|\nabla u|^2}{1+2\lambda u^2}\in C(\overline{\Omega}).$$

Thus, we can consider the operator $K:\mathcal{U}\to C_0^1(\overline{\Omega})$ defined by

$$K(\lambda,u)=(-\Delta)^{-1}\left[\frac{\mu u-u^p+2\lambda u|\nabla u|^2}{1+2\lambda u^2}\right], \tag{3.1.5}$$

where $(-\Delta)^{-1} : C_0(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is the resolvent operator of the linear boundary value problem associated with $-\Delta$ in Ω.

The lemma below will guarantee that K is compact in the open subset $\mathcal{U}$.

Lemma 3.1.2 (Compactness of K)**.** *If C is a bounded closed subset of $\mathcal{U}$ such that* $\operatorname{dist}(C, \partial\mathcal{U}) > 0$, *then the operator $K : C \subset \mathcal{U} \to C_0^1(\overline{\Omega})$ is compact.*

Proof. Since C is bounded, then $\operatorname{proj}_\lambda C$ is bounded. Once that $\operatorname{dist}(C, \partial\mathcal{U}) > 0$, it follows from Lemma 2.3.1 that there exists a $\rho > 0$ such that

$$1 + 2\lambda \|u\|_0^2 > \rho \ \forall\ (\lambda, u) \in C \tag{3.1.6}$$

due to Lemma 2.3.1 applied to $E = C_0^1(\overline{\Omega})$, $F = C(\overline{\Omega})$ and $\alpha = q = 2$. Consequently, the application

$$\begin{array}{rccc} z: & \mathcal{U} & \to & C(\overline{\Omega}) \\ & (\lambda, u) & \mapsto & z(\lambda, u) = \dfrac{\mu u - u^p + 2\lambda u |\nabla u|^2}{1 + 2\lambda u^2} \end{array}$$

Moreover, the boundedness of C combined with (3.1.6), implies that $z(C)$ is bounded subset of $C(\overline{\Omega})$, in particular, bounded in $L^p(\Omega)$ for any $p > 1$. By elliptic regularity, $(-\Delta)^{-1}(z(C))$ is bounded in $W^{2,p}(\Omega)$ for any $p > 1$. By the compact embedding $W_0^{2,p}(\Omega) \stackrel{c}{\hookrightarrow} C_0^1(\overline{\Omega})$, for large p, it follows that compactness of K. This ends the proof of Lemma 3.1.2. □

The below result is crucial in order to apply Theorem 0.0.1.

Lemma 3.1.3. *If $\mu > \mu_1$, then* $\operatorname{i}(\Phi_0, u_0, 0) \neq 0$, *where $\Phi_0(u) = \Phi(0, u)$ with $\Phi_0(\lambda, u) = u - K(\lambda, u)$ for $(\lambda, u) \in \mathcal{U}$.*

Proof. Define the operator $T : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ by $T(u) := K(0, u)$. It is immediate to check that T is differentiable at $u = u_0$ with

$$T'(u_0)h = (-\Delta)^{-1}((\mu - p u_0^{p-1})h).$$

We claim that 1 is not a characteristic value of $T'(u_0)$. On the contrary, the problem

$$\begin{cases} -\Delta\varphi = (\mu - p u_0^{p-1})\varphi & \text{in } \Omega, \\ \quad \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

would admit a non-trivial solution, which implies that

$$\sigma_j^{\Omega}[-\Delta + pu_0^{p-1} - \mu] = 0$$

where $\sigma_j[L]$ stands for the eigenvalue of operator $L := -\Delta + M(x)$ ($M \in L^\infty$) in Ω under homogeneous Dirichlet boundary conditions on $\partial\Omega$. By the dominance of the principal eigenvalue, we obtain that

$$\sigma_1^{\Omega}[-\Delta + pu_0^{p-1} - \mu] \leq 0. \tag{3.1.7}$$

On the other hand,

$$-\Delta u_0 + pu_0^p - \mu u_0 = (p-1)u_0^p > 0 \text{ in } \Omega$$

showing that u_0 is a strictly supersolution of the problem

$$\begin{cases} -\Delta u + pu_0^{p-1} - \mu u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

that would lead to

$$\sigma_1^{\Omega}[-\Delta + pu_0^{p-1} - \mu] > 0,$$

which contradicts (3.1.7). As a consequence, we have that

$$\mathrm{i}(\Phi_0, u_0, 0) \neq 0$$

due to the Leray-Schauder Formula (see, for instance, [7, Theorem 3.20]). This completes the proof. □

Now, we are ready to state and apply Theorem 0.0.1 to obtain a connected set of positive solutions to the problem ($P_{\lambda,\mu}$) contained in $\mathcal{U}$. Before doing this, let us denote $\mathcal{U}^\nu = \mathcal{U} \cap \mathbb{R}_0^\nu$, $\nu \in \{-,+\}$.

As a consequence of the previous results, we deduce the following existence result.

Proposition 3.1.1 (Continuation Theorem for ($P_{\lambda,\mu}$)). *Assume that $\mu > \mu_1$, $p > 1$ and consider the 0-partition of $\mathcal{U}$. Then there exists a pair of connected sets $\mathscr{C}^\nu \subset \mathcal{U}^\nu \cap \mathscr{S}^\nu$ of solutions for the problem ($P_{\lambda,\mu}$) such that $\mathscr{C}^- \cap \mathscr{C}^+ = \{(0, u_0)\}$. In addition, $\mathscr{C}^\nu$ satisfies at least one of the following alternatives:*

i) $\mathscr{C}^{\nu}$ *is unbounded in* $\mathbb{R}_0^{\nu} \times C_0^1(\overline{\Omega})$.

ii) $\operatorname{dist}(\mathscr{C}^{\nu}, \partial\mathcal{U}) = 0$

iii) $\mathscr{C}^{\nu} \cap \{(0, \tilde{u}_0)\} \neq \emptyset$ *for some* $u_0 \neq \tilde{u}_0$

for each $\nu \in \{-, +\}$.

Proof. By Lemma 3.1.2, K is a compact operator in the open subset $\mathcal{U}$, and Lemma 3.1.3 shows that $(0, u_0) \in \mathcal{U}$ is such that $u_0 \in \mathcal{U}_0$ is an isolated solution of $\Phi_0(u) = 0$ with index $i(\Phi_0, u_0, 0) \neq 0$. So, we are in position to apply the Theorem 0.0.1 to obtain the existence of a pair of connected sets $\mathscr{C}^{\nu} \subset \mathcal{U}^{\nu} \cap \mathscr{S}^{\nu}$ of solutions for the problem ($P_{\lambda,\mu}$) satisfying the at least one of the alternatives i), ii) or iii). This ends the proof. □

Remark 3.1.1. *It is worth to mention at this point that by using the qualitative information about* $\mathscr{C}$*, which will be provided in the next subsection, we will be able to prove that the alternative* ii) *of Proposition 3.1.1 must occur (see the proof of Theorem 0.0.2).*

Let us denote by

$$\mathscr{C} = \mathscr{C}^{-} \cup \mathscr{C}^{+}$$

the connected set of solutions for the problem ($P_{\lambda,\mu}$) crossing $\mathbb{R} \times C^1(\overline{\Omega})$ at $(\lambda, u) = (0, u_0)$. At this moment, we do not have any information about the sign of the solutions ($P_{\lambda,\mu}$) belongs to $\mathscr{C}$.

3.1.2 Qualitative information about the connected of positive solutions

We already know that $\mathscr{C}$ is a connected set of solutions of ($P_{\lambda,\mu}$) that contains the point $(0, u_0)$. We begin this section showing estimates of the solutions for the problem ($P_{\lambda,\mu}$) related to parameter $\lambda < 0$ and on its $C(\overline{\Omega})$-norm. These estimates will be useful in completing the proof of Theorem 0.0.2. To do so, we will need the following properties of the first eigenvalue that can be deduced by its variational characterization (see [26]). Given $A(x) \in W^{1,q}(\Omega)$, such that $A(x) \geq A_0 > 0$ for all $x \in \overline{\Omega}$ and $B(x) \in L^q(\Omega)$, $q > N/2$, the first eigenvalue of $-\operatorname{div}[A\nabla] + B$ is well defined and it is increasing with respect to A and B.

Lemma 3.1.4. *Assume that* $\mu > \mu_1$ *and* $p > 1$. *Let* $(\lambda, u) \in \mathcal{U}^{-}$ *be a classical positive solution of the problem (*$P_{\lambda,\mu}$*). Then*

$$\|u\|_0 \geq (\mu - \mu_1)^{1/(p-1)} \quad \text{and} \quad \lambda \geq -\frac{1}{2(\mu - \mu_1)^{2/(p-1)}}.$$

Proof. Since (λ, u) satisfies

$$\begin{aligned} -\operatorname{div}[(1+2\lambda u^2)\nabla u]+2\lambda u|\nabla u|^2 &= -4\lambda u|\nabla u|^2+(1+2\lambda u^2)(-\Delta u)+2\lambda u|\nabla u|^2 \\ &= -2\lambda u|\nabla u|^2+(1+2\lambda u^2)(-\Delta u) \\ &= \mu u - u^p, \end{aligned}$$

we obtain

$$\left[-\operatorname{div}[(1+2\lambda u^2)\nabla(\,\cdot\,)]+2\lambda|\nabla u|^2+u^{p-1}\right]u = \mu u$$

whence follows that

$$\sigma_1\left[-\operatorname{div}[(1+2\lambda u^2)\nabla(\,\cdot\,)]+2\lambda|\nabla u|^2+u^{p-1}\right] = \mu.$$

By using the monotonicity properties of the first eigenvalue and $\lambda \le 0$, we deduce

$$\begin{aligned} \mu &\le \sigma_1\left[-\operatorname{div}[\nabla(\,\cdot\,)]+2\lambda|\nabla u|^2+u^{p-1}\right] \\ &\le \sigma_1\left[-\operatorname{div}[\nabla(\,\cdot\,)]+\|u\|_0^{p-1}\right] \\ &= \mu_1+\|u\|_0^{p-1}, \end{aligned}$$

from which follows that $\|u\|_0 \ge (\mu-\mu_1)^{1/(p-1)}$.

To obtain the estimate on λ, we just note that $(\lambda, u) \in \mathcal{U}^-$ and $\|u\|_0 \ge (\mu-\mu_1)^{1/(p-1)}$ to lead us to

$$0 < 1+2\lambda\|u\|_0^2 \le 1+2\lambda(\mu-\mu_1)^{2/(p-1)}$$

so that

$$\lambda \ge -\frac{1}{2(\mu-\mu_1)^{2/(p-1)}}.$$

This ends the proof. □

The next result will allow us to conclude that the connect $\mathscr{C}$ is far from trivial solutions.

Lemma 3.1.5. *Assume that $p>1$ and $0<\mu\neq\mu_1$. Then there is no bifurcation point of non-negative and non-trivial solutions in $\mathcal{U}$ of $(P_{\lambda,\mu})$ from its trivial curve of solutions, in the $C(\overline{\Omega})$−norm.*

Proof. Suppose by contradiction that there exists a bifurcation point $(\lambda_0, 0)$ of non-negative and non-trivial solutions in $\mathcal{U}$ of $(P_{\lambda,\mu})$. So, there would exist a sequence $(\lambda_n, u_n) \in \mathcal{U}$ of non-negative and non-trivial solutions of $(P_{\lambda,\mu})$ converging to $(\lambda_0, 0)$ in $\mathbb{R} \times C(\overline{\Omega})$. Since $\{(\lambda_n, u_n)\}$ is a convergent sequence, it follows that $\{(\lambda_n, u_n)\}$ is bounded in $\mathbb{R} \times C(\overline{\Omega})$. So we can apply Lemma 2.3.1 with $Y = \mathrm{id}$ and $C := \{(\lambda_n, u_n); n \in \mathbb{N}\}$. Moreover, since $\|u_n\|_0 \to 0$ and (λ_n) is bounded, it follows that $I_C > 0$. Consequently, $\mathrm{dist}(C, \partial\mathcal{U}) > 0$ due to Lemma 2.3.1. Thus, K is compact in C by Lemma 3.1.2, which implies that there exists some $u_0 \in C_0^1(\overline{\Omega})$ such that $u_n \to u_0$, in $C_0^1(\overline{\Omega})$, up to a subsequence. Since $\|u_n\|_0 \to 0$, then $u_0 = 0$.

Now, observe that (λ_n, u_n) satisfies

$$u_n \quad = \quad K(\lambda_n, u_n) = (-\Delta)^{-1}\left(\frac{\mu u_n - u_n^p + 2\lambda_n u_n |\nabla u|^2}{1 + 2\lambda_n u_n^2}\right) \; \forall\, n$$

whence follows that

$$\frac{u_n}{\|u_n\|_1} \quad = \quad (-\Delta)^{-1}\left[\frac{1}{1+2\lambda_n u_n^2}\left(\mu \frac{u_n}{\|u_n\|_1} - \frac{u_n^p}{\|u_n\|_1} + \frac{2\lambda_n u_n |\nabla u_n|^2}{\|u_n\|_1}\right)\right] \; \forall\, n.$$

So, by passing to the limit in the above equality and using the compact embedding $C_0^{1,\alpha}(\overline{\Omega}) \to C(\overline{\Omega})$, for some $0 < \alpha < 1$, we obtain that

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

in weak sense, for some non-trivial and non-negative function $u \in C_0^1(\overline{\Omega})$, implying that $\mu = \mu_1$, which is impossible by assumption. □

The previous results can be combined with Proposition 2.2.1 to prove that $\mathscr{C}$ is composed by strong positive solutions.

Proposition 3.1.2. $\mathscr{C} := \mathscr{C}^- \cup \mathscr{C}^+ \subset \mathbb{R} \times \mathrm{int} P_{C_0^1(\overline{\Omega})}$.

Proof. First, we note that $(0, u_0) \in \mathscr{C} \cap [\mathbb{R} \times \mathrm{int} P]$. Second, since $\mathscr{C} \subset \mathcal{U}$, we obtain from the item *iii*) of Lemma 3.1.1 that $\mathscr{C}$ satisfies the condition (2.2.1) of Proposition 2.2.1. Furthermore, we have from Lemma 3.1.5 that there is no $\lambda \in \mathbb{R}$ such that $(\lambda, 0) \in \overline{\mathscr{C} \cap \mathbb{R} \times (P \setminus \{0\})}$, whence follows by Proposition 2.2.1 that $\mathscr{C} \subset \mathbb{R} \times \mathrm{int} P$. This ends the proof. □

Now, we will apply Theorem A.2.3 in order to obtain additional information about the $C^1_0(\overline{\Omega})$-norm of the solutions of the problem ($P_{\lambda,\mu}$) belonging to $\mathscr{C}$. To do this, we will need to define two smooth extensions of the nonlinear perturbation given in the problem (3.1.4).

For each $0<\rho<1$, define the function $f_{\rho,\lambda}$ by

$$f_{\rho,\lambda}(s,\xi) := \frac{\mu s - s^p + 2\lambda s|\xi|^2}{1+2\lambda s^2},\ s \leq \sqrt{\frac{\rho-1}{2\lambda}} \text{ and } \xi \in \mathbb{R}^N, \tag{3.1.8}$$

extended in a smooth way to $\mathbb{R}\times\mathbb{R}^N$ so that there exists an increasing continuous function $c_1 : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that

$$|f_{\rho,\lambda}(s,\xi)| \leq c_1(|s|)(1+|\xi|^2),\ \forall\,(s,\xi)\in\mathbb{R}\times\mathbb{R}^N,\ \forall\,\lambda\in\Lambda_1$$

for any compact interval $\Lambda_1 \subset (-\infty,0)$ given, where the function c_1 depends only on ρ, the length of μ_1.

To include the end point 0 in $(-\infty,0]$, we proceed in a similar way. Given

$$\Lambda_2 \subset \left(-1/\left(2\mu^{2/(p-1)}\right),0\right]$$

being a compact interval, we define the function

$$h_\lambda(s,\xi) = \frac{\mu s - s^p + 2\lambda s|\xi|^2}{1+2\lambda s^2},\ \forall\, s \leq \mu^{1/(p-1)},\ \xi\in\mathbb{R}^N, \text{ and } \lambda\in\Lambda_2,$$

extended in a smooth way to $\mathbb{R}\times\mathbb{R}^N$, to infer that there exists an increasing continuous function $c_2 : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, depending only on the length of Λ_2, such that

$$|h_\lambda(s,\xi)| \leq c_2(|s|)(1+|\xi|^2),\ \forall\,(s,\xi)\in\mathbb{R}\times\mathbb{R}^N,\ \forall\,\lambda\in\Lambda_2$$

holds.

Based on these facts, we are able to prove the next lemma. Before doing this, let us denote $\mathcal{U}^+_\rho = \mathcal{U}_\rho \cap \mathbb{R}^+_0$.

Lemma 3.1.6. *Let $C \subset \mathcal{U}^-$ be a set of positive solutions of ($P_{\lambda,\mu}$) such that* $\operatorname{dist}(C,\partial\mathcal{U}^-)>0$. *Then*

$$\|u\|_1 \leq M,\ \forall\,(\lambda,u) \text{ in } C,$$

where M depends on $\operatorname{dist}(C,\partial\mathcal{U}^-)$ *and the size of* $\operatorname{proj}_\lambda C$.

Proof. By Lemma 3.1.4, it follows that $\mathrm{proj}_\lambda C$ is bounded. Since $\mathrm{dist}(C,\partial\mathcal{U})>0$, we can apply Lemma 2.3.1 with $Y=\mathrm{Id}$, $E=C_0^1(\overline{\Omega})$, $F=C(\overline{\Omega})$ and $\alpha=s=2$ to infer that there exists a $\rho>0$ such that and

$$1+2\lambda\|u\|_0^2>\rho\ \forall\ (\lambda,u)\in C. \tag{3.1.9}$$

Let

$$\Upsilon<\min\left\{-\sup_{\lambda\in\mathrm{proj}_\lambda C}|\lambda|,-1/\left(4\mu^{2/(p-1)}\right)\right\}.$$

By using the definition of Υ, we have that

$$\Upsilon<-\sup_{\lambda\in\mathrm{proj}_\lambda C}|\lambda| \tag{3.1.10}$$

and

$$\Upsilon<-1/\left(4\mu^{2/(p-1)}\right). \tag{3.1.11}$$

As a consequence of (3.1.10), we have that $\mathrm{proj}_\lambda C\subset[\Upsilon,0]$. By (3.1.11), the interval

$$\left[\Upsilon,-1/\left(4\mu^{2/(p-1)}\right)\right] \tag{3.1.12}$$

is well defined. So

$$\begin{aligned}\mathrm{proj}_\lambda C &\subset [\Upsilon,0]\\ &= \left[\Upsilon,-1/\left(4\mu^{2/(p-1)}\right)\right]\cup\left[-1/\left(4\mu^{2/(p-1)}\right),0\right].\end{aligned} \tag{3.1.13}$$

Let us denote

$$\Lambda_1=\left[\Upsilon,-1/\left(4\mu^{2/(p-1)}\right)\right]\text{ and }\Lambda_2=\left[-1/\left(4\mu^{2/(p-1)}\right),0\right].$$

So rewriting (3.1.13) with these notations, we have

$$\mathrm{proj}_\lambda C\subset\Lambda_1\cup\Lambda_2. \tag{3.1.14}$$

Observe that Λ_1 is a compact interval contained in $(-\infty,0)$ and Λ_2 is a compact interval contained in $\left(-1/\left(2\mu^{2/(p-1)}\right),0\right]$. According to the text introducing Lemma 3.1.6, it follows that there exists a function $f_{\rho,\lambda}$ associated to ρ and defined for each $\lambda\in\Lambda_1$ and a function h_λ defined for each $\lambda\in\Lambda_2$, both satisfying the conditions of Theorem A.2.3.

Since (3.1.9), for even greater reason we have $1+2\lambda\|u\|_0^2 \geq 0$. Then

$$\|u\|_0 \leq \mu^{\frac{1}{p-1}} \ \forall\ (\lambda, u) \in C \tag{3.1.15}$$

due to Lemma 3.1.1.

Moreover,

$$\lambda \in \Lambda_1 \cup \Lambda_2 \ \forall\ (\lambda, u) \in C \tag{3.1.16}$$

due to (3.1.14).

Note that, regardless of whether λ belongs to Λ_1 or Λ_2, we have (3.1.9) and (3.1.15) for each (λ, u). But, in order to use the extensions $f_{\rho,\lambda}$ and h_λ, it will be convenient to state the facts (3.1.9) and (3.1.15) by dividing in the following two cases. Let $(\lambda, u) \in C$. Then either

$$\begin{cases} \lambda \in \Lambda_1 \\ \text{and } 1+2\lambda\|u\|_0^2 > \rho \end{cases} \tag{3.1.17}$$

or

$$\begin{cases} \lambda \in \Lambda_2 \\ \text{and } \|u\|_0 \leq \mu^{\frac{1}{p-1}}. \end{cases} \tag{3.1.18}$$

If the case is (3.1.17), then (λ, u) satisfies

$$\begin{cases} -\Delta u = f_{\rho,\lambda}(u, \nabla u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.19}$$

On the other hand, if the case is (3.1.18), then

$$\begin{cases} -\Delta u = h_\lambda(u, \nabla u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.20}$$

By Theorem A.2.3,

$$\|u\|_{W^{2,s}(\Omega)} \leq \gamma_1(\|u\|_0) \leq \gamma_1(\mu^{1/(p-1)}),\ \forall\ (\lambda, u) \in C,\ \lambda \in \Lambda_1 \tag{3.1.21}$$

and

$$\|u\|_{W^{2,s}(\Omega)} \leq \gamma_2(\|u\|_0) \leq \gamma_2(\mu^{1/(p-1)}),\ \forall\ (\lambda, u) \in C,\ \lambda \in \Lambda_2 \tag{3.1.22}$$

where we used in the last inequalities of (3.1.21) and (3.1.22) a consequence of Lemma 3.1.1 together with the fact that $\gamma_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an increasing function, for each $i = 1, 2$, that depend only on Ω, p and the function c_i. So, for $s > 1$ large enough, we obtain the claim due to the embedding $W^{2,s}(\Omega) \hookrightarrow C^1(\overline{\Omega})$. So by combining this embedding with (3.1.21) we deduce that there exists a constant $M_1 > 0$ such that

$$\|u\|_1 \leq M_1, \text{ for all solution } (\lambda, u) \in C, \; \lambda \in \Lambda_1$$

and a constant $M_2 > 0$ such that

$$\|u\|_1 \leq M_2, \text{ for all solution } (\lambda, u) \in C, \; \lambda \in \Lambda_2$$

due to (3.1.22), whence follows the claim with $M = M_1 + M_2$. □

About the connected set $\mathscr{C}^+$, we have the following result.

Lemma 3.1.7. *Let $0 < \Gamma < +\infty$. Then there exists a constant $M > 0$ (depending on Γ) such that*

$$\|u\|_1 \leq M, \textit{ for all positive solution } (\lambda, u) \textit{ in } \overline{\mathscr{U}^+} \cap \left([0, \Gamma] \times C_0^1(\overline{\Omega})\right).$$

Proof. Let us define for each $0 \leq \lambda \leq \Gamma$ the function

$$f_{\lambda,\Gamma}(s, \xi) := \frac{\mu s - s^p + 2\lambda s|\xi|^2}{1 + 2\lambda s^2}, \; \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

Clearly, there exists an increasing continuous function $c : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$|f_{\lambda,\Gamma}(s, \xi)| \leq c(|s|)(1 + |\xi|^2), \; \forall \, (s, \xi) \in \mathbb{R} \times \mathbb{R}^N, \; \forall \, \lambda \in [0, \Gamma].$$

So the proof follows by using the $W^{2,p}(\Omega)$-estimate given in Theorem A.2.3 and the embedding $W^{2,s}(\Omega) \hookrightarrow C^1(\overline{\Omega})$. □

Below, let us provide more qualitative information about global behavior of $\mathscr{C}^+$.

Lemma 3.1.8. *Assume that $p > 1$ and $\mu > \mu_1$. Then:*

i) $\text{proj}_\lambda \mathscr{C}^+ = \mathbb{R}_0^+$,

ii) there is a unique positive solution $u_\lambda \in C_0^1(\overline{\Omega})$ of $(P_{\lambda,\mu})$ for each $\lambda \geq 0$. Moreover the set of these solutions coincides with $\mathscr{C}^+$, *iii)* *the curve* $\mathbb{R}_0^+ \ni \lambda \mapsto (\lambda, u_\lambda) \in \mathbb{R}_0^+ \times C_0^1(\overline{\Omega})$ *is continuous,*

iv) $\|u_\lambda\|_0 \to 0$ *as* $\lambda \to \infty$.

Proof. Let us begin proving the item $i)$. Assume, by contradiction, that $\mathrm{proj}_\lambda \mathscr{C}^+$ were bounded. Then, by Lemma 3.1.7, $\mathrm{proj}_{C_0^1(\overline{\Omega})} \mathscr{C}^+$ would be bounded in $\mathbb{R}$ implying that $\mathscr{C}^+$ would also be bounded in $\mathbb{R} \times C_0^1(\overline{\Omega})$. However, since $\mathscr{C}^+$ neither satisfies the alternatives $ii)$ nor $iii)$ of Theorem 3.1.1, it must be unbounded, which is a contradiction. The claim of items $ii)$ and $iv)$ follows from the uniqueness and the behaviour of the solutions u_λ of $(P_{\lambda,\mu})$ proved in [21]. Finally, the claim of the item $iii)$ follows from the fact that the operator K is compact in the open set $\mathcal{U}$ (see Lemma 3.1.2). This ends the proof. □

Now, we are ready to complete the proof of the Theorem 0.0.2.

Proof of Theorem 0.0.2: First note that Proposition 3.1.1 implies that there exists a connected set $\mathscr{C} = \mathscr{C}^- \cup \mathscr{C}^+ \subset \mathcal{U}$ of solutions for the problem $(P_{\lambda,\mu})$ crossing $\mathbb{R} \times C^1(\overline{\Omega})$ at $(\lambda, u) = (0, u_0)$ with $\mathscr{C}^- \subset \mathcal{U}^-$ and $\mathscr{C}^+ \subset \mathcal{U}^+$ satisfying, each one, at least one alternative of Theorem 0.0.1. Furthermore, Proposition 3.1.2 implies that $\mathscr{C} \subset \mathbb{R} \times \mathrm{int} P_{C_0^1(\overline{\Omega})}$, that is, $\mathscr{C}$ is a connected set of strong-positive solutions of the problem $(P_{\lambda,\mu})$. This finishes the proof of the first part of Theorem.

First, we will prove item (0.0.3). We claim that

$$\mathrm{dist}\left(\mathscr{C}^-, \partial\mathcal{U}\right) = 0. \tag{3.1.23}$$

The proof will be by using contradiction. So assume that (3.1.23) is false. By Lemma 3.1.6, $\mathscr{C}^-$ would be bounded in $\mathbb{R} \times C_0^1(\overline{\Omega})$ implying that $\mathscr{C}^-$ does not satisfy either alternatives $i)$ or $ii)$ of Proposition 3.1.1. Note that $\mathscr{C}^-$ does not satisfy the alternative $iii)$ of Proposition 3.1.1 due to the uniqueness of positive solutions for the problem $(P_{0,\mu})$ with $\mu > \mu_1$. So we have just concluded that $\mathscr{C}^-$ does not satisfy any of the alternatives $i), ii)$ or $iii)$ of Proposition 3.1.1, but this contradicts the proposition and we just proved (3.1.23). Noting that

$$\partial\mathcal{U} = \left\{(\lambda, u) \in \mathbb{R} \times C_0^1(\overline{\Omega}); 1 + 2\lambda \|u\|_0^2 = 0\right\},$$

we just proved (0.0.3).

Let us prove (0.0.4). Since $\mathscr{C}^- \subset \mathcal{U}$, then we can apply Lemma 3.1.1 to obtain that $\|u\|_0 \leq \mu^{\frac{1}{p-1}}$ for all $(\lambda, u) \in \mathscr{C}^-$. Then the continuous inclusion $\mathrm{i} : C_0^1(\overline{\Omega}) \to C_0(\overline{\Omega})$ is such that $\mathrm{i}(\mathscr{C}^-)$ is bounded. So we can apply Lemma 2.3.1 with $E = C_0^1(\overline{\Omega})$, $F = C(\overline{\Omega})$, $\alpha = q = 2$ and

$Y = \mathrm{i} : C_0^1(\overline{\Omega}) \to C_0(\overline{\Omega})$, we deduce that

$$I_{\mathscr{C}^-} = \inf\{1 + 2\lambda \|u\|_0^2;\ (\lambda, u) \in \mathscr{C}^-\} = 0 \tag{3.1.24}$$

and we just proved (0.0.4).

Let us prove (0.0.1). Due to (3.1.24), there exists a sequence $(\lambda_n, u_n) \in \mathscr{C}^-$ such that

$$\rho_n := 1 + 2\lambda_n \|u_n\|_0^2 \to 0. \tag{3.1.25}$$

From Lemma 3.1.1, we know that $\|u_n\|_0 \leq \mu^{1/(p-1)}$. This estimate together with $\lambda \leq 0$ imply that

$$\lambda_n \leq \frac{\rho_n - 1}{2\mu^{2/(p-1)}}.$$

Since $\rho_n \to 0$, then

$$\lim_{n\to+\infty} \lambda_n \leq -\frac{1}{2\mu^{2/(p-1)}}.$$

Being $\mathscr{C}^-$ connected and $(0, u_0) \in \mathscr{C}^-$, we infer that $\mathscr{C}^-$ satisfies

$$\left(-\frac{1}{2\mu^{2/(p-1)}}, 0\right] \subset \mathrm{proj}_\lambda \mathscr{C}^-.$$

To complete the proof of (0.0.1), we just note Lemma 3.1.4 implies that

$$\mathrm{proj}_\lambda \mathscr{C}^- \subset \left(-\frac{1}{2(\mu - \mu_1)^{2/(p-1)}}, 0\right],$$

completing the proof of the item (0.0.1).

The inclusions (0.0.2) follow by Lemma 3.1.1-*ii*) and Lemma 3.1.4.

Item *a*) is a direct consequence of (0.0.1) combined with the connectedness of $\mathscr{C}^-$. Item *b*) follows from Lemma 3.1.4. Item *i*) follows from Lemma 3.1.8 and item *ii*) follows from Lemma 3.1.5. This ends the proof. □

3.1.3 Non-existence results for the case $\mu < \mu_1$

For the sake of completeness, we collect here some non-existence results for the problem ($P_{\lambda,\mu}$) when $\mu \leq \mu_1$.

Proposition 3.1.3. *Assume that* $p > 1$, $\mu \le \mu_1$, *and* $(\lambda, u) \in \mathcal{U}$ *be a positive solution for the problem* $(P_{\lambda,\mu})$. *Then:*

(i) $\lambda < 0$, *that is, the problem* $(P_{\lambda,\mu})$ *does not admit positive solutions for any* $\lambda \ge 0$ *and* $\mu \le \mu_1$,

(ii) $\mu > 0$. *In particular, the problem* $(P_{\lambda,\mu})$ *has no positive solutions for any* $\mu \le 0$ *and* $\lambda \in \mathbb{R}$,

(iii) *one has*

$$\lambda < \frac{\mu - \mu_1}{2\mu_1 \mu^{2/(p-1)}}$$

so that the problem $(P_{\lambda,\mu})$ *does not admit any positive solution for any*

$$\lambda \ge \frac{\mu - \mu_1}{2\mu_1 \mu^{2/(p-1)}} \text{ and } 0 \le \mu \le \mu_1.$$

In particular,

$$\lambda \le -1/2\mu^{2/(p-1)} \text{ if } 1 < p \le 3,$$

showing that the problem $(P_{\lambda,\mu})$ *does not admit any positive solution for any*

$$\lambda \ge -1/2\mu^{2/(p-1)} \text{ and } 0 \le \mu \le \mu_1.$$

Proof. Let $(\lambda, u) \in \mathcal{U}$ be a positive solution of $(P_{\lambda,\mu})$.

Let us prove (i). Assume that $\lambda \ge 0$. Then, we would obtain from the monotonicity properties of the principal eigenvalue, that

$$\begin{aligned} \mu_1 \quad &\ge \quad \mu = \sigma_1^\Omega[-\operatorname{div}[(1+2\lambda u^2)\nabla] + 2\lambda|\nabla u|^2 + u^{p-1}] \\ &> \quad \sigma_1^\Omega[-\operatorname{div}(\nabla)] = \mu_1, \end{aligned}$$

showing that we must have $\lambda < 0$.

Proof of (ii). It follows from (i) that $(\lambda, u) \in \mathcal{U}$ being a positive solution of $(P_{\lambda,\mu})$, we have

$$\begin{cases} -\Delta u - \dfrac{\mu}{1+2\lambda u^2} u = -\dfrac{u^p}{1+2\lambda u^2} + \dfrac{2\lambda u|\nabla u|^2}{1+2\lambda u^2} \le 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If were $\mu \leq 0$, then we could apply the Maximum Principle to obtain $u \leq 0$, leading to a contradiction with the positivity of u. This ends the proof of the item (ii).

Proof of (iii). We already know from (i) that $\lambda < 0$. Then the pair (λ, u) should satisfy

$$\mu_1 \int_\Omega u^2 \leq \int_\Omega |\nabla u|^2 < \int_\Omega \frac{|\nabla u|^2}{1+2\lambda u^2} = \int_\Omega u^2 \frac{\mu - u^{p-1}}{1+2\lambda u^2},$$

that is,

$$\int_\Omega \left[\mu_1 - \frac{\mu - u^{p-1}}{1+2\lambda u^2}\right] u^2 < 0. \tag{3.1.26}$$

So

$$\mu_1 < \frac{\mu - u(x_0)^{p-1}}{1+2\lambda u(x_0)^2} \text{ for some } x_0 \in \Omega,$$

leading us to

$$2\lambda \mu_1 \mu^{2/(p-1)} \leq 2\lambda \mu_1 \|u\|_0^2 \leq 2\lambda \mu_1 u(x_0)^2 < \mu - u(x_0)^{p-1} - \mu_1 < \mu - \mu_1 \leq 0,$$

after using Lemma 3.1.1-ii). This proves the first part of the item (iii). To complete the proof, assume, by contradiction, that were

$$\lambda > -\frac{1}{2\mu^{2/(p-1)}}. \tag{3.1.27}$$

After the inequality (3.1.27), we have well-defined the function

$$g(s) = \frac{\mu - s^{p-1}}{1+2\lambda s^2} \quad \forall\, s \in \left[0, \mu^{\frac{1}{p-1}}\right]. \tag{3.1.28}$$

Let us consider two cases: First case $p = 3$. Here, we obtain from (3.1.27), and (3.1.28) that $g'(s) < 0$ for all $0 < s < \mu^{\frac{1}{p-1}}$. Since $0 < u \leq \mu^{1/(p-1)}$ (see Lemma 3.1.1), we obtain from the monotonicity of g, that

$$\mu = g(0) > g(u(x_0))$$

whence follows, together with (3.1.26) and $\mu \leq \mu_1$, that

$$0 \leq \int_\Omega (\mu_1 - \mu) u^2 = \int_\Omega (\mu_1 - g(0)) u^2 < \int_\Omega (\mu_1 - g(u(x_0))) u^2 < 0,$$

which is impossible.

Second case, $1 < p < 3$. So, it is straightforward from from (3.1.28) that $g'(s) = 0$ if and only if $h(s) = 4\lambda\mu$, where

$$h(s) = 2\lambda(3-p)s^{p-1} - (p-1)s^{p-3} \text{ for all } 0 < s < \mu^{\frac{1}{p-1}}$$

so that

$$h'(s) = (p-1)(3-p)s^{p-4}\left[1+2\lambda s^2\right] > 0 \text{ for all } 0 < s < \left(-\frac{1}{2\lambda}\right)^{1/2}.$$

Since $\lambda > -1/(2\mu^{2/(p-1)})$, we have that

$$\mu^{\frac{1}{p-1}} \leq \left(-\frac{1}{2\lambda}\right)^{1/2},$$

showing that

$$h'(s) > 0 \text{ for all } 0 < s < \mu^{\frac{1}{p-1}}$$

that leads to

$$\begin{aligned} \max_{0\leq s\leq \mu^{\frac{1}{p-1}}} h &= h\left(\mu^{\frac{1}{p-1}}\right) = \mu\left[2\lambda(3-p) - (p-1)\mu^{-\frac{2}{p-1}}\right] \\ &< \mu\left[2\lambda(3-p) + (p-1)2\lambda\right] = 4\lambda\mu, \end{aligned}$$

where the last inequality follows from (3.1.27). That is, $g'(s) < 0$ for all $s \in [0, \mu^{1/(p-1)}]$. As done in the case $p = 3$, we obtain a contradiction again. This ends the proof. □

3.2 Carrier-Type problem with logistic perturbation

In this section, we will be inspired by the arguments used in the previous section to apply Theorem (0.0.1) again to study existence and qualitative information about the positive solutions of

$$\begin{cases} -(1+\lambda|u|_r^q)\Delta u = \mu u - u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \qquad (Q_{\lambda,\mu})$$

where $\lambda \in \mathbb{R}$ is a parameter, $q > 0$, $p > 1$, $r \geq 1$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 1$. At Section 3.2.2, we will prove Theorem 0.0.3.

3.2.1 Establishing the sufficient conditions for continuation

To do this, first we note that, depending on the size of $p > 1$ and $r \geq 1$, we are not able to set a compact operator on any open subset of $H_0^1(\Omega)$. Motivated by this technical challenging, let us define a truncation $T : H_0^1(\Omega) \to L^\kappa(\Omega)$ by

$$(T(u))(x) = \begin{cases} u(x) & \text{if } u(x) \leq \mu^{\frac{1}{p-1}}, \\ \mu^{\frac{1}{p-1}} & \text{if } u(x) > \mu^{\frac{1}{p-1}}, \end{cases}$$

for any $\kappa > \max\{pN, r\}$ given, that is, $T(u)$ is the truncation of u at the level $\mu^{\frac{1}{p-1}}$.

From this, and motivated by the case $T(u) = u$ for $\|u\|_0 \leq \mu^{\frac{1}{p-1}}$, let us consider the open set

$$\hat{\mathcal{V}} = \hat{\mathcal{V}}^- \cup \hat{\mathcal{V}}^+,$$

where

$$\hat{\mathcal{V}}^\nu := \{(\lambda, u) \in \mathbb{R}_0^\nu \times H_0^1(\Omega); 1 + \lambda \|T(u)\|_r^q > 0\},$$

for each $\nu \in \{-,+\}$, and define the operators $K, \Phi : \hat{\mathcal{V}} \to H_0^1(\Omega)$ by

$$K(\lambda, u) = (-\Delta)^{-1} \left(\frac{\mu T(u) - (T(u))^p}{1 + \lambda \|T(u)\|_r^q} \right), \tag{3.2.1}$$

and $\Phi(\lambda, u) := u - K(\lambda, u)$, taking advantage of the results and notations in Section 2.

The next lemma will guarantee us that the zeros of Φ are actually classical positive solutions of ($Q_{\lambda,\mu}$).

Lemma 3.2.1. *Let $(\lambda, u) \in \hat{\mathcal{V}}$, with u being non negative and non zero, such that $K(\lambda, u) = u$. Then (λ, u) is a classical positive solution of ($Q_{\lambda,\mu}$), that is, $0 < u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

Proof. By using $(u-\mu^{\frac{1}{p-1}})^+$ as a test function in $K(\lambda,u)=u$, we have

$$\begin{aligned}\int_\Omega |\nabla(u-\mu^{\frac{1}{p-1}})^+|^2 &= \int_\Omega \nabla(u-\mu^{\frac{1}{p-1}})\nabla((u-\mu^{\frac{1}{p-1}})^+)\\ &= \int_\Omega \nabla u \nabla((u-\mu^{\frac{1}{p-1}})^+)\\ &= \int_\Omega \frac{\mu T(u)-(T(u))^p}{1+\lambda|u|_r^q}(u-\mu^{\frac{1}{p-1}})^+\\ &= \int_{\{u(x)>\mu^{\frac{1}{p-1}}\}} \frac{\mu T(u)-(T(u))^p}{1+\lambda|u|_r^q}(u-\mu^{\frac{1}{p-1}})^+\\ &= \int_{\{u(x)>\mu^{\frac{1}{p-1}}\}} \frac{\mu\mu-\mu^p}{1+\lambda|u|_r^q}(u-\mu^{\frac{1}{p-1}})^+ = 0.\end{aligned}$$

That is, $u(x)\le \mu^{\frac{1}{p-1}}$ for a.e. in Ω and consequently $T(u)=u$. This just proves that u is a solution of ($Q_{\lambda,\mu}$). Since $u(x)\le\mu^{\frac{1}{p-1}}$, then $u\in L^t(\Omega)$ for any $t\ge 1$ and consequently u is classical by Theorem A.2.1 combined with Sobolev embedding. The positiveness of u follows by applying Theorem A.1.2. □

Now we are ready to state the lemma below, whose proof follows the same steps as those used to prove Lemma 3.1.1.

Lemma 3.2.2. *Let $(\lambda,u)\in\mathbb{R}\times(C^2(\Omega)\times C(\overline{\Omega}))$ be a non-negative and non-trivial solution of ($Q_{\lambda,\mu}$):*

i) if $\|u\|_0<\mu^{1/(p-1)}$, then $1+\lambda|u|_r^q>0$,

ii) if $1+\lambda\|u\|_r^q\ge 0$, then $\|u\|_0\le\mu^{1/(p-1)}$,

iii) if $1+\lambda\|u\|_r^q>0$, then $u\in \operatorname{int}P_{C_0^1(\overline{\Omega})}$,

iv) if $p\ge 2$ and $1+2\lambda\|u\|_r^q>0$, then $\|u\|_0<\mu^{1/(p-1)}$.

The proof of the Lemma below follows the same arguments as those used to prove the Lemma 3.1.3.

Lemma 3.2.3. *Assume that $\mu>\mu_1$ and $p>1$. Then $\mathrm{i}(\Phi_0,u_0,0)\neq 0$.*

The following result can be proved similarly to the proof of the Lemma 3.1.5:

Lemma 3.2.4. *Assume that $p>1$ and $0<\mu\neq\mu_1$. Then there is no bifurcation point of non-negative and non-trivial solutions in $\mathcal{U}$ of ($Q_{\lambda,\mu}$) from its trivial curve of solutions.*

Before proving Theorem 0.0.3, we will prove the next proposition that shows, in particular, that there is no bifurcation point $\lambda \leq 0$ from the trivial solutions in the $C(\overline{\Omega})$-norm.

Proposition 3.2.1. *Suppose that* $(\underline{\lambda}, u)$ *and* $(\overline{\lambda}, v)$ *are classical positive solutions of* $(Q_{\lambda,\mu})$ *in* $\mathcal{U}(F)$ *with* $\underline{\lambda} \leq 0 \leq \overline{\lambda}$. *Then:*

$$v \leq u_0 \leq u. \tag{3.2.2}$$

and

$$(\mu - \mu_1(1 + \underline{\lambda}|u|_r^q))^{1/(p-1)}\varphi_1 \leq u \tag{3.2.3}$$

In particular, if (λ, u) *is a positive solution of* $(Q_{\lambda,\mu})$ *satisfying* $1 + \lambda|u|_r^q > 0$, *then*

$$|u|_r \geq (\mu - \mu_1)^{\frac{1}{p-1}}|\varphi_1|_r \text{ and } \underline{\lambda} > -\frac{1}{(\mu - \mu_1)^{\frac{q}{p-1}}|\varphi_1|_r^q}.$$

Proof. We start proving the first inequality of (3.2.2). Since $\overline{\lambda} \geq 0$ and $v \leq \mu^{1/(p-1)}$ (Lemma 3.2.2-*ii*)), it is apparent that

$$-\Delta v = \frac{v}{1 + \overline{\lambda}|v|_r^q}(\mu - v^{p-1}) \leq v(\mu - v^{p-1}) \quad \text{in } \Omega,$$

whence follows that v is a subsolution of $(P_{0,\mu}^F)$. Moreover, it is straightforward to verify that positive constants large enough are supersolutions of $(P_{0,\mu}^F)$. Thus, by sub and supersolution methods, there exists a positive solution of $(P_{0,\mu}^F)$ between v and $K > 0$ large. Since u_0 is the unique positive solution of $(P_{0,\mu}^F)$, we obtain that $v \leq u_0$.

To prove that $u_0 \leq u$, we observe that $\varepsilon\varphi_1$ and u are a pair of sub and supersolution of $(P_{0,\mu}^F)$ such that $\varepsilon\varphi_1 \leq u$ for $\varepsilon > 0$ small enough. Again, by using sub and supersolution methods, combined with the uniqueness of positive solutions for the problem $(P_{0,\mu}^F)$, we get the claimed inequality.

To prove (3.2.3), consider the problem, in w, given by

$$\begin{cases} -\Delta w = \dfrac{w}{1 + \underline{\lambda}|u|_r^q}(\mu - w^{p-1}) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2.4}$$

This problem has a unique positive solution. But, by the definition of u, we know that u is a positive solution of (3.2.4) and consequently the unique positive solution of (3.2.4).

On the other hand, given $\varepsilon > 0$, $\varepsilon\varphi_1$ will be a subsolution of (3.2.4) if

$$\begin{aligned} -\Delta(\varepsilon\varphi_1) &\leq \frac{\varepsilon\varphi_1}{1+\underline{\lambda}|u|_r^q}(\mu-(\varepsilon\varphi_1)^{p-1}) \quad \text{in } \Omega, \\ \varepsilon\varphi_1 &\leq (\mu-\mu_1(1+\underline{\lambda}|u|_r^q))^{1/(p-1)} \quad \text{in } \Omega. \end{aligned}$$

Taking into account that $\|\varphi_1\|_0 = 1$, the above inequality holds for $\varepsilon = (\mu-\mu_1(1+\underline{\lambda}|u|_r^q))^{1/(p-1)} > 0$. Furthermore, large constants are supersolution for (3.2.4), which implies by the sub and supersolution methods that there exists a (positive) solution of (3.2.4) between $(\mu-\mu_1(1+\overline{\lambda}|v|_r^q))^{1/(p-1)}\varphi_1$ and $K > 0$ large enough. Once that u is the unique solution of (3.2.4), we obtain (3.2.3).

Let us prove the second part of the theorem. First note that by applying (3.2.3) for $\underline{\lambda} = 0$, we have

$$(\mu-\mu_1)^{1/(p-1)}\varphi_1 \leq u_0 \leq u. \tag{3.2.5}$$

Powering the above inequality by r, integrating and then powering again by q, we deduce that

$$|u|_r^q \geq (\mu-\mu_1)^{\frac{q}{p-1}}|\varphi_1|_r^q. \tag{3.2.6}$$

Since $1+\underline{\lambda}|u|_r^q \geq 0$, then $\|u\|_0 \leq \mu^{\frac{1}{p-1}}$ due to item *ii*) of Lemma 3.2.2. By using this a priori bound of u, in (3.2.3), we obtain

$$\mu^{\frac{1}{p-1}} \geq (\mu-\mu_1(1+\underline{\lambda}|u|_r^q))^{1/(p-1)}\varphi_1, \tag{3.2.7}$$

which implies

$$\mu \geq (\mu-\mu_1(1+\underline{\lambda}|u|_r^q))\varphi_1^{p-1}. \tag{3.2.8}$$

By using (3.2.6) in (3.2.8), and $\|\varphi_1\|_\infty = 1$, we obtain

$$(\mu-\mu_1)-\underline{\lambda}\mu_1(\mu-\mu_1)^{\frac{q}{p-1}}|\varphi_1|_r^q \leq (\mu-\mu_1)+\mu_1, \tag{3.2.9}$$

which implies by $\mu-\mu_1 > 0$, and (3.2.9), that

$$-\underline{\lambda}\mu_1(\mu-\mu_1)^{\frac{q}{p-1}}|\varphi_1|_r^q < \mu_1. \tag{3.2.10}$$

That is,

$$\underline{\lambda} > -\frac{1}{(\mu-\mu_1)^{\frac{q}{p-1}}|\varphi_1|_r^q}.$$

This ends the proof of the Proposition.

□

Before starting the proof of Theorem 0.0.3, we will need the above notations and lemmas.

3.2.2 Connected set of positive solutions

In this section we will use the above results and apply Theorem 0.0.1 to the operator K introduced in (3.2.1) to obtain a connected of positive solutions of ($Q_{\lambda,\mu}$).

Proof of Theorem 0.0.3-Conclusion. Consider the operator K introduced in (3.2.1).

Let us show that K is compact in the open subset $\mathcal{U}$. The idea is to apply Lemma 2.3.1. First, we claim that T is uniformly continuous. Indeed, let $u, v \in H_0^1(\Omega)$. We claim that

$$|T(u)(x) - T(v)(x)| = |u(x) - v(x)| \text{ for a.e. } x \in \Omega. \tag{3.2.11}$$

Divide in two cases. If x is such that $u(x) \le \mu^{\frac{1}{p-1}}$ and $v(x) \le \mu^{\frac{1}{p-1}}$, then $T(u)(x) = u(x)$ and $T(v)(x) = v(x)$ and so whence $|T(u)(x) - T(v)(x)| = |u(x) - v(x)|$. If x is such that $u(x) \le \mu^{\frac{1}{p-1}}$ and $v(x) > \mu^{\frac{1}{p-1}}$, then $T(u)(x) = u(x)$ and $T(v)(x) = \mu^{\frac{1}{p-1}}$. Note that $u(x) \le v(x)$ and so $|u(x) - v(x)| = v(x) - u(x) = \mu^{\frac{1}{p-1}} - u(x)$. Then

$$|T(u)(x) - T(v)(x)| = |u(x) - \mu^{\frac{1}{p-1}}| = \mu^{\frac{1}{p-1}} - u(x) = |u(x) - v(x)|$$

and we just proved (3.2.11).

Let (u_n) and (v_n) be sequences in $H_0^1(\Omega)$ such that $u_n - v_n \to 0$ in $H_0^1(\Omega)$. Observe that

$$|T(u_n)(x) - T(v_n)(x)| \le 2\mu^{\frac{1}{p-1}} \ \forall\, n \text{ for a.e. } x \in \Omega. \tag{3.2.12}$$

Moreover, since $u_n - v_n \to 0$ in $H_0^1(\Omega)$ then $u_n(x) - v_n(x) \to 0$ a.e. in Ω. By combining this with (3.2.11), we deduce that

$$T(u_n)(x) - T(v_n)(x) \to 0 \text{ a.e. in } \Omega.$$

By Dominated Convergence Lebesgue Theorem, $T(u_n)-T(v_n)$ converges to 0 in $L^\kappa(\Omega)$ up to a subsequence and the uniformly continuity of $T:H_0^1(\Omega)\to L^s(\Omega)$ is proved. Now, let C be a closed and bounded subset of $\mathcal{U}$ such that $\text{dist}(C,\partial\mathcal{U})>0$. Let us apply Lemma 2.3.1 with $E=H_0^1(\Omega)$, $F=L^r(\Omega)$, $\alpha=1$, $s=q$ and $Y=i\circ T:H_0^1(\Omega)\to L^r(\Omega)$, where $i:L^\kappa(\Omega)\to L^r(\Omega)$ is the inclusion which is continuous since $\kappa>r$. So Y is continuous we deduce that there exists a $\rho>0$ such that

$$1+\lambda|u|_r^q>\rho\ \forall\ (\lambda,u)\in C. \tag{3.2.13}$$

Moreover, if $u\in H_0^1(\Omega)$, then $T(u)\in L^\kappa(\Omega)$ and consequently $(T(u))^p\in L^{\kappa/p}(\Omega)$. By elliptic regularity theory, we obtain that

$$\begin{aligned}\|K(\lambda,u)\|_{W^{2,\kappa/p}(\Omega)} &\le M\left|\frac{\mu T(u)-(T(u))^p}{1+\alpha\lambda|u|_r^q}\right|_{\kappa/p}\\ &\le \frac{M}{\rho}|\mu T(u)-(T(u))^p|_{\kappa/p}\\ &\le \frac{M}{\rho}\left(\mu\,|T(u)|_{\kappa/p}+|T(u)|_\kappa^p\right)\\ &\le \frac{\tilde{M}}{\rho}(\mu\,|T(u)|_1+|T(u)|_\kappa^p)\ \forall\ (\lambda,u)\in C,\end{aligned} \tag{3.2.14}$$

where in the second inequality we used (3.2.13). Let (λ_n,u_n) be a bounded sequence in C. So

$$|T(u_n)|_1+|T(u_n)|_\kappa^p$$

is a bounded sequence. Then the sequence $K(\lambda_n,u_n)$ is bounded in $W^{2,p/\kappa}(\Omega)$ due to (3.2.14). Since $k>Np$, then $W^{2,p/\kappa}(\Omega)$ is compactly embedded in $C_0^1(\overline{\Omega})$, in particular, in $H_0^1(\Omega)$ and consequently $K(\lambda_n,u_n)$ converges up to a subsequence in $H_0^1(\Omega)$. We just proved that K is compact in C and consequently, is compact in the the open subset $\hat{\mathcal{V}}$.

Afterward, we will apply Theorem 0.0.1 to $K:\mathcal{U}\to H_0^1(\overline{\Omega})$, considering the 0-partition of $\mathcal{U}$, to deduce that there exists a pair of connected sets $\mathscr{C}^\nu$, $(\nu\in\{-,+\})$ such that

$$u=K(\lambda,u)\ \forall\ (\lambda,u)\in\mathscr{C}^-\cup\mathscr{C}^-$$

and satisfying at least one of the alternatives

i) $\text{dist}\left(\mathscr{C}^\nu,\partial\hat{\mathcal{V}}^\nu\right)=0.$

ii) $\mathscr{C}^\nu$ is unbounded in $\mathbb{R}\times H_0^1(\Omega)$.

iii) $\mathscr{C}^\nu$ meets $(\lambda_0,u_1)\in\hat{\mathscr{V}}$ with $u_1\neq u_0$.

Now, observe that the alternative *iii*) does not occurs both for $\mathscr{C}^-$ and $\mathscr{C}^+$ due to the uniqueness of strongly positive solutions for the problem $(Q_{0,\mu})$.

Taking into account $\partial\hat{\mathscr{V}}^+=\emptyset$, then i) cannot occurs for $\mathscr{C}^+$. Consequently, $\mathscr{C}^+$ is unbounded. Moreover, by Lemma 3.2.2 combined with elliptic regularity, we obtain that $\text{proj}_{H_0^1(\Omega)}\mathscr{C}^+$ is bounded and so $\text{proj}_\lambda\mathscr{C}^+$ must be unbounded. The strong positiveness of the solutions is guaranteed by Lemmas 3.2.2 and 3.2.4. So, gathering these information, we have that

$$\mathscr{C}:=\mathscr{C}^-\cup\mathscr{C}^+$$

is an unbounded connected set of positive solutions for the problem ($Q_{\lambda,\mu}$) containing $(0,u_0)$.

Below, let us prove more qualitative information about $\mathscr{C}$. First, let us prove (0.0.7). To do so, we claim that alternative *i*) must occur. Indeed, suppose that *i*) were not true, that is, $\text{dist}\left(\mathscr{C}^-,\partial\hat{\mathscr{V}}^-\right)>0$. By Proposition 3.2.1, we know that $\text{proj}_\lambda\mathscr{C}^-$ is bounded. So we can apply Lemma 2.3.1 with $\mathscr{C}^-$ in order to deduce that there exists some $\rho_1>0$ such that $1+\lambda|u|_r^q>\rho_1$ for all $(\lambda,u)\in\mathscr{C}^-$. Consequently, given $t>1$, the elliptic regularity theory gives

$$\|u\|_{W^{2,t}(\Omega)}\leq\frac{\tilde{M}_1}{\rho_1}\|\mu u-u^p\|_t\ \forall\,(\lambda,u)\in\mathscr{C}^-$$

for some positive constant $\tilde{M}_1$. Since $\mathscr{C}^-\subset\hat{\mathscr{V}}$, we can apply Lemma 3.2.2 to imply that $\|u\|_0\leq\mu^{\frac{1}{p-1}}$, whence follows that $(\|u\|_{W^{2,t}(\Omega)})$ is bounded. By the Sobolev embedding of $W^{2,t}(\Omega)$ into $H_0^1(\Omega)$, for sufficiently large $t>1$, we obtain that $\text{proj}_{H_0^1(\Omega)}\mathscr{C}^-$ is bounded.

As we already know that $\text{proj}_\lambda\mathscr{C}^-$ is bounded, the last conclusion leads $\mathscr{C}^-$ be bounded. That is, $\mathscr{C}^-$ does not satisfy alternative *ii*). We already had that $\mathscr{C}^-$ does not satisfy either alternative *i*) or *iii*) so $\mathscr{C}^-$ does not satisfy any of the alternatives *i*), *ii*) or *iii*) and this contradicts Theorem 0.0.1. We just proved that the alternative *i*) must occurs, that is,

$$\text{dist}\left(\mathscr{C}^\nu,\partial\hat{\mathscr{V}}^\nu\right)=0.$$

Now we are in position to complete the proof (0.0.7). Let us split the proof in two cases. First, assume that $\text{proj}_{H_0^1(\Omega)}\mathscr{C}^-$ is unbounded. Let $(\lambda,u)\in\mathscr{C}^-$. By taking u as a test function and using $\|u\|_0\leq\mu^{1/(p-1)}$ (see Lemma 3.2.2), we obtain that $(1+\lambda|u|_r^q)\|u\|^2$ is bounded. Since we are assuming that $\text{proj}_{H_0^1(\Omega)}\mathscr{C}^-$ is unbounded, we obtain that

$$I_{\mathscr{C}^-} := \inf\{1+\lambda|u|_r^q; (\lambda,u)\in\mathscr{C}^-\} = 0.$$

For the second case, we assume that $\text{proj}_{H_0^1(\Omega)}\mathscr{C}^-$ be bounded. So, we can apply Lemma 2.3.1 with $\mathscr{C}^-$ to deduce that $I_{\mathscr{C}^-}=0$ and the second case is proved. This completes the proof of (0.0.7). In particular, we obtain from $\text{proj}_\lambda\mathscr{C}^-$ be bounded and Lemma 2.3.1, that

$$\text{dist}_{\mathbb{R}\times H_0^1(\Omega)}\left(\mathscr{C},\{(\lambda,u)\in\mathbb{R}\times H_0^1(\Omega); 1+\lambda|T(u)|_r^q=0\}\right)=0. \tag{3.2.15}$$

As a consequence of $I_{\mathscr{C}^-}=0$, we will prove below the first inclusion of (0.0.5), that is,

$$\left(-\frac{1}{\mu^{\frac{q}{p-1}}|\Omega|^{\frac{q}{r}}},+\infty\right)\subset \text{proj}_\lambda\mathscr{C}. \tag{3.2.16}$$

Since $I_{\mathscr{C}^-}=0$, then there exists a sequence $(\lambda_n,u_n)\in\mathscr{C}^-$ such that $\rho_n := 1+\lambda_n|u_n|_r^q\to 0$, whence follows that

$$\begin{aligned}\lambda_n &= \frac{\rho_n-1}{|u_n|_r^q}\\ &\le \frac{\rho_n-1}{\mu^{\frac{q}{p-1}}|\Omega|^{\frac{q}{r}}},\end{aligned}$$

where the last inequality follows from $\|u_n\|_0\le\mu^{\frac{1}{p-1}}$. By passing to the limit, we get that

$$\lim_{n\to+\infty}\lambda_n\le-\frac{1}{\mu^{\frac{q}{p-1}}|\Omega|^{\frac{q}{r}}}. \tag{3.2.17}$$

So (3.2.16) follows from the connectedness of $\mathscr{C}$ combined with (3.2.17). The other inclusion in (0.0.5) follows directly from Proposition 3.2.1. The inclusions in (0.0.6) follow from Lemmas 3.2.2 and 3.2.1.

Let us prove item i). Consider $(\lambda,u_\lambda)\in\mathscr{C}$ and define $\rho_\lambda=1+\lambda|u_\lambda|_r^q$. Note that u_λ satisfies, in the classical sense, the problem

$$\begin{cases}-\dfrac{\rho_\lambda}{\mu}\Delta u_\lambda = u_\lambda\left(1-\dfrac{u_\lambda^{p-1}}{\mu}\right) & \text{in } \Omega,\\ u_\lambda=0 & \text{on } \partial\Omega,\end{cases}$$

for each $\lambda \in \text{proj}_\lambda \mathscr{C}$. So, we are able to apply Proposition 6.6 of [31] to infer that

$$u_\lambda(x) \to \mu^{1/(p-1)} \text{ uniformly in } K \text{ as } \rho_\lambda \to 0,$$

for any compact subset $K \subset \Omega$ given. This ends the proof of item i).

Item ii) follows immediately from Proposition 3.2.1. Let us prove item iii). By using Sobolev embeddings and elliptic regularity combined with the fact that $\|u_n\|_0 \leq \mu^{\frac{1}{p-1}}$ (see item ii) of Lemma 3.2.1), we deduce that there exists constants $R, D, M > 0$ such that

$$\begin{aligned}
\|u_\lambda\|_0 &\leq D\|u_\lambda\|_{W^{2,t}(\Omega)} \\
&\leq R\left|\frac{\mu u_\lambda - u_\lambda^p}{1+\lambda|u_\lambda|_r^q}\right|_t \\
&\leq R\left|\frac{\mu u_\lambda - u_\lambda^p}{1+\lambda_n|u_\lambda|_r^q}\right|_t \\
&\leq \frac{RM}{1+\lambda|u_\lambda|_r^q} \\
&\leq \frac{RM}{1+\lambda(\mu-\mu_1)^{\frac{q}{p-1}}|\varphi_1|_r^q},
\end{aligned}$$

holds for sufficiently large $t > 1$, where in the last inequality, we used Proposition 3.2.1. So $\|u_\lambda\|_0 \to 0$ as $\lambda \to +\infty$.

3.2.3 Two special cases

We begin this subsection noting that we were not able to prove that

$$\text{dist}_{\mathbb{R}\times H_0^1(\Omega)}\left(\mathscr{C}, \{(\lambda, u) \in \mathbb{R}\times H_0^1(\Omega); 1+\lambda|u|_r^q = 0\}\right) = 0$$

holds in Theorem 0.0.3. In fact, we just prove

$$\text{dist}_{\mathbb{R}\times H_0^1(\Omega)}\left(\mathscr{C}, \{(\lambda, u) \in \mathbb{R}\times H_0^1(\Omega); 1+\lambda|T(u)|_r^q = 0\}\right) = 0$$

holds, see (3.2.15). The reason of the adoption of the truncation T is due to the technical problem of the well definition and the compactness of the operator K for p and r large. However, if we impose some additional restrictions on p, r and N, we are able to prove a version of Theorem 0.0.3 without the need of the truncation T. This will give us Corollary

3.2.1. Another alternative is to choose a different ambient space for the operator K and then obtain Corollary 3.2.2. In the following, we have the aforementioned corollaries.

Corollary 3.2.1 (of the proof of Theorem 0.0.3)**.** *Additionally to the hypotheses of Theorem 0.0.3, assume that $r \leq 2^*$, $1 < p < 2^*$ and one of the following arrangements of p and N hold:*

1) $p(N-2) > 2$ *and* $p < 2^* - 1$,

2) $p(N-2) \leq 2$,

3) $1 + N/2 < p = 2^* - 1$.

Then there exists an unbounded connected set of strongly positive classical solutions $\mathscr{C}_1 \subset \mathbb{R} \times H_0^1(\Omega)$ of ($Q_{\lambda,\mu}$) satisfying, additionally to all the stated in Theorem 0.0.3, the property

$$\operatorname{dist}_{\mathbb{R}\times H_0^1(\Omega)}\left(\mathscr{C}_1, \{(\lambda,u) \in \mathbb{R} \times H_0^1(\Omega);\ 1+\lambda|u|_r^q = 0\}\right) = 0.$$

Proof. Let us follow the same strategy as in last subsection. Define the operator $K_1 : \mathcal{V}_1 \to H_0^1(\Omega)$ by

$$K_1(\lambda,u) = (-\Delta)^{-1}\left(\frac{\mu u - u^p}{1+\lambda|u|_r^q}\right),$$

where

$$\mathcal{V}_1 := \{(\lambda,u) \in \mathbb{R} \times H_0^1(\Omega); 1+\lambda|u|_r^q > 0\}.$$

So, under the additional assumptions of Corollary 3.2.1, we are able to prove that K_1 is compact in the open subset $\mathcal{V}_1$. In this case, instead of (3.2.15), we have

$$\operatorname{dist}_{\mathbb{R}\times H_0^1(\Omega)}\left(\mathscr{C}_1, \{(\lambda,u) \in \mathbb{R} \times H_0^1(\Omega) 1+\lambda|u|_r^q = 0\}\right) = 0.$$

So let us prove the compactness of K_1 in $\mathcal{V}_1$. Let C be a bounded subset of $\mathcal{V}_1$ such that $\operatorname{dist}(C, \partial\mathcal{V}_1) > 0$. By Lemma 2.3.1 with $E = H_0^1(\Omega)$, $F = L^r(\Omega)$, $\alpha = 1$, $s = q$ and $Y = \mathrm{Id}$, there exists some $\rho > 0$ such that $1+\lambda|u|_r^q > \rho$ for all $(\lambda,u) \in C$.

Set $\beta := 2^*/p < 2^*$, and take $u \in H_0^1(\Omega)$. So $u^p \in L^\beta(\Omega)$. Since $\beta > 1$, then $K_1(\lambda,u) \in W_0^{2,\beta}(\Omega)$ is well defined due to the classic elliptic regularity theory. Moreover,

$$\begin{aligned}
\|K_1(\lambda,u)\|_{W_0^{2,\beta}(\Omega)} &\leq \frac{M}{\rho}|\mu u - u^p|_\beta \\
&\leq \frac{M}{\rho}(\mu|u|_\beta + |u^p|_\beta) \\
&= \frac{M}{\rho}\left(\mu|u|_\beta + |u|_{2^*}^{2^*/\beta}\right) \\
&\leq \frac{M_1}{\rho}\left(\|u\| + \|u\|^{2^*/\beta}\right),
\end{aligned}$$

that is,

$$\|K_1(\lambda,u)\|_{W_0^{2,\beta}(\Omega)} \leq \frac{M_1}{\rho}\left(\|u\| + \|u\|^{2^*/\beta}\right) \ \forall\,(\lambda,u) \in C \tag{3.2.18}$$

for some $M_1 > 0$.

After (3.2.18), let us consider the cases 1) and 2) and 3). First assume that p and N satisfy 1). Note that $p(N-2) > 2$ is equivalent to $\beta < N$. Consequently, the embedding $W_0^{2,\beta}(\Omega) \hookrightarrow W_0^{1,t}(\Omega)$ is compact for any $1 \leq t < \beta^*$. Moreover, $p < 2^* - 1$ is equivalent to $\beta^* > 2$. So we can take $t = 2$ to obtain that the embedding

$$W_0^{2,\beta}(\Omega) \hookrightarrow H_0^1(\Omega) \tag{3.2.19}$$

is compact as well. Let (λ_n, u_n) be a bounded sequence in $H_0^1(\Omega)$. By (3.2.18), we have that $K_1(\lambda_n,u_n)$ is bounded in $W_0^{2,\beta}(\Omega)$. Using (3.2.19), we deduce that $K_1(\lambda_n,u_n)$ converges in $H_0^1(\Omega)$ up to a subsequence.

Now assume that p and N satisfy 2). Note that $p(N-2) \leq 2$ is equivalent to $\beta \geq N$ and so $W_0^{2,\beta}(\Omega) \hookrightarrow W_0^{1,t}(\Omega)$ is a compact embedding for any $1 \leq t < +\infty$. In particular, by taking $t = 2$, we deduce that

$$W_0^{2,\beta}(\Omega) \hookrightarrow H_0^1(\Omega)$$

is a compact embedding. Similarly as in case 1), the compactness of K_1 is a consequence of (3.2.18).

Finally, assume that p and N satisfy 3). Let (u_n) be a bounded sequence in $H_0^1(\Omega)$ and $y_n = K_1(\lambda_n,u_n)$. Observe that $p > 1 + N/2$ implies that $H_0^1(\Omega) \hookrightarrow L^{2\tau'}(\Omega)$ is a compact embedding, where $\tau = p-1$. By testing $y_n = K_1(\lambda_n,u_n)$ against u_n, and applying Hölder

Inequality twice, we deduce that

$$\begin{aligned}\|y_n\|^2 &\leq \frac{1}{\rho}\left(\int_\Omega \mu|u_n||y_n| + \int_\Omega |u_n|^{p-1}|u_n||y_n|\right)\\ &\leq \frac{M}{\rho}(|u_n\|_2|y_n|_2 + |u_n|_\tau^{\tau/(t(2\tau')')}|u_n|_{2\tau'}^{2\tau'/(t'(2\tau')')}|y_n|_{2\tau'})\end{aligned}$$

for some constant $M>0$. So (y_n) is bounded in $H_0^1(\Omega)$, which implies that there exists some $y\in H_0^1(\Omega)$ such that

$$y_n \rightharpoonup y \text{ in } H_0^1(\Omega)$$

and

$$y_n \to y \text{ in } L^{2\tau'}(\Omega). \tag{3.2.20}$$

Now, by testing $y_n = K_1(\lambda_n, u_n)$ against $y_n - y$, subtracting

$$\int_\Omega \nabla y \nabla (y_n - y),$$

and applying Hölder inequality twice, we get that

$$\begin{aligned}\|y_n - y\|^2 &= \int_\Omega \frac{\mu u_n - u_n^p}{1+\lambda|u_n|_r^q}(y_n - y) - \int_\Omega \nabla y\nabla(y_n - y)\\ &\leq \frac{M}{\rho}\int_\Omega (\mu|u_n| + |u_n|^p)|y_n - y| + o(1)\\ &= \frac{M}{\rho}\int_\Omega \left(|u_n||y_n - y| + |u_n|^{p-1}|u_n||y_n - y|\right) + o(1)\\ &\leq \frac{M}{\rho}\left(|u_n|_2|y_n - y|_2 + |u_n|_\tau^{\tau/(t(2\tau')')}|u_n|_{2\tau'}^{2\tau'/(t'(2\tau')')}|y_n - y|_{2\tau'}\right) + o(1),\end{aligned}$$

for some constant $M>0$, where the term $o(1)$ comes from using the convergences (3.2.20) and $y_n \rightharpoonup y$ in $H_0^1(\Omega)$ weakly. Consequently, $y_n \to y$ strongly in $H_0^1(\Omega)$ and the compactness of K_1 is proved for the case 3). This ends the proof. □

Corollary 3.2.2 (of the proof of Theorem 0.0.3)**.** *Assume all the hypotheses of Theorem 0.0.3. Then there exist an unbounded connected set of strongly positive classical solutions $\mathscr{C}_2 \subset \mathbb{R}\times C(\overline{\Omega})$ of ($Q_{\lambda,\mu}$) satisfying, additionally to all the stated in Theorem 0.0.3, the property*

$$\operatorname{dist}_{\mathbb{R}\times C(\Omega)}(\mathscr{C}_2, \{(\lambda,u)\in\mathbb{R}\times C(\Omega);\ 1+\lambda|u|_r^q = 0\}) = 0.$$

Proof. It suffices to follow the same steps as in the proof of Theorem 0.0.3 for the the operator $K_2 : \mathcal{V}_2 \subset \mathbb{R} \times C(\overline{\Omega}) \to C(\overline{\Omega})$ defined by

$$K_2(\lambda, u) = (-\Delta)^{-1} \left(\frac{\mu u - u^p}{1 + \lambda |u|_r^q} \right),$$

where

$$\mathcal{V}_2 = \{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega});\ 1 + \lambda |u|_r^q > 0\},$$

and apply Theorem 0.0.1 to K_2. □

Conclusion of Chapter 3 Applying Theorem 0.0.1, we prove Theorems 0.0.2 and 0.0.3, which provide qualitative information about the positive solutions (λ, u) of ($P_{\lambda,\mu}$) and ($Q_{\lambda,\mu}$), respectively. Specifically, for Theorem 0.0.2, we obtained precise information regarding the range of λ and the $C(\overline{\Omega})$-norm of the solutions (λ, u). Similarly, for Theorem 0.0.3, we determined precise information about the range of λ and the pointwise behavior of the solutions. However, due to the focus of the assertions of Theorem 0.0.1 on a single Banach space E, as noted in the conclusion of Chapter 2, the boundedness of $\|\nabla u_\lambda\|_0$ for problem ($P_{\lambda,\mu}$) (and, respectively, $\|u_\lambda\|_{H_0^1(\Omega)}$ for problem ($Q_{\lambda,\mu}$)) remains an open question.

Chapter 4

Two parameters logistic problem with degradation and refuge zone

This chapter is dedicated to provide result on existence and behavior of positive solutions of

$$\begin{cases} -\Delta u+\lambda V(x)=\mu m(\lambda,x)u-b(\lambda,x)u^{p} & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \\ u>0 & \text{in } \Omega, \end{cases} \tag{4.0.1}$$

$0\leq V(x)\in L^{\infty}(\Omega)$, $0\lneq b(\lambda,\,\cdot\,)\in C(\overline{\Omega})$, $0\neq m(\lambda,\,\cdot\,)\in C(\overline{\Omega})$ possibly changing sign and $\emptyset\neq M_{\lambda}^{+}:=\{x\in\Omega;\ m(\lambda,x)>0\}$, $B_{0,\lambda}:=\text{int}\{x\in\overline{\Omega};b(\lambda,x)=0\}$.

In Section 4.1, we will prove the existence result. In Section 4.2, we provide a condition for uniform boundedness of the positive solutions of (4.0.1). Finally, in Section 4.3, we exhibit a subsolution of (4.0.1) that will drive to blow-up behaviors of positive solutions of ($R_{\lambda,\mu}$) and ($S_{\lambda,\mu}$).

In this chapter we will denote

$$\mathcal{S}=\{(\lambda,\mu)\in\mathbb{R}^{2};\sigma_{1}^{\Omega}[-\Delta+\lambda V-\mu m(\lambda,x)]<0<\sigma_{1}^{B_{0,\lambda}}[-\Delta+\lambda V(x)-\mu m(\lambda,x)]\},$$

$$\mathcal{C}_{0}:=\{\mu\in C([0,\Lambda]),\ 0<\Lambda<\infty;(\lambda,\mu(\lambda))\in\mathcal{S}\ \forall\,\lambda\in[0,\Lambda)\}$$

and

$$\mathcal{C}_\infty := \left\{ \mu \in C([0,+\infty));\ \exists \mu(\infty) := \lim_{\lambda\to+\infty} \mu(\lambda) < \infty \text{ and } (\lambda, \mu(\lambda)) \in \mathcal{S}\ \forall\, \lambda \in [0,+\infty) \right\}.$$

4.1 Existence theorem of positive solutions

Let us denote

$$b(\infty, x) := \liminf_{\lambda\to\infty} b(\lambda, x),\ B_{0,\infty} := \operatorname{int}\{x \in \overline{\Omega}; b(\infty, x) = 0\},$$

We will assume that

$$B_{0,\lambda} \subsetneq \Omega\ \forall\, \lambda \geq 0$$

and $B_{0,\infty} \subsetneq \Omega$.

Proof of Theorem 0.0.4. Since $(\lambda, \mu) \in \mathcal{S}$, then

$$\sigma_1^\Omega[-\Delta + \lambda V - \mu m(\lambda, x)] < 0 < \sigma_1^{B_{0,\lambda}}[-\Delta + \lambda V - \mu m(\lambda, x)]. \tag{4.1.1}$$

First, prove that (4.1.1) is a necessary condition for the existence of a positive solution of (0.0.13). Assume that there exists a positive solution u of (0.0.13). By using the monotonicity of the principal eigenvalue we deduce that

$$0 = \sigma_1^\Omega[-\Delta + \lambda V(x) - \mu m(\lambda, x) + b(\lambda, x)u^{p-1}] > \sigma_1^\Omega[-\Delta + \lambda V(x) - \mu m(\lambda, x)].$$

On the other hand, by the monotonicity of the principal eigenvalue with respect to the domain, we deduce that

$$0 = \sigma_1^\Omega[-\Delta + \lambda V(x) - \mu m(\lambda, x) + b(\lambda, x)u^{p-1}] < \sigma_1^{B_{0,\lambda}}[-\Delta + \lambda V(x) - \mu m(\lambda, x)].$$

We just proved that (4.1.1) is a necessary condition.

In order to prove the existence, we use the sub-supersolution method, see [3] for instance. Let $\underline{\varphi}$ be the positive eigenfunction associated with $\sigma_1^\Omega[-\Delta + \lambda V(x) - \mu m(\lambda, x)]$ normalized such that $\|\underline{\varphi}\|_\infty = 1$. It is not difficult to verify that $\underline{u} := \varepsilon \underline{\varphi}$ is a subsolution of (0.0.13) if

$$\varepsilon \leq (-\sigma_1^\Omega[-\Delta + \lambda V(x) - \mu m(\lambda, x)] / \|b\|_\infty)^{1/(p-1)}.$$

Now, we build the supersolution. Let us define

$$K_\delta := \{x \in \Omega \setminus \overline{B}_{0,\lambda}; \mathrm{dist}(x, \partial(\Omega \setminus B_{0,\lambda})) \geq \delta\}.$$

Observe that K_δ is a compact subset of $\Omega \setminus \overline{B}_{0,\lambda}$. Consider $v_\delta \in \mathrm{int}_{C_0^1(\overline{\Omega})}$ such that $v_\delta(x) \geq \delta^{-1}$ for all $x \in K_\delta$. Given any compact subset $K \subset \Omega \setminus \overline{B}_{0,\lambda}$, there exists $\delta = \delta(K)$ such that $K \subset K_\delta$ and consequently

$$\lim_{\delta \mapsto 0} \min_K b(\lambda, x) v_\delta(x) = +\infty$$

for each compact subset $K \subset \Omega \setminus \overline{B}_{0,\lambda}$. By Corollary 2.5 of [32], we obtain that

$$\lim_{\delta \downarrow 0} \sigma_1^\Omega[-\Delta + \lambda V(x) + b(\lambda, x) v_\delta(x)^{p-1} - \mu m(\lambda, x)] = \sigma_1^{B_{0,\lambda}}[-\Delta + \lambda V(x) - \mu m(\lambda, x)] > 0.$$

Let $\delta_0 > 0$ sufficiently small such that $\sigma_1^\Omega[-\Delta + \lambda V(x) + b(\lambda, x) v_{\delta_0}(x)^{p-1} - \mu m(\lambda, x)] \geq 0$. Consider the positive eigenfunction $\overline{\varphi}$ associated with $\sigma_1^\Omega[-\Delta + \lambda V(x) + b(\lambda, x) v_{\delta_0}(x)^{p-1} - \mu m(\lambda, x)]$. It is easy to check that if $k\overline{\varphi} \geq v_{\delta_0}$ in Ω, then

$$\overline{u} := k\overline{\varphi}$$

is a supersolution of (0.0.13). Observe that the conditions in ε and k allow us to take ε arbitrarily small and k arbitrarily large. So we can assume that

$$\underline{u} = \varepsilon \underline{\varphi} \leq k\overline{\varphi} = \overline{u}. \tag{4.1.2}$$

Thus, we deduce that there exists a positive solution of (0.0.13) which is minimal in the interval $[\underline{u}, \overline{u}]$.

Now, let us prove the uniqueness. So assume that there exist two distinct positive solutions u_1 and u_2 of (0.0.13). If necessary, we can assume that ε is even smaller and k is even larger so that $\varepsilon\underline{\varphi} \leq \min[u_1, u_2] \leq \max[u_1, u_2] \leq k\overline{\varphi}$. Again, by applying Theorem 6.1 of [1], we can deduce that there exists a positive solution u_0 of (0.0.13) which is minimal in the interval $[\underline{u}, \overline{u}]$. Since u_0 is minimal in this interval, it follows that $u_0 \leq u_1$ and $u_0 \leq u_2$. Thus, we can suppose, without loss of generality, that $u_0 \leq u_1$. Let us prove that $u_0 < u_1$. Observe that

$$(-\Delta + b\upsilon(u_0, u_1) + \lambda V - \mu(\lambda) m(\lambda, x))(u_1 - u_0) = 0,$$

where

$$\upsilon(u_0,u_1)(x) = (u_1(x) - u_0(x))pv(x)^{p-1}$$

for some $u_0(x) \le v(x) \le u_1(x)$. Since $u_0 \ne u_1$, then

$$0 = \sigma_1^{\Omega}[-\Delta + b\upsilon(u_0,u_1) + \lambda V - \mu(\lambda)m] \ge \sigma_1^{\Omega}[-\Delta + bu_0^{p-1} + \lambda V - \mu(\lambda)m(\lambda,x)] > 0,$$

which is an absurd and we just proved that $u_0 < u_1$.

Finally, since u_0 and u_1 are positive, then

$$\begin{aligned} 0 &= \sigma_1^{\Omega}[-\Delta + b(\lambda,x)u_1^{p-1} + \lambda V(x) - \mu m(\lambda,x)] \\ &> \sigma_1^{\Omega}[-\Delta + b(\lambda,x)u_0^{p-1} + \lambda V(x) - \mu m(\lambda,x)] = 0, \end{aligned}$$

which is an absurd and we just proved the uniqueness. The proof of Theorem 0.0.4 is concluded. □

We point out that the condition of existence of the limit of μ_∞ is adopted to simplify the statements of Theorem 0.0.6. However, an alternative statement in terms of superior and inferior limits of μ_∞ and dismissing this condition also suit. For each given $\mu \in \mathcal{C}_0 \cup \mathcal{C}_\infty$, we can associate a family of positive solutions $\{u_\lambda\}$, where $u_\lambda = u_{\lambda,\mu(\lambda)}$ is the unique positive solution of (0.0.13) with $\mu = \mu(\lambda)$, for $\lambda \in [0,\Lambda)$ (respectively $\lambda \in [0,+\infty)$) if $\mu \in \mathcal{C}_0$ (respectively $\mu \in \mathcal{C}_\infty$). It should be noted that the union $\mathcal{C}_0 \cup \mathcal{C}_\infty$ covers all the existence and uniqueness region $\mathcal{S}$ of positive solution $u_{\lambda,\mu}$.

4.2 Results on uniform boundedness of positive solutions

The following theorem has several implications as we can see in Sections 4.1, 5.1 and 5.2.

Theorem 4.2.1. *Let $\mu \in \mathcal{C}_0 \cup \mathcal{C}_\infty$. Assume that there exists some $L \ge 0$ such that $m \in \mathcal{C}(\overline{\mathrm{Dom}\,\mu} \times \overline{\Omega};\mathbb{R})$*

$$M_1 := \sup_{\lambda \ge L} \mu(\lambda) < +\infty,\; M_2 := \sup_{\lambda \ge L} \|m(\lambda,\cdot)\|_\infty < +\infty.$$

If

$$\inf_{\lambda \ge L} \left| \sigma_1^{B_{0,\infty}}[-\Delta + \lambda V(x); m(\lambda,x)] - \mu(\lambda) \right| > 0, \tag{4.2.1}$$

then

$$\sup_{\lambda \geq 0} \|u_\lambda\|_\infty < +\infty. \tag{4.2.2}$$

Proof. First we will prove that there exists some $\tilde{L} > 0$ such that

$$\sup_{\lambda \geq \tilde{L}} \|u_\lambda\|_2 < +\infty \tag{4.2.3}$$

and then we will use Lemma 1.1.1 to infer (4.2.2). Assume by absurd that there is a sequence $(\lambda_n) \subset \operatorname{Dom}\mu$ such that $\|u_{\lambda_n}\|_2 \to +\infty$. By the definition of u_{λ_n}, we have that

$$\int_\Omega \nabla u_{\lambda_n} \nabla \varphi = \int_\Omega (\mu(\lambda_n) m(\lambda_n, x) - \lambda_n V(x)) u_{\lambda_n} \varphi - \int_\Omega b(\lambda_n, x) u_{\lambda_n}^p \varphi \;\forall\; \varphi \in H_0^1(\Omega). \tag{4.2.4}$$

Dividing (4.2.4) by $\|u_{\lambda_n}\|_2$, testing against $\varphi = v_{\lambda_n} = u_{\lambda_n}/\|u_{\lambda_n}\|_2$ and using that μ is bounded, we deduce that

$$\int_\Omega |\nabla v_{\lambda_n}|^2 \leq \int_\Omega |\nabla v_{\lambda_n}|^2 + \int_\Omega b(\lambda_n, x) v_{\lambda_n}^{p+1} \|u_{\lambda_n}\|_2^{p-1} \tag{4.2.5}$$

$$= \int_\Omega (\mu(\lambda_n) m(\lambda_n, x) - \lambda_n V(x)) v_{\lambda_n}^2 \leq M_1 M_2 |\Omega|, \tag{4.2.6}$$

that is, v_{λ_n} is bounded in the reflexive space $H_0^1(\Omega)$. So there exists $0 \lneq v_\infty \in H_0^1(\Omega)$, $\|v_\infty\|_2 = 1$, such that $v_{\lambda_n} \rightharpoonup v_\infty \in H_0^1(\Omega)$. Moreover,

$$\|u_{\lambda_n}\|_2^{p-1} \int_\Omega b(\lambda_n, x) v_{\lambda_n}^{p+1} + \int_\Omega \lambda_n V(x) v_{\lambda_n}^2 \leq \int_\Omega |\nabla v_{\lambda_n}|^2 + \int_\Omega b(\lambda_n, x) v_{\lambda_n}^{p+1} \|u_{\lambda_n}\|_2^{p-1} + \int_\Omega \lambda_n V(x) v_{\lambda_n}^2 \tag{4.2.7}$$

$$= \int_\Omega \mu(\lambda_n) m(\lambda, x) v_{\lambda_n}^2 \leq M_1 M_2 |\Omega|. \tag{4.2.8}$$

Consequently,

$$\liminf_{n \to +\infty} \|u_{\lambda_n}\|_2^{p-1} \int_\Omega b(\lambda_n, x) v_{\lambda_n}^{p+1} < +\infty \tag{4.2.9}$$

$$\liminf_{n \to +\infty} \int_\Omega \lambda_n V(x) v_{\lambda_n}^2 < +\infty. \tag{4.2.10}$$

Applying Lemma 1.1.1 with $\mathcal{D} = \{\lambda_n\}$, $\varphi_{\lambda_n} = v_{\lambda_n}$, $h_{\lambda_n} = \|u_{\lambda_n}\|_2^{p-1} b(\lambda_n, \cdot\,) u_{\lambda_n}^{p-1} + \lambda_n V + \mu(\lambda_n) m_-(\lambda_n, \cdot\,)$ and $m_+(\lambda_n, \cdot\,) = \mu(\lambda_n) m_+(\lambda_n, \cdot\,)$, we would obtain that v_{λ_n} is the eigenfunction associated with $\sigma_1^{\Omega}[-\Delta+h_{\lambda_n};m_+(\lambda_n,x))=1$ and so

$$\|v_{\lambda_n}\|_\infty\leq C(\|v_{\lambda_n}\|_2+1)=2C.$$

Hence

$$\liminf_{n\to+\infty}\int_\Omega b(\lambda_n,x)v_{\lambda_n}^{p+1}\geq\int_\Omega b(\infty,x)v_\infty^{p+1},$$

but by combining the assumption $\|u_{\lambda_n}\|_2\to+\infty$ with (4.2.9), we deduce that

$$\int_\Omega b(\infty,x)v_\infty^{p+1}=0,$$

whence we imply

$$v_\infty=0 \text{ in } \Omega\setminus B_{0,\infty} \tag{4.2.11}$$

and so $v_\infty\in H_0^1(B_{0,\infty})$. Now we will analyze two cases separately, that is,

$$\limsup_{n\to+\infty}\lambda_n<\infty \text{ or } \limsup_{n\to+\infty}\lambda_n=\infty. \tag{4.2.12}$$

1. Assume that $\limsup_{n\to+\infty}\lambda_n<\infty$. Let $\psi\in H_0^1(B_{0,\infty})\subset H_0^1(\Omega)$. If we take $\varphi=\psi/\|u_{\lambda_n}\|_2$ in (4.2.4) and pass to the limit, we deduce that v_∞ satisfies

$$\begin{cases}-\Delta v_\infty+\lambda_* V(x)v_\infty=\mu(\lambda_*)m(\lambda_*,x)v_\infty & \text{in } B_{0,\infty}\\ v_\infty=0 & \text{on } \partial B_{0,\infty},\\ v_\infty>0 & \text{in } B_{0,\infty},\end{cases}$$

   where $\lambda_*=\lim_{k\to+\infty}\lambda_{n_k}<\infty$ for some subsequence λ_{n_k} of λ_n. Since $v_\infty\neq 0$, by the strong maximum principle we deduce that v_∞ is positive in $B_{0,\infty}$, and then $\mu(\lambda_*)=\sigma_1^{B_{0,\infty}}[-\Delta+\lambda_* V(x);m(\lambda_*,x)]$. Hence,

$$\lim_{k\to+\infty}\mu(\lambda_{n_k})=\lim_{k\to+\infty}\sigma_1^{B_{0,\infty}}[-\Delta+\lambda_{n_k}V(x);m(\lambda_{n_k},x)],$$

   but this contradicts (4.2.1).

2. Assume now that $\limsup_{n\to+\infty}\lambda_n=+\infty$. Observe that

$$\liminf_{n\to+\infty}\int_\Omega V(x)v_n^2\geq\int_\Omega V(x)v_\infty^2,$$

but since (4.2.10) and $\limsup_{n\to+\infty}\lambda_n = +\infty$, then it follows that

$$\int_\Omega V(x)v_\infty^2 = 0.$$

So $v_\infty = 0$ in $\Omega\setminus V_0$. Combining this with (4.2.11) and noting that $v_\infty \neq 0$ (because $\|v_\infty\|_2 = 1$), we deduce that $V_0\cap B_{0,\infty}\neq\emptyset$ and $v_\infty\in H_0^1(V_0\cap B_{0,\infty})$. Consider

$$\psi\in H_0^1(V_0\cap B_{0,\infty})\subset H_0^1(\Omega).$$

If we make $\varphi = \psi/\|u_{\lambda_n}\|_2$ in (4.2.4) and pass to the limit, we deduce that v_∞ satisfies

$$\begin{cases} -\Delta v_\infty = \mu_* m(\infty,x)v_\infty & \text{in } V_0\cap B_{0,\infty},\\ v_\infty = 0 & \text{on } \partial(V_0\cap B_{0,\infty}),\\ v_\infty > 0 & \text{in } V_0\cap B_{0,\infty}, \end{cases}$$

where $0\le\mu_* = \lim_{k\to+\infty}\mu(\lambda_{n_k}) < \infty$, for some subsequence λ_{n_k} of λ_n. Again by the strong maximum principle, we deduce that $v_\infty > 0$, by whence $\sigma_1^{V_0\cap B_{0,\infty}}[-\Delta;m(\infty,x)]$ would be finite and

$$\lim_{k\to+\infty}\mu(\lambda_{n_k}) = \sigma_1^{V_0\cap B_{0,\infty}}[-\Delta;m(\infty,x)] = \lim_{k\to+\infty}\sigma_1^{B_{0,\infty}}[-\Delta+\lambda_{n_k}V(x);m(\lambda_{n_k},x)],$$

but this contradicts (4.2.1).

Hence in both cases stated in (4.2.12), we have obtained a contradiction, by whence we just proved (4.2.3). Applying Lemma 1.1.1 with $\mathcal{D} = \{\lambda_n\}$, $\varphi_{\lambda_n} = u_{\lambda_n}$,

$$h_{\lambda_n} = b(\lambda_n,\,\cdot\,)u_{\lambda_n}^{p-1} + \lambda_n V + \mu(\lambda_n)m_-(\lambda_n,x),$$

$m_+(\lambda_n,x) = \mu(\lambda_n)m_+(\lambda_n,x)$, we would obtain $\sigma_1^\Omega[-\Delta+h_{\lambda_n};m_+(\lambda_n,x)] = 1$ and so

$$\|u_{\lambda_n}\|_\infty \le C(\|u_{\lambda_n}\|_2+1) < +\infty$$

and the proof of (4.2.3) is concluded.

Using similar arguments, one can prove that

$$\sup_{\lambda\in[0,\tilde{L}]} \|u_\lambda\|_2 < \infty$$

and the proof Theorem 4.2.1 is concluded. □

The benefit of Theorem 4.2.1 is that it gives a priori boundedness of positive solutions of a problem that admits terms that can possibly be not monotone with respect to μ or λ and even terms that changes sign with respect to x, as $m(\lambda,x)$.

Theorem 4.2.2. *Let $\mu \in \mathcal{C}_0 \cap \mathcal{C}([0,\Lambda])$. Assume that*

$$\lim_{\lambda\to\Lambda} \mu(\lambda) = \sigma_1^\Omega[-\Delta+\Lambda V; m(\Lambda,\,\cdot\,)]. \tag{4.2.13}$$

Then

$$\lim_{\lambda\uparrow\Lambda} \|u_\lambda\|_{C_0^1(\overline{\Omega})} = 0.$$

Proof. Combining (4.2.13) and the monotonicity of the principal eigenvalue with respect to the domain, we get

$$0 = \sigma_1^\Omega[-\Delta+\Lambda V-\mu(\Lambda)m(\Lambda,\,\cdot\,)] < \sigma_1^{B_{0,\infty}}[-\Delta+\Lambda V-\mu_0(\Lambda)m(\Lambda,\,\cdot\,)].$$

Consequently,

$$\inf_{\lambda\in[\Lambda-\varepsilon,\Lambda]} \left|\sigma_1^{B_{0,\infty}}[-\Delta+\lambda V; m(\lambda,\,\cdot\,)] - \mu(\lambda)\right| > 0,$$

for some $\varepsilon > 0$ due to the continuity of the first eigenvalue with respect to the potential. Then we can apply Theorem 4.2.1 to deduce that

$$\sup_{\lambda\in[\Lambda-\varepsilon,\Lambda]} \|u_\lambda\|_\infty < +\infty. \tag{4.2.14}$$

By elliptic regularity, it follows that u_λ is bounded in $W_0^{2,s}(\Omega)$ for arbitrarily large $s > 1$. So let $\lambda_n \uparrow \Lambda$. Consider $s > 1$ sufficiently large such that $W_0^{2,s}(\Omega)$ is compactly embedded in $C_0^1(\overline{\Omega})$. So $\|u_{\lambda_n} - u_\Lambda\|_{C_0^1(\overline{\Omega})} \to 0$, up to a subsequence, where u_Λ satisfies

$$\begin{cases} -\Delta u_\Lambda + \Lambda V(x) u_\Lambda = \sigma_1^\Omega[-\Delta+\Lambda V; m] m(\Lambda,x) u_\Lambda - b(\Lambda,x) u_\Lambda^p & \text{in } \Omega, \\ u_\Lambda = 0 & \text{on } \partial\Omega, \\ u_\Lambda > 0 & \text{in } \Omega, \end{cases}$$

So it must be $u_\Lambda = 0$, on the contrary, we would contradict the condition of existence of positive solution stated in Theorem 0.0.4. We just proved that u_{λ_n} converges to 0 in $C_0^1(\overline{\Omega})$. By the arbitrariness of the sequence λ_n converging to Λ, we imply that

$$\lim_{\lambda\to\Lambda} \|u_\lambda\|_{C_0^1(\overline{\Omega})} = 0.$$

This ends the proof. □

Theorem 4.2.3. *Let $\mu \in \mathcal{C}_\infty$. Assume that*

$$\sigma_1^{V_0}[-\Delta; m(\infty,\,\cdot\,)] < \sigma_1^{V_0\cap B_{0,\infty}}[-\Delta; m(\infty,\,\cdot\,)]$$

and $\sigma_1^{V_0}[-\Delta; m(\infty,\,\cdot\,)] \leq \mu(\infty) := \lim_{\lambda\to\infty} \mu(\lambda) < \sigma_1^{V_0\cap B_{0,\infty}}[-\Delta; m(\infty,\,\cdot\,)]$.

Then

$$\lim_{\lambda\to\infty} \|u_\lambda\|_{C^{1,\gamma}(\overline{\Omega})} < +\infty, \text{ for some } 0<\gamma<1$$

and $u_\lambda \to u_\infty$ in $C^1(\overline{\Omega})$, where $u_\infty \equiv 0$ in the case $\mu(\infty) = \sigma_1^{\Omega}[-\Delta; m(\infty,\,\cdot\,)]$ and u_∞ is the null extension to Ω of the unique positive solution of the problem

$$\begin{cases} -\Delta u = \mu(\infty) m(\infty,x)u - b(\infty,x)u^p & \text{in } V_0, \\ u = 0 & \text{on } \partial V_0, \\ u > 0 & \text{in } V_0, \end{cases} \tag{4.2.15}$$

for $\mu(\infty) > \sigma_1^{\Omega}[-\Delta; m(\infty,x)]$.

Proof. Since $\mu(\infty) < \sigma_1^{V_0\cap B_{0,\infty}}[-\Delta; m(\infty,\,\cdot\,)]$, then

$$\inf_{\lambda\geq L} \left| \sigma_1^{B_{0,\infty}}[-\Delta + \lambda V(x); m(\lambda,x)] - \mu_\infty(\lambda) \right| > 0,$$

for some large $L > 0$. Therefore

$$\sup_{\lambda\geq L} \|u_\lambda\|_\infty < \infty$$

due to Theorem 4.2.1. Then the boundedness in $C^{1,\gamma}(\overline{\Omega})$ follows by classic elliptic regularity.

Now we will divide the proof in the cases $\mu(\infty) = \sigma_1^{\Omega}[-\Delta; m(\infty,x)]$ and $\mu(\infty) > \sigma_1^{\Omega}[-\Delta; m(\infty,x)]$.

1. Assume that $\mu(\infty) = \sigma_1^{V_0}[-\Delta; m(\infty,x)]$. By arguing as we done in Theorem 4.2.1, we can prove that
$$\lim_{\lambda\to\infty} u_\lambda(x) = 0 \quad \text{for all } x \in \Omega \setminus \bar{V}_0.$$
Testing against $\varphi \in H_0^1(V_0)$ in the definition of u_λ and passing to the limit when $\lambda \to +\infty$, we can deduce that there exists a non negative $u_\infty \in H_0^1(V_0)$ such that $\lim_{\lambda\to+\infty} \|u_\lambda - \tilde{u}_\infty\|_{C_0^1(\overline{\Omega})} = 0$, where $\tilde{u}_\infty$ is the null extension (to $\bar{\Omega}$) of u_∞ and u_∞ satisfies
$$\begin{cases} -\Delta u_\infty = \sigma_1^{V_0}[-\Delta; m(\infty,x)]m(\infty,x)u_\infty - b(\infty,x)u_\infty^p & \text{in } V_0, \\ u_\infty = 0 & \text{on } \partial V_0, \\ u_\infty > 0 & \text{in } V_0. \end{cases}$$
Suppose by absurd that $u_\infty \neq 0$. By summing a sufficiently large $C > 0$ in both sides of the above equation, we would be able to able to apply the strong maximum principle in order to conclude that $u_\infty > 0$. Since $V_0 \not\subset B_{0,\infty}$, then $0 \lneq u_\infty^{p-1} b(\infty,x)$ in V_0 and consequently
$$\begin{aligned} 0 &= \sigma_1^{V_0}\left[-\Delta + u_\infty^{p-1} b(\infty,x) - \left(\lim_{\lambda\to+\infty} \mu_\infty(\lambda)\right) m(\infty,x)\right] \\ &> \sigma_1^{V_0}\left[-\Delta - \lim_{\lambda\to+\infty} \mu_\infty(\lambda) m(\infty,x)\right] = 0 \end{aligned}$$
which is an absurd and so $u_\infty = 0$.

2. Assume that $\mu(\infty) > \sigma_1^{V_0}[-\Delta; m(\infty,x)]$. By arguing similar as we done in the first case, we deduce that there exists a non negative $u_\infty \in H_0^1(V_0)$ such that $\lim_{\lambda\to+\infty} \|u_\lambda - \tilde{u}_\infty\|_{C_0^1(\overline{\Omega})} = 0$, where $\tilde{u}_\infty$ is the null extension (to $\bar{\Omega}$) of u_∞ and u_∞ satisfies
$$\begin{cases} -\Delta u_\infty = \mu(\infty)m(\infty,x)u_\infty - b(\infty,x)u_\infty^p & \text{in } V_0, \\ u_\infty = 0 & \text{on } \partial V_0, \\ u_\infty > 0 & \text{in } V_0. \end{cases}$$
We claim that u_∞ is non null. In fact, assume by absurd that $u_\infty = 0$. Let $v_\lambda = u_\lambda / \|u_\lambda\|_\infty$, take any $\varphi \in H_0^1(\Omega)$ and test against $\varphi / \|u_\lambda\|_2$ in the definition of u_λ. By passing to the limit as $\lambda \to +\infty$, we deduce that there exists $0 \lneq v_\lambda \in H_0^1(\Omega)$ satisfying

$$\begin{cases} -\Delta v_\infty = \mu(\infty)m(\infty,x)v_\infty & \text{in } V_0, \\ \quad u_\infty = 0 & \text{on } \partial V_0, \\ \quad u_\infty > 0 & \text{in } V_0. \end{cases}$$

By summing a sufficiently large $C > 0$ in both sides, we can apply the strong maximum principle to conclude that $v_\infty > 0$ is positive and consequently

$$\mu(\infty) = \sigma_1^{V_0}[-\Delta; m(\infty,x)].$$

But $\mu(\infty) > \sigma_1^{V_0}[-\Delta; m(\infty,x)]$ by hypothesis.

This ends the proof. □

4.3 Subsolution driving blow-up

In the previous sections, we established conditions for uniform boundedness of positive solutions and presented their limits. In this section, we will study the complementary case. Precisely, the next theorem provides a subsolution that will lead to the blow-up phenomenum for two special cases of the problem (0.0.13) in the next chapter.

Theorem 4.3.1. *Let* $(\lambda, \sigma(\lambda)) \in \mathcal{S}$, $\overline{b}(\lambda,\,\cdot\,) \in \mathcal{C}(\overline{\Omega})$ *be such that* $\overline{b}(\lambda,x) \geq b(\lambda,x)$ *for all* (λ,x) *and let* $\sigma(\lambda) \in \mathbb{R}$ *such that*

$$(\mu - \sigma(\lambda))m(\lambda,x) \geq 0 \;\forall\, x \in \Omega.$$

Then there exists a unique $\beta(\lambda) > 0$ *such that*

$$\sigma(\lambda) = \sigma_1^{\Omega}[-\Delta + \lambda V(x) + \beta(\lambda)\overline{b}(\lambda,x); m(\lambda,x)].$$

Moreover,

$$\beta(\lambda)^{1/(p-1)} \frac{\varphi_\lambda}{\|\varphi_\lambda\|_\infty} \leq u_\lambda,$$

where φ_λ *is the positive eigenfunction associated to* $\sigma(\lambda)$.

Proof. Let $(\lambda, \sigma(\lambda)) \in \mathcal{S}$ and define

$$g(\beta) := \sigma_1^{\Omega}[-\Delta + \lambda V(x) + \beta\overline{b}(\lambda,x) - \sigma(\lambda)m(\lambda,x)]. \tag{4.3.1}$$

By using the definition of $\mathcal{S}$ and the fact that $(\lambda,\sigma(\lambda)) \in \mathcal{S}$, we deduce that $g(0) < 0$. Again by using the definition of $\mathcal{S}$ and Theorem 0.0.4, we imply that there exists a unique positive solution $u_{\lambda,\mu}$ of (0.0.13). Let $\beta > \sup_{\overline{\Omega}} u_{\lambda,\mu}^{p-1}$ and observe that

$$\begin{aligned}
&[-\Delta+\lambda V(x)+\beta\overline{b}(\lambda,x)-\sigma(\lambda)m(x)]u_{\lambda,\mu} \geq \\
&[-\Delta+\lambda V(x)+\beta b(\lambda,x)+(\mu-\sigma(\lambda))m(x)-\mu m(x)]u_{\lambda,\mu} \\
&\geq [-\Delta+\lambda V(x)+\beta b(\lambda,x)-\mu m(x)]u_{\lambda,\mu} \\
&= \beta b(\lambda,x)u_{\lambda,\mu}-b(\lambda,x)u_{\lambda,\mu}^{p} = (\beta-u_{\lambda,\mu}^{p-1})u_{\lambda,\mu}b(\lambda,x).
\end{aligned}$$

So u_λ is a strict positive super-solution of the operator

$$L_\beta := -\Delta+\lambda V(x)+\beta\overline{b}(\lambda,x)-\sigma(\lambda)m(\lambda,x)$$

and consequently $g(\beta) > 0$ by the characterization of the strong maximum principle (see Theorem 7.5.2 of [44]). Since g is continuous and increasing, the existence and uniqueness of $\beta(\lambda) > 0$ is proved.

Let us prove the second part of the theorem. Observe that φ_λ is a subsolution of (0.0.13) if and only if

$$\beta(\lambda)\left[\left(\frac{\varphi_\lambda}{\|\varphi_\lambda\|_\infty}\right)^{p-1} b(\lambda,x)-\overline{b}(\lambda,x)\right] \leq (\mu-\sigma(\lambda))m(\lambda,x). \tag{4.3.2}$$

By combining the hypotheses (4.3.2) and $\overline{b}(\lambda,\,\cdot\,) \geq b(\lambda,\,\cdot\,)$, we deduce that the above inequality holds and the theorem is proved. □

Conclusion of Chapter 4

Theorem 4.2.1 is an a priori boundedness result that extends to positive solutions of problem (0.0.13), the well-known behavior.

$$\sup_{\mu\in\mathcal{M}} \|u_\mu\|_\infty < \infty \Leftrightarrow \inf_{\mu\in\mathcal{M}} (d\sigma_1^{B_0}(-\Delta)-\mu) > 0$$

of the positive solutions u_μ of (0.0.9), where $\mathcal{M}$ is any subset of $(d\sigma_1(-\Delta), d\sigma_1^{B_0}(-\Delta))$. We point out that the case $\lambda \to +\infty$ is included and is the most interesting aspect of this extension.

The a priori boundedness provided by Theorem 4.2.1 naturally leads to the question of the limit of these solutions, which is addressed by Theorems 4.2.2 and 4.2.3.

Theorem 4.3.1 does not directly provide information about positive solutions of (0.0.13). This technical theorem's benefit will become apparent in the Corollaries 5.1.1 and 5.2.1, that present a blow-up phenomena to ($R_{\lambda,\mu}$) and ($S_{\lambda,\mu}$), respectively.

Chapter 5

Fast diffusion and strong degradation for logistic problem with refuge zone

In this chapter, we will apply the results of Sections 1.2, 1.3 and Chapter 4 to determine fine qualitative information about the positive solutions of ($R_{\lambda,\mu}$) and ($S_{\lambda,\mu}$).

5.1 Fast diffusion

In this section, we will study the positive solutions of the problem

$$\begin{cases} -(1+\lambda a(x))\Delta u = \mu u - b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$R_{\lambda,\mu}$}$$

where $0 \leq a \in C(\overline{\Omega})$ and $0 \lneq b \in C(\overline{\Omega})$.

Clearly, ($R_{\lambda,\mu}$) is a special case of (0.0.13) by making

$$V \equiv 0,\ b(\lambda,x) = b(x)/(1+\lambda a(x)) \text{ and } m(\lambda,x) = (1+\lambda a(x))^{-1}.$$

Consequently, we can apply Theorem 0.0.4 to imply that there exists a unique positive solution $u_{\lambda,\mu}$ of ($R_{\lambda,\mu}$) if and only if

$$\sigma_1^{\Omega}\left(-\Delta; \frac{1}{1+\lambda a(x)}\right) < \mu < \sigma_1^{B_0}\left(-\Delta; \frac{1}{1+\lambda a(x)}\right). \tag{5.1.1}$$

As said in the introduction of this work, we are interested in the behavior of the positive solutions at the extremes of the interval of existence. In order to properly determine which are these extremes, will state the next two propositions. To do so, we will analyze two essential maps. Consider $\lambda > -1/\|a\|_0$. Observe that

$$1+\lambda a(x) > 0 \text{ for all } x \in \overline{\Omega},$$

and then the following maps are well defined

$$\begin{array}{rcl} h,H : (-1/\|a\|_0, +\infty) & \to & \mathbb{R} \\ \lambda & \mapsto & h(\lambda) := \sigma_1^{\Omega}\left(-\Delta; \dfrac{1}{1+\lambda a(x)}\right), \\ \lambda & \mapsto & H(\lambda) := \sigma_1^{B_0}\left(-\Delta; \dfrac{1}{1+\lambda a(x)}\right). \end{array} \tag{5.1.2}$$

With these notations, we can equivalently write the existence condition in (5.1.1) as

$$h(\lambda) < \mu < H(\lambda). \tag{5.1.3}$$

In the following result we prove the main properties of both maps.

Proposition 5.1.1. *One has:*

1. *The maps $\lambda \mapsto h(\lambda), H(\lambda)$ are continuous, increasing and*

$$0 < h(\lambda) < H(\lambda) \quad \textit{for all } \lambda > -1/\|a\|_0.$$

2. *It holds*

$$0 \le \underline{\mu} := \lim_{\lambda \downarrow -1/\|a\|_0} h(\lambda) \le \lim_{\lambda \downarrow -1/\|a\|_0} H(\lambda) := \overline{\mu} < \infty.$$

3. *It holds*

$$\lim_{\lambda \to +\infty} h(\lambda) = \begin{cases} \sigma_1^{\Omega}[-\Delta; \chi_{A_0}] & \textit{if } A_0 \neq \emptyset, \\ +\infty & \textit{if } A_0 = \emptyset. \end{cases}$$

and

$$\lim_{\lambda \to +\infty} H(\lambda) = \begin{cases} \sigma_1^{B_0}[-\Delta; \chi_{A_0}] & \textit{if } B_0 \cap A_0 \neq \emptyset, \\ +\infty & \textit{if } B_0 \cap A_0 = \emptyset. \end{cases}$$

Proof. The maps $\lambda \mapsto h(\lambda), H(\lambda)$ are continuous and increasing by Proposition 1.0.3. Moreover, $h(\lambda) < H(\lambda)$ by Proposition 1.0.3. The existence of $\overline{\mu}$ and $\underline{\mu}$ is also a consequence of the monotony and continuity of the maps and because $h(\lambda), H(\lambda) > 0$.

Now suppose that $A_0 \neq \emptyset$. Note that

$$h(\lambda) \leq \sigma_1^{A_0}\left(-\Delta; \frac{1}{1+\lambda a}\right) = \sigma_1^{A_0}(-\Delta). \tag{5.1.4}$$

Consequently,

$$\lim_{\lambda \to +\infty} h(\lambda) = \sup_{\lambda \geq 0} h(\lambda) = h_\infty < \infty. \tag{5.1.5}$$

Let $\lambda_n \to +\infty$, φ_{λ_n} be the positive eigenfunction associated to $h(\lambda_n)$ such that $\|\varphi_{\lambda_n}\|_2 = 1$. Then $\|\varphi_{\lambda_n}\|_{H_0^1(\Omega)}$ is a bounded sequence. Indeed, note that

$$\int_\Omega |\nabla \varphi_{\lambda_n}|^2 = h(\lambda_n) \int_\Omega \frac{\varphi_{\lambda_n}^2}{1+\lambda_n a(x)} \leq h(\lambda_n) \int_\Omega \varphi_{\lambda_n}^2 \overset{(5.1.4)}{\leq} \sigma_1^{A_0}(-\Delta).$$

Hence, $\|\varphi_{\lambda_n}\|_{H_0^1(\Omega)}$ is bounded and so there exists $\varphi_\infty \in H_0^1(\Omega)$ such that $\varphi_{\lambda_n} \rightharpoonup \varphi_\infty$ in $H_0^1(\Omega)$ and $\varphi_{\lambda_n} \to \varphi_\infty$ in $L^2(\Omega)$ with $\varphi_\infty \geq 0$ and $\varphi_\infty \neq 0$ in Ω.

Moreover,

$$\frac{1}{1+\lambda_n a(x)} \to \chi_{A_0} \text{ in } L^2(\Omega) \tag{5.1.6}$$

due to Lebesgue's Dominated Convergence Theorem. Consequently, if $\varphi \in C_c^\infty(\Omega)$, then by passing to the limit (up to a subsequence) in the equality

$$\int_\Omega \nabla \varphi_{\lambda_n} \cdot \nabla \varphi = h(\lambda_n) \int_\Omega \frac{\varphi_{\lambda_n} \varphi}{1+\lambda_n a(x)}, \tag{5.1.7}$$

we deduce that

$$\int_\Omega \nabla \varphi_\infty \cdot \nabla \varphi = h_\infty \int_\Omega \chi_{A_0} \varphi_\infty \varphi.$$

Consequently, since $\varphi_\infty \gneq 0$ in Ω, we deduce that $h_\infty = \sigma_1^\Omega[-\Delta; \chi_{A_0}]$ and φ_∞ is the positive eigenfunction associated to $\sigma_1^\Omega[-\Delta; \chi_{A_0}]$.

Now, assume that $A_0 = \emptyset$ and (5.1.5). We can argue exactly as in the previous case, using (5.1.6), and conclude that

$$\int_\Omega \nabla \varphi_\infty \cdot \nabla \varphi = 0, \quad \forall \varphi \in H_0^1(\Omega),$$

and then, $\varphi_\infty = 0$ in Ω, a contradiction due to $\|\varphi_\infty\|_2 = 1$. Thus, $\lim_{\lambda\to+\infty} h(\lambda) = \infty$.

Analogously, we can study the function $H(\lambda)$ and the $\lim_{\lambda\to+\infty} H(\lambda)$. This concludes the proof. □

Proposition 5.1.2. *Let $\overline{\mu} < \mu < \sigma_1^{B_0}(-\Delta;\chi_{A_0})$. Then there exists a unique $-1/\|a\|_0 < \lambda_*(\mu) < +\infty$ such that*

$$\mu \begin{cases} = H(\lambda) & \textit{for } \lambda = \lambda_*(\mu), \\ < H(\lambda) & \textit{for } \lambda > \lambda_*(\mu), \\ > H(\lambda) & \textit{for } \lambda < \lambda_*(\mu). \end{cases}$$

The map $\mu \mapsto \lambda_*(\mu)$ is continuous, increasing and

$$\lim_{\mu\downarrow\overline{\mu}} \lambda_*(\mu) = -\frac{1}{\|a\|_0}, \qquad \lim_{\mu\uparrow\sigma_1^{B_0}[-\Delta;\chi_{A_0}]} \lambda_*(\mu) = +\infty.$$

Analogously, given $\underline{\mu} < \mu < \sigma_1^{\Omega}[-\Delta;\chi_{A_0}]$, there exists a unique $-1/\|a\|_0 < \lambda^*(\mu) < +\infty$ such that

$$\mu \begin{cases} = h(\lambda) & \textit{for } \lambda = \lambda^*(\mu), \\ < h(\lambda) & \textit{for } \lambda > \lambda^*(\mu), \\ > h(\lambda) & \textit{for } \lambda < \lambda^*(\mu). \end{cases}$$

Moreover, the map $\mu \mapsto \lambda^*(\mu)$ is continuous, increasing and

$$\lim_{\mu\downarrow\underline{\mu}} \lambda^*(\mu) = -\frac{1}{\|a\|_0}, \qquad \lim_{\mu\uparrow\sigma_1^{\Omega}[-\Delta;\chi_{A_0}]} \lambda^*(\mu) = +\infty.$$

Furthermore,

$$\lambda_*(\mu) < \lambda^*(\mu). \tag{5.1.8}$$

Proof. Take $\overline{\mu} < \mu < \sigma_1^{B_0}[-\Delta;\chi_{A_0}]$. Thanks to the properties of $H(\lambda)$, see Proposition 5.1.1, the existence, uniqueness of $\lambda_*(\mu)$, as well as its increasing character, follow.

On the other hand, assume that

$$\lim_{\mu\uparrow\sigma_1^{B_0}[-\Delta;\chi_{A_0}]} \lambda_*(\mu) = r < \infty.$$

By definition

$$\mu = H(\lambda_*(\mu)), \tag{5.1.9}$$

then we get by Proposition 5.1.1 that

$$\sigma_1^{B_0}(-\Delta;\chi_{A_0}) = H(r) = \sigma_1^{B_0}\left(-\Delta;\frac{1}{1+ra}\right) < \sigma_1^{B_0}(-\Delta;\chi_{A_0}) < \infty,$$

a contradiction if $B_0 \cap A_0 \neq \emptyset$. If $B_0 \cap A_0 = \emptyset$, then $\sigma_1^{B_0}(-\Delta;\chi_{A_0}) = \infty$, and taking limit in (5.1.9) as $\mu \to \infty$, we have by Proposition 5.1.1, that

$$\infty = \lim_{\mu\to\infty} H(\lambda_*(\mu)) = \sigma_1^{B_0}\left(-\Delta;\frac{1}{1+ra}\right) < \infty,$$

a contradiction again.

On the other hand, suppose that

$$\lim_{\mu\downarrow\overline{\mu}} \lambda_*(\mu) = r > -\frac{1}{\|a\|_0},$$

so taking limit in (5.1.9) we get

$$\overline{\mu} = H(r) > \lim_{\lambda\downarrow -1/\|a\|_0} H(\lambda) = \overline{\mu},$$

a contradiction.

Finally, since $h(\lambda) < H(\lambda)$, we deduce (5.1.8). This ends the proof. □

Before enunciating the next result, let us remember the following notation given in the Introduction. Define

$$A_+ \cup B_0 = \bigcup_{i=1}^{d} D_i \bigcup_{i=1}^{m} \bigcup_{j=1}^{k_i} C_i^j,$$

where $m,d,k_i \in \mathbb{N}$, $m,d,k_i \geq 1$, C_i^j, D_i are the connected components of $A_+ \cup B_0$ such that

$H_1)$ $\quad D_i \subseteq A_+, \quad C_i^j \not\subset A_+,$

$H_2)$ $\quad C_i^j$ is isolated from any other component of $A_+ \cup B_0$

and

$$\sigma_1^i[-\Delta;\chi_{A_0}] = \sigma_1^{C_i^j}[-\Delta;\chi_{A_0}],\ j = 1,\ldots,k_i.$$

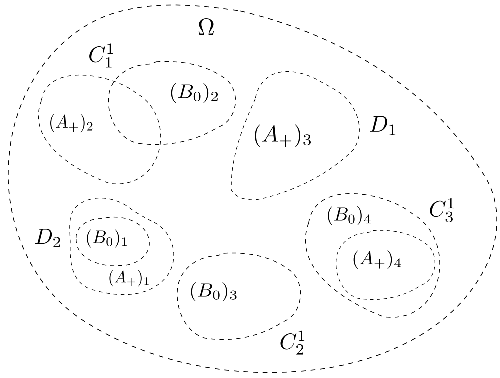

Fig. 5.1 A possible distribution of the set $A_+ \cup B_0$ with $m = 3$, $d = 2$ and $k_1 = k_2 = k_3 = 1$.

That is, by H_1), we have separated the connected components only of A_+, denoted D_i, and C_i^j that are not contained in A_+. Then,

$$\sigma_1^{D_i}[-\Delta;\chi_{A_0}] = +\infty,$$

because $D_i \subseteq A_+$. Hence, since

$$\sigma_1^{A_+\cup B_0}[-\Delta;\chi_{A_0}] = \min\left\{\min_{1\le i\le d}\left\{\sigma_1^{D_i}[-\Delta;\chi_{A_0}]\right\}, \min_{1\le i\le m}\left\{\sigma_1^i[-\Delta;\chi_{A_0}]\right\}\right\} = \min_{1\le i\le m}\sigma_1^i[-\Delta;\chi_{A_0}],$$

we can order the sets C_i^j according to

$$\sigma_1^{A_+\cup B_0}[-\Delta;\chi_{A_0}] = \sigma_1^1[-\Delta;\chi_{A_0}] < ... < \sigma_1^m[-\Delta;\chi_{A_0}].$$

In Figure 5.1 we have represented a possible distribution of $A_+ \cup B_0$. In such case, there exist two connected components included in A_+, D_1, D_2 and three different connected components C_i^j.

Observe that as a consequence of H_2), we can take $\delta > 0$ such that the set

$$C_i^j(\delta) := \left\{x \in \Omega; \mathrm{dist}\left(x, C_i^j\right) \le \delta\right\}$$

be isolated from any other component of $A_+ \cup B_0$. Hence, we can define b_{ij} be a smooth extension of $b|_{C_i^j}$ satisfying

$$b_{ij} \ge b \quad \text{in } \Omega,\ b_{ij} = 0 \text{ in } C_i^j \cap B_0,$$

and

$$b_{ij}(x) \geq b_0 > 0 \quad \forall x \in \Omega \setminus \overline{C_i^j(\delta)}, \quad \text{for some } b_0 > 0.$$

Besides this, let us denote by a_{ij} the continuous function defined by $a_{ij}(x) = 0$ for $x \in \Omega \setminus (C_i^j \cap A_+)$, and $a_{ij}(x) = a(x)$ for $x \in C_i^j \cap A_+$ so that $a_{ij} \leq a$ in Ω and $a_{ij} \equiv 0$ in Ω if $C_i^j \cap A_+ = \emptyset$.

In order to prove a blow up result for u_λ, we will need the following corollary.

Corollary 5.1.1 (of Theorem 4.3.1). *If* $\sigma_1^{C_i^j}[-\Delta; \chi_{A_0}] < \mu < \sigma_1^{B_0}[-\Delta; \chi_{A_0}]$, *then*

$$\lim_{\lambda \to \infty} u_\lambda(x) = \infty \; \forall \, x \in C_i^j.$$

Proof. Let $\varphi_{\lambda,i}$ be the positive eigenfunction associated to the eigenvalue

$$\sigma_1^\Omega \left[-\Delta + \frac{\lambda^r b_i(x)}{1 + \lambda a_i(x)}; \; \frac{1}{1 + \lambda a(x)} \right]$$

normalized in $L^2(\Omega)$.

By hypothesis, there exists some $M > 0$ such that

$$a(x)^r \geq M \, \mathrm{dist}(x, \partial C_i^j)$$

for all $x \in A_+ \setminus B_0$ in a neighborhood of ∂C_i^j. Consequently,

$$\sup_{\lambda \geq 0} \frac{\lambda^r b_i(x)}{1 + \lambda a_i(x)} = r^r (1-r)^{1-r} \frac{b_i(x)}{a_i(x)^r} \leq C r^r (1-r)^{1-r} \frac{b_i(x)}{d(x)}. \tag{5.1.10}$$

Moreover, since $0 < r < 1$, we deduce by either $x \in C_i^j \cap B_0$ or $x \in C_i^j \cap A_+$, that

$$\lim_{\lambda_n \to +\infty} \frac{\lambda_n^r b_{ij}(x) \varphi}{1 + \lambda_n a_{ij}(x)} = \lim_{\lambda_n \to +\infty} \frac{b_{ij}(x) \varphi}{\lambda_n^{-r} + \lambda_n^{1-r} a_{ij}(x)} = 0 \quad \forall \, x \in C_i^j. \tag{5.1.11}$$

Due to (5.1.10) and (5.1.11), we can apply Theorem 1.2.1 with

$$q_\lambda(x) = \frac{\lambda^r b_i(x)}{1 + \lambda a_i(x)}$$

and $V \equiv 0$ to deduce that

$$\lim_{\lambda \to \infty} \sigma_1^\Omega \left[-\Delta + \frac{\lambda^r b_i(x)}{1 + \lambda a_i(x)}; \; \frac{1}{1 + \lambda a(x)} \right] = \sigma_1^{C_i^j} [-\Delta; \; \chi_{A_0}] \tag{5.1.12}$$

and

$$\lim_{\lambda\to\infty}\|\varphi_\lambda-\varphi_\infty\|_{H_0^1(\Omega)}=0, \tag{5.1.13}$$

where φ_∞ is the null extension to Ω of the positive eigenfunction associated to $\sigma_1^{C_i^j}(-\Delta;\ \chi_{A_0})$.

Combining the hypothesis $\mu > \sigma_1^{C_i^j}[-\Delta;\chi_{A_0}]$ with (5.1.12), we deduce that there exists some large $L>0$ such that

$$\mu \geq \sigma_1^\Omega\left[-\Delta+\frac{\lambda^r b_{ij}(x)}{1+\lambda a_{ij}(x)};\ \frac{1}{1+\lambda a(x)}\right]\ \forall\,\lambda\geq L.$$

So

$$\frac{\mu-\sigma_1^\Omega\left[-\Delta+\frac{\lambda^r b_{ij}(x)}{1+\lambda a_{ij}(x)};\ \frac{1}{1+\lambda a(x)}\right]}{1+\lambda a(x)}\geq 0\ \forall\,\lambda\geq L\ \forall\,x\in\Omega. \tag{5.1.14}$$

Moreover, since $b_{ij}\geq b$ and $a_{ij}\leq a$, then

$$\frac{b_{ij}(x)}{1+\lambda a_{ij}(x)}\geq\frac{b(x)}{1+\lambda a(x)}\ \forall\,\lambda\geq 0,\ \forall\,x\in\Omega. \tag{5.1.15}$$

Since (5.1.14) and (5.1.15), then the hypothesis of Theorem 4.3.1 is satisfied by making $V\equiv 0$, $m(\lambda,x)=(1+\lambda a(x))^{-1}$ and $\bar{b}(\lambda,x)=b_{ij}(x)/(1+\lambda a_{ij}(x))$ and

$$\sigma(\lambda):=\sigma_1^\Omega\left[-\Delta+\frac{\lambda^r b_{ij}(x)}{1+\lambda a_{ij}(x)};\frac{1}{1+\lambda a(x)}\right].$$

Theorem 4.3.1 then states that there exists a unique $\beta(\lambda)>0$ such that

$$\sigma(\lambda)=\sigma_1^\Omega\left[-\Delta+\frac{\beta(\lambda)b_{ij}(x)}{1+\lambda a_{ij}(x)};\frac{1}{1+\lambda a(x)}\right].$$

Since $\beta(\lambda)$ is unique, it follows that $\beta(\lambda)=\lambda^r$. then

$$\lambda^{r/(p-1)}\frac{\varphi_\lambda}{\|\varphi_\lambda\|_\infty}\leq u_\lambda. \tag{5.1.16}$$

Note that

$$\sigma_1^\Omega\left[-\Delta+\frac{\lambda^r b_{ij}(x)}{1+\lambda a_{ij}(x)};\frac{1}{1+\lambda a(x)}\right]<\sigma_1^{C_i^j\cap B_0}[-\Delta;\ \chi_{A_0}].$$

So we can apply Lemma 1.1.1 to deduce that $\|\varphi_\lambda\|_\infty$ is bounded. Since (5.1.13), then the (LHS) of (5.1.16) explodes in C_i^j when $\lambda\to\infty$ and the proof is concluded. □

Remark 5.1.1. *Observe that since*
$$\frac{\lambda^r b_i(x)}{1+\lambda a_i(x)}$$
is not identically null in C_i^j, we can not apply Theorem 2.4 of [32] to deduce (5.1.12). This technical issue was overcame with Theorem 1.2.1.

Corollary 5.1.2 (of Theorem 4.3.1)**.** *Let $\underline{\mu} < \mu < \sigma_1^{B_0}[-\Delta;\chi_{A_0}]$. Then*
$$\lim_{\lambda\to\lambda_*(\mu)} u_\lambda(x) = +\infty \quad \textit{for all } x \in B_0.$$

Proof. By the definition of $\lambda_*(\mu)$, we imply that there exists some $\varepsilon > 0$ such that
$$\mu < \sigma_1^{B_0}\left[-\Delta;\frac{1}{1+\lambda a(x)}\right] \;\forall\, \lambda_*(\mu) < \lambda < \lambda_*(\mu)+\varepsilon.$$

Fix $\lambda_*(\mu) < \lambda < \lambda_*(\mu)+\varepsilon$. Since
$$\mu < \sigma_1^{B_0}\left[-\Delta;\frac{1}{1+\lambda a(x)}\right] = \lim_{\beta\to\infty}\sigma_1^{B_0}\left[-\Delta+\beta b(x);\frac{1}{1+\lambda a(x)}\right],$$

Then there exists a unique $\beta(\lambda) > 0$ such that
$$\mu = \sigma_1^{\Omega}\left[-\Delta+\beta(\lambda)b(x);\frac{1}{1+\lambda a(x)}\right], \tag{5.1.17}$$

due to the continuity of the function
$$\beta \mapsto \sigma_1^{B_0}\left[-\Delta+\beta b(x);\frac{1}{1+\lambda a(x)}\right].$$

Define
$$\sigma(\lambda) := \sigma_1^{\Omega}\left[-\Delta+\beta(\lambda)b(x);\frac{1}{1+\lambda a(x)}\right],\; \forall\lambda \in (\lambda_*(\mu),\lambda_*(\mu)+\varepsilon).$$

Then
$$\frac{\mu-\sigma(\lambda)}{1+\lambda a(x)} = 0 \;\forall\, \lambda \in (\lambda_*(\mu),\lambda_*(\mu)+\varepsilon), \tag{5.1.18}$$

due to (5.1.17).

Again, by the definition of $\lambda_*(\mu)$,
$$(\lambda,\mu) \in \mathcal{S} \;\forall\, \lambda \in (\lambda_*(\mu),\lambda_*(\mu)+\varepsilon). \tag{5.1.19}$$

Since (5.1.18) and (5.1.19), then the hypotheses of Theorem 4.3.1 are satisfied with $\underline{b}(\lambda,x) = b(x)/(1+\lambda a(x))$, $V \equiv 0$ and $m(\lambda,x) = (1+\lambda a(x))^{-1}$. Then Theorem 4.3.1 states that

$$\beta(\lambda)^{1/(p-1)} \frac{\varphi_\lambda}{\|\varphi_\lambda\|_\infty} \leq u_\lambda \forall\, \lambda \in (\lambda_*(\mu), \lambda_*(\mu)+\varepsilon), \tag{5.1.20}$$

where φ_λ is the positive eigenfunction associated with $\sigma(\lambda)$ and u_λ is the unique positive solution of ($R_{\lambda,\mu}$). Assume without loss of generality that φ_λ is normalized in $L^2(\Omega)$. Since $(\lambda_*(\mu), \lambda_*(\mu)+\varepsilon) \mapsto \sigma(\lambda)$ is constant, in particular, $(\lambda_*(\mu), \lambda_*(\mu)+\varepsilon) \mapsto \sigma(\lambda)$ is bounded, so we can apply the Lemma 1.1.1 to deduce that

$$\|\varphi_\lambda\|_\infty \text{ is bounded.} \tag{5.1.21}$$

We claim that

$$\lim_{\lambda \to \lambda_*(\mu)} \beta(\lambda) = \infty. \tag{5.1.22}$$

Indeed, let us suppose by contradiction that there exists a sequence $\lambda_n \downarrow \lambda_*(\mu)$ and $0 < \beta_0 < \infty$ such that such that

$$\begin{aligned}
0 &= \lim_{\lambda_n \downarrow \lambda_*(\mu)} \sigma_1^\Omega \left[-\Delta + \frac{\beta(\lambda_n) b}{1+\lambda_n a} - \frac{\mu}{1+\lambda_n a} \right] \\
&= \sigma_1^\Omega \left[-\Delta + \frac{\beta_0 b}{1+\lambda_*(\mu) a} - \frac{\mu}{1+\lambda_*(\mu) a} \right] \\
&< \sigma_1^{B_0} \left[-\Delta - \frac{\mu}{1+\lambda_*(\mu) a} \right] = \sigma_1^{B_0} \left[-\Delta - \frac{\sigma_1^{B_0}\left[-\Delta; \frac{1}{1+\lambda_*(\mu) a(x)}\right]}{1+\lambda_*(\mu) a} \right] = 0,
\end{aligned}$$

which is an absurd, and we just proved (5.1.22).

By applying Theorem 1.2.1 with

$$q_\lambda(x) = \beta(\lambda) b(x),$$

we have that

$$\lim_{\lambda \to \lambda_*(\mu)} \|\varphi_\lambda - \varphi_{\lambda_*(\mu)}\|_{H_0^1(\Omega)} = 0, \tag{5.1.23}$$

where $\varphi_{\lambda_*(\mu)}$ is the extension to Ω of the positive eigenfunction associated with

$$\sigma_1^{B_0} \left[-\Delta; \frac{1}{1+\lambda a(x)} \right].$$

□

Finally, using (5.1.21), (5.1.22) and (5.1.23) in (5.1.20), we conclude the proof of the corollary.

Below we list in ascendant order the proofs of the items of Theorem 0.0.5.

Proof of item 1.1) of Theorem 0.0.5: It follows directly by applying Theorem 4.2.2 and the definition of $\lambda^*(\mu)$.

Proof of item 1.2) of Theorem 0.0.5: Since $V \equiv 0$ and $b(\lambda,x) = b(x)/(1+\lambda a(x))$ in ($R_{\lambda,\mu}$), then

$$V_0 = \Omega,\ b(\infty,x) = \chi_{A_0} b(x),\ m(\infty,x) = \chi_{A_0}(x) \text{ and } B_{0,\infty} = A_+ \cup B_0.$$

So the statement of 1.2 of Theorem 0.0.5 follows directly by Theorem 4.2.3.

Proof of item 1.3) of Theorem 0.0.5:

We will argue by contradiction. So, let us assume that $\mu = \sigma_1^{A^+\cup B_0}(-\Delta;\chi_{A_0})$ and that there is a sequence $\lambda_n \to +\infty$ such that $\|u_{\lambda_n}\|_2$ is a bounded sequence.

By testing against u_{λ_n} in the definition of u_{λ_n}, we deduce that u_{λ_n} is bounded in $H_0^1(\Omega)$. So, there is $0 \le v \in H_0^1(\Omega)$ such that

$$u_{\lambda_n} \rightharpoonup v \quad \text{in } H_0^1(\Omega) \textit{ and } \quad u_{\lambda_n} \to v \quad \text{in } L^2(\Omega) \text{ up to a subsequence.}$$

Using Lemma 1.1.1 with $h_\lambda = u_\lambda^{p-1} b(x)$ and $m_\lambda(x) = (1+\lambda a(x))^{-1}$ we deduce that u_{λ_n} is bounded in $L^\infty(\Omega)$, and by elliptic regularity we obtain that

$$u_{\lambda_n} \to v \quad \text{in } C^1(\overline{\Omega})$$

being v a solution of (0.0.16).

Assume that $v=0$. Then by dividing by $\|u_{\lambda_n}\|_2$ in the definition of u_λ and passing to the limit, we get that $\mu = \sigma_1[-\Delta;\chi_{A_0}]$ which is an absurd since $\sigma_1[-\Delta;\chi_{A_0}] < \sigma_1^{A^+\cup B_0}[-\Delta;\chi_{A_0}] = \mu$ by hypothesis. On the other hand, if $v \neq 0$, since (0.0.16) possesses a positive solution if and only if $\sigma_1[-\Delta;\chi_{A_0}] < \mu < \sigma_1^{A^+\cup B_0}[-\Delta;\chi_{A_0}]$, then we get $\mu < \sigma_1^{A^+\cup B_0}[-\Delta;\chi_{A_0}]$, which is an absurd.

Proof of item 1.4) of Theorem 0.0.5: Follows directly by Corollary 5.1.1.

Proof of item 1.5) of Theorem 0.0.5:

First, assume that $\Lambda < 0$ and let us prove that

$$\sup_{\Lambda \le \lambda \le 0} \|u_\lambda\|_\infty < +\infty.$$

Observe that

$$\mu = \sigma_1^{B_0}\left[-\Delta, \frac{1}{1+\lambda_*(\mu)a}\right] < \sigma_1^{B_0}\left[-\Delta, \frac{1}{1+\Lambda a}\right], \tag{5.1.24}$$

and

$$\mu = \sigma_1^{\Omega}\left[-\Delta, \frac{1}{1+\lambda^*(\mu)a}\right] > \sigma_1^{\Omega}\left[-\Delta, \frac{1}{1+\Lambda a}\right]. \tag{5.1.25}$$

By (5.1.24) and (5.1.25), we deduce that there exists the positive solution $\overline{u}$ of

$$\begin{cases} -\Delta u = \dfrac{\mu}{1+\Lambda a} u - bu^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, observe that

$$\frac{1}{1+\lambda a} \le \frac{1}{1+\Lambda a} \quad \text{and} \quad \frac{1}{1+\lambda a} \ge 1 \qquad \forall\, \Lambda \le \lambda \le 0.$$

Hence, u_λ is a subsolution of the above problem and consequently,

$$u_\lambda \le \overline{u} \quad \forall\, \Lambda \le \lambda \le 0.$$

Now assume that $\lambda > 0$. Using

$$\frac{1}{1+\lambda a} \le 1 \quad \text{and} \quad \chi_{A_0} \le \frac{1}{1+\lambda a},$$

we obtain that since u_λ is a positive solution of ($R_{\lambda,\mu}$), then

$$-\Delta u_\lambda \le \mu u_\lambda - \chi_{A_0} b(x) u_\lambda^p,$$

and then u_λ is subsolution of

$$\begin{cases} -\Delta u = \mu u - \chi_{A_0} b u^p & \text{in } D_{j_0-1}, \\ u = 0 & \text{on } \partial D_{j_0-1} \cap \partial\Omega, \\ u = +\infty & \text{on } \partial D_{j_0-1}, \end{cases} \tag{5.1.26}$$

where

$$D_k := \Omega \setminus \cup_{i=1}^{k} \cup_{j=1}^{\ell_i} \overline{\mathcal{C}}_{ij}.$$

Since $\mu < \sigma_1^{\mathcal{C}_{j_0}}(-\Delta)$, it follows by Theorem 4.7 of [45] that there exists the large solution L_1 of (5.1.26), and then

$$u_\lambda \le L_1 \quad \text{in } D_{j_0-1}.$$

Analogously, u_λ is subsolution of

$$\begin{cases} -\Delta u = \mu u - \chi_{A_0} b u^p & \text{in } D_{m+d}, \\ u = 0 & \text{on } \partial D_{m+d} \cap \partial\Omega, \\ u = +\infty & \text{on } \partial D_{m+d}, \\ u > 0 & \text{on } D_{m+d} \end{cases} \tag{5.1.27}$$

Equation (5.1.27) possesses a large solution L_2 for any μ and $u_\lambda \le L_2$ in D_{m+d}. This completes the proof.

Proof of item 2) of Theorem 0.0.5:

The first part follows directly from Corollary 5.1.2.

Now, consider $D \subset\subset D_1 \subset\subset \Omega \setminus \overline{B_0}$ an open subset and Λ be a number such that $\lambda_*(\mu) < \Lambda < \lambda^*(\mu)$. Then, for $\lambda \in [\lambda_*(\mu), \Lambda]$ we have that

$$\frac{b(x)}{1+\lambda a(x)} \ge \frac{b_0}{1+\Lambda a_M},$$

where $a_M = \max_{x \in \overline{D_1}} a(x)$, $b_0 = \min_{x \in \overline{D_1}} b(x) > 0$, and then u_λ is a subsolution of the equation

$$\begin{cases} -\Delta u = \mu u - \dfrac{b_0}{1+\Lambda a_M} u^p & \text{in } D_1, \\ u = +\infty & \text{on } \partial D_1. \end{cases} \tag{5.1.28}$$

By [45] there exists a large solution of (5.1.28). This implies that u_λ is bounded in D. We just concluded the proof of item 2) of Theorem 0.0.5.

5.2 Strong degradation

In this section, we will study the problem

$$\begin{cases} -\Delta u + \lambda V(x)u = \mu m(x)u - b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{$S_{\lambda,\mu}$}$$

$0 \lneq V(x) \in L^\infty(\Omega)$, $0 \lneq b \in C(\overline{\Omega})$, $0 \neq m \in C(\overline{\Omega})$ possibly changing sign.

We can see that ($S_{\lambda,\mu}$) is a special case of (0.0.13) by making $b(\lambda,x) = b(x)$ and $m(\lambda,x) = m(x)$. Consequently, we can apply Theorem 0.0.4 to imply that there exists a unique positive solution $u_{\lambda,\mu}$ of ($R_{\lambda,\mu}$) if and only if

$$\sigma_1^\Omega[-\Delta + \lambda V; m] < \mu < \sigma_1^{B_0}[-\Delta + \lambda V; m].$$

Corollary 5.2.1 (of Theorem 4.3.1)**.** *Let $\mu \in (\mathcal{C}_0 \cup \mathcal{C}_\infty) \cap \mathcal{C}([0,\overline{\Lambda}))$, $\overline{\Lambda} \leq \infty$. If*

$$\lim_{\lambda \uparrow \overline{\Lambda}} \left(\sigma_1^{B_0}[-\Delta + \lambda V(x); m(x)] - \mu(\lambda) \right) = 0, \tag{5.2.1}$$

then

$$\begin{cases} \lim_{\lambda \to \overline{\Lambda}} u_\lambda(x) = +\infty \ \forall\, x \in B_0 \text{ if } \mu \in \mathcal{C}_0,\ \overline{\Lambda} < \infty & (5.2.2) \\ \lim_{\lambda \to +\infty} u_\lambda(x) = +\infty \ \forall\, x \in D_0 \text{ if } \mu \in \mathcal{C}_\infty,\ \overline{\Lambda} = \infty,\ V_0 \not\subset B_0. & (5.2.3) \end{cases}$$

Additionally, if $m \in \mathcal{C}^r(B_0)$ ($m \in \mathcal{C}^r(D_0)$, respectively), then the convergence in (5.2.2) (in (5.2.3), respectively) holds uniformly in compact subsets of B_0 (V_0, respectively).

Proof. Let $\mu \in \mathcal{C}_0 \cup \mathcal{C}_\infty$. Define $\sigma(\lambda) := \mu(\lambda)$. Then

$$(\mu(\lambda) - \sigma(\lambda))m(x) = 0 \ \forall\, \lambda \in \operatorname{Dom}\mu. \tag{5.2.4}$$

Moreover, since $\mu \in \mathcal{C}_0 \cup \mathcal{C}_\infty \subset \mathcal{S}$, then

$$(\lambda, \mu(\lambda)) \in \mathcal{S}. \tag{5.2.5}$$

Since (5.2.4) and (5.2.5), then the hypotheses of Theorem 4.3.1 are satisfied with $m(\lambda,x) = m(x)$, $\overline{b}(\lambda,x) = b(x)$. Thus, Theorem 4.3.1 state that there exists a $\beta(\lambda) > 0$ such that

$$\mu(\lambda) = \sigma_1^\Omega[-\Delta + \lambda V + \beta(\lambda)b; m] \tag{5.2.6}$$

and

$$\beta(\lambda)^{1/(p-1)}\frac{\varphi_\lambda}{\|\varphi_\lambda\|_\infty} \leq u_\lambda, \tag{5.2.7}$$

where φ_λ is the positive eigenfunction associated to (5.2.6).

Since $\mu \in \mathcal{C}_0 \cup \mathcal{C}_\infty$, then μ is bounded. By (5.2.6), it follows that $\sigma_1^\Omega[-\Delta+\lambda V+\beta(\lambda)b;m]$ is bounded and so we can apply Theorem (1.1.1) to deduce that

$$\|\varphi_\lambda\|_\infty \text{ is bounded.} \tag{5.2.8}$$

Now we claim that

$$\lim_{\lambda\to\overline{\Lambda}} \beta(\lambda) = \infty. \tag{5.2.9}$$

We will argue by absurd assuming (5.2.11) and $\lim\limits_{\lambda\uparrow\overline{\Lambda}} \beta(\lambda) < +\infty$. Then, there exist a sequence $\lambda_n \uparrow \overline{\Lambda}$ and a positive number $\beta_* > 0$ such that

$$\lim_{n\to+\infty} \beta(\lambda_n) = \beta_* < \infty.$$

By combining Theorem 1.2.1 with the hypothesis (5.2.1), we deduce that

$$\begin{aligned}\lim_{n\to+\infty} \mu(\lambda_n) &= \lim_{n\to+\infty} \sigma_1^{B_0}[-\Delta+\lambda_n V(x);m(x)]\\ &= \begin{cases} \sigma_1^{B_0}[-\Delta+\overline{\Lambda}V(x);m(x)], & \text{if } \overline{\Lambda}<\infty,\\ \sigma_1^{D_0}[-\Delta;m(x)], & \text{if } \overline{\Lambda}=\infty.\end{cases}\end{aligned} \tag{5.2.10}$$

On the other hand, by using the definition of $\beta(\lambda_n)$ and Theorem 1.2.1, we obtain

$$\lim_{n\to+\infty} \mu(\lambda_n) = \begin{cases} \sigma_1^{\Omega}[-\Delta+\beta_* b(x)+\overline{\Lambda}V(x);m(x)], & \text{if } \overline{\Lambda}<\infty,\\ \sigma_1^{V_0}[-\Delta+\beta_* b(x);m(x)], & \text{if } \overline{\Lambda}=\infty.\end{cases} \tag{5.2.11}$$

By combining (5.2.10) and (5.2.10), we have that

$$\begin{cases} \sigma_1^{B_0}[-\Delta+\overline{\Lambda}V(x);m(x)] = \sigma_1^{\Omega}[-\Delta+\beta_* b(x)+\overline{\Lambda}V(x);m(x)], & \text{if } \overline{\Lambda}<\infty\\ \sigma_1^{D_0}[-\Delta;m(x)] = \sigma_1^{V_0}[-\Delta+\beta_* b(x);m(x)] & \text{if } \overline{\Lambda}=\infty.\end{cases} \tag{5.2.12}$$

Both equalities lead to a contradiction; in the second one, we used that $V_0 \not\subset B_0$. We just proved that $\lim_{\lambda\to\overline{\Lambda}} \beta(\lambda) = \infty$. Then Theorem 1.2.1 implies that

$$\lim_{\lambda\to\overline{\Lambda}} \|\varphi_\lambda - \varphi_{\overline{\Lambda}}\|_{H_0^1(\Omega)} = 0, \tag{5.2.13}$$

where $\varphi_{\overline{\Lambda}}$ is the extension to Ω of the positive eigenfunction associated with $\sigma_1^{B_0}[-\Delta + \overline{\Lambda}V(x); m(x)]$ (if $\overline{\Lambda} < \infty$, respectively) or with $\sigma_1^{D_0}[-\Delta; m(x)]$ (if $\overline{\Lambda} = \infty$, respectively). Finally, using (5.2.8), (5.2.9) and (5.2.13) in (5.2.7), we conclude the proof. □

Theorem 5.2.1. *Let $\lambda \geq 0$. Then*

$$\lim_{\mu\downarrow\sigma_1^{\Omega}[-\Delta+\lambda V;m]} \|u_{\lambda,\mu}\|_{C_0^1(\bar{\Omega})} = 0 \tag{5.2.14}$$

and

$$\lim_{\mu\uparrow\sigma_1^{B_0}[-\Delta+\lambda V;m]} u_{\lambda,\mu}(x) = +\infty \;\forall\, x \in B_0 \tag{5.2.15}$$

Proof. (5.2.14) (respectively, (5.2.15)) can be obtained by following similar arguments as done in Theorem 4.2.2 (respectively, Corollary 5.2.1). □

The next lemma will be useful in order to prove Theorem 5.2.2.

Lemma 5.2.1. *Let $\mu \in \mathcal{C}_0 \cap \mathcal{C}([0,\Lambda])$. Let f and B_δ as defined in Section 2.3. Assume that μ is analytic for $\lambda < \Lambda$ in a neighborhood of Λ and consider any analytic extension of μ to $[0, \Lambda+\eta)$. Assume all the hypotheses of Theorem 1.3.2. Then there exists $\eta \geq \eta_2 > 0$ and a function $\delta(\lambda) > 0$, defined for each $\lambda \in (\Lambda - \eta_2, \Lambda)$, such that*

$$\lim_{\lambda\uparrow\Lambda} \delta(\lambda) = 0, \tag{5.2.16}$$

$$\sigma_1^{B_{\delta(\lambda)}}[-\Delta + \lambda V(x) - \mu(\lambda)m(x)] < 0 \tag{5.2.17}$$

and

$$\lim_{\lambda\uparrow\Lambda} \frac{\sigma_1^{B_{\delta(\lambda)}}[-\Delta + \lambda V(x) - \mu(\lambda)m(x)]}{\delta(\lambda)} = \begin{cases} \lambda_1^{(0,1)} \text{ if } \lambda_1^{(1,0)} = 0, \\ \dfrac{\lambda_1^{(0,1)}}{2} \text{ if } \lambda_1^{(1,0)} \neq 0. \end{cases} \tag{5.2.18}$$

In particular, the limit above is negative by Theorem 1.3.2.

Proof. By Theorem 1.3.2,
$$\frac{\partial f}{\partial \delta}(\Lambda,0)=\lambda_1^{(0,1)}<0$$
and so we can apply the theorem of the implicit function to imply that there exist $\varepsilon>0$, $0<\eta_1\leq\eta$ and a derivable function $\tilde{\delta}:(\Lambda-\eta_1,\Lambda+\eta_1)\to(-\varepsilon,\varepsilon)$ such that $f(\lambda,\tilde{\delta}(\lambda))=0$ for all $\lambda\in(\Lambda-\eta_1,\Lambda+\eta_1)$ and
$$\tilde{\delta}'(\Lambda)=-\frac{\frac{\partial f}{\partial \lambda}}{\frac{\partial f}{\partial \delta}}=-\frac{\lambda_1^{(1,0)}}{\lambda_1^{(0,1)}}.$$

Note that since $\mu\in\mathcal{C}_0$, then $(\lambda,\mu(\lambda))\in\mathcal{S}$. By using this fact and the definition of $\tilde{\delta}$, we get
$$\sigma_1^{B_0}[-\Delta-\mu(\lambda)m+\lambda V]>0=\sigma_1^{B_{\tilde{\delta}(\lambda)}}[-\Delta-\mu(\lambda)m+\lambda V]\ \forall\,\lambda\in(\Lambda-\eta_1,\Lambda)$$
and so $\tilde{\delta}(\lambda)>0$ for all $\lambda\in(\Lambda-\eta_1,\Lambda)$. Let us define the function $\delta=2\tilde{\delta}$ and note that δ satisfies
$$\begin{aligned}0&=f(\lambda,\tilde{\delta}(\lambda))\\&=\sigma_1^{B_{\delta(\lambda)/2}}[-\Delta+\lambda V(x)-\mu(\lambda)m(x)]\\&>\sigma_1^{B_{\delta(\lambda)}}[-\Delta+\lambda V(x)-\mu(\lambda)m(x)]\ \forall\lambda\in(\Lambda-\eta_1,\Lambda).\end{aligned}$$

We claim that $\delta(\lambda)\to 0$ as $\lambda\uparrow\Lambda$. Indeed, assume by absurd that there exists a sequence $\lambda_n\uparrow\Lambda$ such that $\lim_{n\to\infty}\delta(\lambda_n)=\delta_*>0$. By passing to the limit in
$$\sigma_1^{B_{\delta(\lambda_n)/2}}[-\Delta+\lambda_n V(x)-\mu(\lambda_n)m(x)]=0,$$
using the monotonicity of the principal eigenvalue with respect to the domain and the hypothesis (1.3.8), we obtain
$$0=\sigma_1^{B_0}[-\Delta+\Lambda V(x)-\mu(\Lambda)m(x)]>\sigma_1^{B_{\delta_*/2}}[-\Delta+\Lambda V(x)-\mu(\Lambda)m(x)]=0,$$
which is an absurd. So $\delta(\lambda)$ satisfies (5.2.16) and (5.2.17). To prove (5.2.18) we will analyze two cases:

1. Assume that $\lambda_1^{(1,0)} = 0$. By using (1.3.9), we get

$$\lim_{\lambda\uparrow\Lambda} \frac{\sigma_1^{B_{\delta(\lambda)}}[-\Delta+\lambda V(x)-\mu(\lambda)m(x)]}{\delta(\lambda)} = \lim_{\lambda\uparrow\Lambda}\left(\lambda_1^{(0,1)} + \frac{g(\lambda,\delta(\lambda))}{\delta(\lambda)}\right) = \lambda_1^{(0,1)},$$

2. Assume that $\lambda_1^{(1,0)} \neq 0$. That is,

$$\frac{\partial f}{\partial\lambda}(\Lambda,0) \neq 0$$

Observe that

$$\begin{aligned}
\lim_{\lambda\uparrow\Lambda}\frac{\lambda-\Lambda}{\delta(\lambda)} &= \left(\delta'(\Lambda)\right)^{-1} \\
&= \left(2\tilde{\delta}'(\Lambda)\right)^{-1} \\
&= \frac{1}{2}\left(-\frac{\frac{\partial f}{\partial\lambda}}{\frac{\partial f}{\partial\delta}}(\Lambda)\right)^{-1} \\
&= -\frac{1}{2}\frac{\lambda_1^{(0,1)}}{\lambda_1^{(1,0)}}
\end{aligned}$$

and so

$$\begin{aligned}
\lim_{\lambda\uparrow\Lambda} \frac{\sigma_1^{B_{\delta(\lambda)}}[-\Delta+\lambda V(x)-\mu(\lambda)m(x)]}{\delta(\lambda)} &= \lim_{\lambda\uparrow\Lambda}\left(\lambda_1^{(0,1)} + \frac{\lambda-\Lambda}{\delta(\lambda)}\lambda_1^{(1,0)} + \frac{g(\lambda,\delta(\lambda))}{\delta(\lambda)}\right) \\
&= \lambda_1^{(0,1)} - \frac{\lambda_1^{(0,1)}}{2\lambda_1^{(1,0)}}\lambda_1^{(1,0)} \\
&= \frac{\lambda_1^{(0,1)}}{2}
\end{aligned}$$

and we just proved the lemma.

□

Lemma 5.2.2. *Suppose that $m(x_0) > 0$ for some $x_0 \in B_0$. Then, the curve*

$$\overline{\mu}(\lambda) := \sigma_1^{B_0}[-\Delta+\lambda V; m], \;\; \lambda \geq 0,$$

is well defined and it is real analytic as a function of the parameter λ. Moreover, if μ is derivable in Λ, then the condition (0.0.23) is equivalent to

$$\mu'(\Lambda)\int_{B_0} m\varphi_\Lambda^2 - \int_{B_0} V\varphi_\Lambda^2 \neq 0.$$

Proof. According to Theorem 9.1 of [44], the principal eigenvalue

$$\Sigma(\lambda,\mu) := \sigma_1^{B_0}[-\Delta+\lambda V-\mu m], \qquad (\lambda,\mu)\in\mathbb{R}^2$$

is real analytic in both parameters, λ and μ. Let $\varphi_{\lambda,\mu}$ denote the principal eigenfunction associated to $\Sigma(\lambda,\mu)$ normalized so that $\|\varphi_{\lambda,\mu}\|_{L^2(B_0)}=1$. Then, $\varphi_{\lambda,\mu}=0$ on B_0, and

$$-\Delta\varphi_{\lambda,\mu}+\lambda V\varphi_{\lambda,\mu}-\mu m\varphi_{\lambda,\mu}=\Sigma(\lambda,\mu)\varphi_{\lambda,\mu} \quad \text{in } B_0. \tag{5.2.19}$$

Thus, differentiating with respect to μ yields to

$$(-\Delta+\lambda V-\mu m-\Sigma(\lambda,\mu))\frac{\partial\varphi_{\lambda,\mu}}{\partial\mu}=m\varphi_{\lambda,\mu}+\frac{\partial\Sigma}{\partial\mu}(\lambda,\mu)\varphi_{\lambda,\mu}.$$

Hence, multiplying this identity by $\varphi_{\lambda,\mu}$ and integrating by parts in B_0, we find from the definition of $\varphi_{\lambda,\mu}$ that

$$\frac{\partial\Sigma}{\partial\mu}(\lambda,\mu)=-\int_{B_0} m\varphi_{\lambda,\mu}^2. \tag{5.2.20}$$

On the other hand, multiplying (5.2.19) by $\varphi_{\lambda,\mu}$ and integrating in B_0 shows that

$$\int_{B_0}|\nabla\varphi_{\lambda,\mu}|^2+\lambda\int_{B_0} V\varphi_{\lambda,\mu}^2-\mu\int_{B_0} m\varphi_{\lambda,\mu}^2=\Sigma(\lambda,\mu).$$

So, thanks to (5.2.20), it becomes apparent that

$$\int_{B_0}|\nabla\varphi_{\lambda,\mu}|^2+\lambda\int_{B_0} V\varphi_{\lambda,\mu}^2+\mu\frac{\partial\Sigma}{\partial\mu}(\lambda,\mu)=\Sigma(\lambda,\mu).$$

Therefore, since $\Sigma(\lambda,\overline{\mu}(\lambda))=0$ for all $\lambda\geq 0$, we find that

$$\int_{B_0}|\nabla\varphi_{\lambda,\overline{\mu}(\lambda)}|^2+\lambda\int_{B_0} V\varphi_{\lambda,\overline{\mu}(\lambda)}^2+\overline{\mu}(\lambda)\frac{\partial\Sigma}{\partial\mu}(\lambda,\overline{\mu}(\lambda))=0.$$

In particular, $\dfrac{\partial\Sigma}{\partial\mu}(\lambda,\overline{\mu}(\lambda))\neq 0$ for all $\lambda\geq 0$. Consequently, by the implicit function theorem, $\overline{\mu}(\lambda)$ must be analytic in λ.

Next, for every $\lambda\geq 0$, we denote by φ_λ the principal eigenfunction, normalized by $\|\varphi_\lambda\|_{L^2(B_0)}=1$, of the equation

$$-\Delta\varphi_\lambda+\lambda V\varphi_\lambda=\overline{\mu}(\lambda)m\varphi_\lambda \quad \text{in } B_0$$

under Dirichlet boundary conditions on ∂B_0. Then, differentiating with respect to λ, multiplying by φ_λ, and integrating in B_0, yields to

$$\int_{B_0}\varphi_\lambda(-\Delta+\lambda V-\overline{\mu}(\lambda)m)\varphi_\lambda'=\overline{\mu}'(\lambda)\int_{B_0}m\varphi_\lambda^2-\int_{B_0}V\varphi_\lambda^2.$$

Therefore, for every $\lambda\geq 0$,

$$\overline{\mu}'(\lambda)\int_{B_0}m\varphi_\lambda^2-\int_{B_0}V\varphi_\lambda^2=0.$$

In particular, condition (0.0.23) can be equivalently expressed as $\overline{\mu}'(\Lambda)\neq\mu'(\Lambda)$. This ends the proof. □

Theorem 5.2.2. *Additionally to the hypotheses of Lemma 5.2.1, assume that $\nabla b=0$ in ∂B_0. Then*

$$\lim_{\lambda\uparrow\Lambda}u_\lambda(x)=+\infty\ \forall\, x\in\partial B_0.$$

Proof. We claim that

$$\underline{u}(x):=\begin{cases} C(\lambda,\delta(\lambda))\varphi_{\lambda,\delta(\lambda)}(x) & x\in B_\delta,\\ 0 & x\in\Omega\setminus\overline{B_\delta},\end{cases}$$

is a subsolution of ($S_{\lambda,\mu}$) with $\mu=\mu(\lambda)$, where

$$C(\lambda,\delta(\lambda)):=\frac{1}{\sup_{B_{\delta(\lambda)}\setminus B_0}\varphi_{\lambda,\delta(\lambda)}}\left(\frac{-\sigma_1^{B_{\delta(\lambda)}}[-\Delta+\lambda V(x)-\mu(\lambda)m(x)]}{\sup_{B_{\delta(\lambda)}\setminus B_0}b}\right)^{\frac{1}{p-1}} \tag{5.2.21}$$

and $\delta(\lambda)$ is the function given by Lemma 5.2.1. Indeed, $\underline{u}$ is subsolution if

$$b(x)\left(C(\lambda_n,\delta_n)\varphi_{\lambda,\delta(\lambda)}\right)^{p-1}\leq-\sigma_1^{B_{\delta(\lambda)}}[-\Delta-\mu(\lambda)m+\lambda V]\quad x\in B_\delta,$$

which is true if $C(\lambda,\delta(\lambda))$ is the one defined in (5.2.21).

We claim that there exists a constant $C_0 > 0$ such that

$$\inf_{\partial B_0} \varphi_{\lambda,\delta(\lambda)} \sim \delta(\lambda) C_0 \quad \lambda \sim \Lambda. \tag{5.2.22}$$

Indeed, since ∂B_0 is a compact set, there exists $y_\lambda \in \partial B_0$ such that

$$\inf_{\partial B_0} \varphi_{\lambda,\delta(\lambda)} = \varphi_{\lambda,\delta(\lambda)}(y_\lambda).$$

Thus, since $T_{\delta(\lambda)} : \bar{B}_0 \to \bar{B}_{\delta(\lambda)}$ is a bijection, there exists $z_\lambda \in \bar{B}_0$ such that $y_\lambda = T_{\delta(\lambda)}(z_\lambda)$. By (1.3.1),

$$T_{\delta(\lambda)}(y_\lambda - \delta(\lambda)n(y_\lambda)) = y_\lambda - \delta(\lambda)n(y_\lambda) + \delta(\lambda)R(y_\lambda - \delta(\lambda)n(y_\lambda)). \tag{5.2.23}$$

Moreover, since $\delta(\Lambda) = 0$, we have that $\delta(\lambda) \in (0, \varepsilon_0/4)$ for λ sufficiently close to Λ. So,

$$\operatorname{dist}(y_\lambda - \delta(\lambda)n(y_\lambda), \partial B_0) = \operatorname{dist}(y_\lambda - \delta(\lambda)n(y_\lambda), y_\lambda) = \delta(\lambda) < \frac{\varepsilon_0}{4},$$

and, thanks to Theorem 1.3.1 (iii),

$$R(y_\lambda - \delta(\lambda)n(y_\lambda)) = n(\pi(y_\lambda - \delta(\lambda)n(y_\lambda))) = n(y_\lambda).$$

Consequently, substituting in (5.2.23) shows that

$$T_{\delta(\lambda)}(y_\lambda - \delta(\lambda)n(y_\lambda)) = y_\lambda - \delta(\lambda)n(y_\lambda) + \delta(\lambda)n(y_\lambda) = y_\lambda. \tag{5.2.24}$$

Therefore, since $T_{\delta(\lambda)}$ is a bijection, it becomes apparent that

$$z_\lambda = y_\lambda - \delta(\lambda)n(y_\lambda).$$

On the other hand, by the definition of $\psi_{\lambda,\delta(\lambda)}$, we have that

$$\psi_{\lambda,\delta(\lambda)}(z_\lambda) = \varphi_{\lambda,\delta(\lambda)}(T_{\delta(\lambda)}(z_\lambda)) = \varphi_{\lambda,\delta(\lambda)}(y_\lambda).$$

So, since $\varphi_{\lambda,\varphi(\lambda)} = 0$ on $\partial B_{\delta(\lambda)}$ and $y_\lambda \in \partial B_0$ entails $T_{\delta(\lambda)}(y_\lambda) \in \partial B_{\delta(\lambda)}$, we have that

$$\psi_{\lambda,\delta(\lambda)}(y_\lambda) = \varphi_{\lambda,\delta(\lambda)}(T_{\delta(\lambda)}(y_\lambda)) = 0.$$

Consequently, by the fundamental theorem of calculus, we find that

$$\begin{aligned}
\varphi_{\lambda,\delta(\lambda)}(y_\lambda) &= \varphi_{\lambda,\delta(\lambda)}(y_\lambda) - 0 \\
&= \varphi_{\lambda,\delta(\lambda)}(y_\lambda) - \varphi_{\lambda,\delta(\lambda)}(T_{\delta(\lambda)}(x_\lambda)) \\
&= \psi_{\lambda,\delta(\lambda)}(z_\lambda) - \psi_{\lambda,\delta(\lambda)}(x_\lambda) \\
&= \psi_{\lambda,\delta(\lambda)}(x_\lambda - s\delta(\lambda)n(x_\lambda))\big|_{s=0}^{s=1} \\
&= \int_{s=0}^{s=1} \frac{d}{ds}\psi_{\lambda,\delta(\lambda)}(x_\lambda - s\delta(\lambda)n(x_\lambda))ds \\
&= -\delta(\lambda)\int_{s=0}^{s=1} \langle \nabla\psi_{\lambda,\delta(\lambda)}(x_\lambda - s\delta n(x_\lambda)), n(x_\lambda)\rangle ds.
\end{aligned}$$

By passing to the limit in the above equation we find that

$$\inf_{\partial B_0} \varphi_{\lambda,\delta(\lambda)} = -\delta(\lambda)\frac{\partial \varphi_{\lambda,\delta(\lambda)}}{\partial n}(x_\Lambda) + o(\lambda - \Lambda),$$

for some $x_\Lambda \in \partial B_0$ as $\lambda \uparrow \Lambda$ and we just proved (5.2.22). Similarly, it can obtained a constant C_1 such that

$$\sup_{B_{\delta(\lambda)} \setminus B_0} \varphi_{\lambda,\delta(\lambda)} \sim C_1\delta(\lambda), \quad \lambda \sim \Lambda. \tag{5.2.25}$$

Now observe that

$$C(\lambda,\delta(\lambda)) \inf_{\partial B_0} \varphi_{\lambda,\delta(\lambda)} \leq C(\lambda,\delta(\lambda))\varphi_{\lambda,\delta(\lambda)} \leq u_\lambda(x)\ \forall\, x \in \partial B_0. \tag{5.2.26}$$

Combining (5.2.26) and the definition of $C(\lambda,\delta(\lambda))$, we find

$$u_\lambda(x) \geq C(\lambda,\delta(\lambda)) \inf_{\partial B_0} \varphi_{\lambda,\delta(\lambda)} \tag{5.2.27}$$

$$= \frac{\inf_{\partial B_0} \varphi_{\lambda,\delta(\lambda)}}{\sup_{B_{\delta(\lambda)} \setminus B_0} \varphi_{\lambda,\delta(\lambda)}} \left(\frac{-\sigma_1^{B_{\delta(\lambda)}}[-\Delta + \lambda V(x) - \mu(\lambda)m(x)]/\delta(\lambda)}{\left(\sup_{B_{\delta(\lambda)} \setminus B_0} b\right)/\delta(\lambda)} \right)^{\frac{1}{p-1}} \tag{5.2.28}$$

for all $x \in \partial B_0$.

We claim that $\delta'(\Lambda) \neq 0$. Indeed, since $\delta = 2\tilde{\delta}$ (see the proof of Lemma 5.2.1), then it is sufficient to prove that $\tilde{\delta}'(\Lambda) \neq 0$. By the definition of $\tilde{\delta}$, we have that $f(\lambda, \tilde{\delta}(\lambda)) = 0$. Differentiating implicitly with respect to λ, we deduce that

$$\tilde{\delta}'(\Lambda) = \left(-\frac{\frac{\partial f}{\partial \delta}}{\frac{\partial f}{\partial \lambda}}(\Lambda) \right)^{-1} = -\frac{\lambda_1^{(1,0)}}{\lambda_1^{(0,1)}} \neq 0,$$

where the last inequality follows from Lemma 5.2.2 combined with hypothesis

$$\mu'(\Lambda) \neq \frac{d}{d\lambda}\bigg|_{\lambda=\Lambda} \sigma_1^{B_0}[-\Delta + \lambda V; m]$$

and Theorem 1.3.1. We just proved that $\delta'(\Lambda) \neq 0$. Combining this fact with the hypothesis that $\nabla b = 0$ in ∂B_0, we deduce that

$$\frac{1}{\left(\sup_{B_{\delta(\lambda)} \setminus B_0} b\right) / \delta(\lambda)} \to \infty \text{ as } \lambda \uparrow \Lambda. \tag{5.2.29}$$

Finally, combining (5.2.22), (5.2.25), (5.2.18) and (5.2.29) we deduce that the limit of the (RHS) of (5.2.27), when $\lambda \to \Lambda$, is infinity and consequently

$$\lim_{\lambda \uparrow \Lambda} u_\lambda(x) = \infty \; \forall\, x \in \partial B_0.$$

□

Assume that $D_0 \in \mathcal{C}^2$ and that there exists a component Γ of ∂D_0) such that Γ satisfies:

1. $\Gamma \cap \partial\Omega = \emptyset$,
2. $\Gamma \subset M_+ = \{x \in \Omega, m(x) > 0\}$ and
3. $\Gamma \subset V_0$

Theorem 5.2.3.

$$\lim_{\lambda \to +\infty} u_\lambda(x) = +\infty \; \forall\, x \in \Gamma \tag{5.2.30}$$

Proof. The idea is to prove that $\lim_{\lambda \to +\infty} \min_\Gamma u_\lambda = +\infty$. We will argue by absurd. So assume that

$$\lim_{\lambda \to +\infty} \min_\Gamma u_\lambda < +\infty. \tag{5.2.31}$$

Since $\Gamma \subset M_+$, then there exists $\delta > 0$ such that

$$m(x) > 0 \;\forall\, x \in \{x \in D_0;\; \mathrm{dist}(x,\Gamma) \leq \delta\}. \tag{5.2.32}$$

Since $D_0 \in \mathcal{C}^2$, then Γ satisfies the uniform interior sphere property. Moreover $\Gamma \cap \partial\Omega = \emptyset$. So there exist a $R > 0$ and a function $Y : \Gamma \to D_0$ that satisfies

$$B_R(Y(x)) \subset D_0, \tag{5.2.33}$$

$$\overline{B}_R(Y(x)) \cap \partial\Omega = \emptyset, \tag{5.2.34}$$

$$\overline{B}_R(Y(x)) \cap \Gamma = \{x\}. \tag{5.2.35}$$

Without loss of generality, we can assume that

$$2R < \delta. \tag{5.2.36}$$

Define

$$(D_0)_R = \{y \in D_0;\; \mathrm{dist}(y,\Gamma) < 2R\}.$$

By combining (5.2.36) and (5.2.32), we deduce that

$$m(x) > 0 \;\forall\, x \in (D_0)_R. \tag{5.2.37}$$

Let $x_\lambda \in \overline{(D_0)_R}$ be such that $\min\limits_{\overline{(D_0)_R}} u_\lambda = u_\lambda(x_\lambda)$. Define

$$\Gamma_R = \{y \in D_0; \mathrm{dist}(y,\Gamma) = 2R\}.$$

Due to Corollary 5.2.1, $\lim\limits_{\lambda \to +\infty} u_\lambda(x) \to +\infty$ uniformly in compact subsets of D_0 and so

$$\lim_{\lambda \to +\infty} u_\lambda(x) = +\infty \text{ uniformly in } \Gamma_R. \tag{5.2.38}$$

We claim $x_\lambda \in \partial(D_0)_R$. In fact, suppose that $x_\lambda \in (D_0)_R$. By the definition of $(D_0)_R$, we can deduce that

$$\overline{(D_0)}_R = \cup_{x \in \Gamma} \overline{B}_R(Y(x)). \tag{5.2.39}$$

Consequently, there exists some $x_0 \in \Gamma$ such that $x_\lambda \in \overline{B}_R(Y(x_0))$. By (5.2.34), $\overline{B}_R(Y(x_0)) \cap \partial\Omega = \emptyset$ and consequently $u_\lambda(x_\lambda) > 0$. By (5.2.37), we have that $m(x_\lambda) > 0$. Now, we have

$$\nabla u_\lambda(x_\lambda) = 0, \;\; \Delta u_\lambda(x_\lambda) \geq 0$$

and

$$0 \geq -\Delta u_\lambda(x_\lambda) = \mu(\lambda) m(x_\lambda) u_\lambda(x_\lambda) > 0,$$

which is an absurd. So $x_\lambda \in \partial(D_0)_R$. But $\partial(D_0)_R$ has two components, Γ and Γ_R. We claim $x_\lambda \in \Gamma$. Observe that since $\Gamma \subset \overline{(D_0)_R}$, then

$$u_\lambda(x_\lambda) = \min_{\overline{(D_0)_R}} u_\lambda \leq \min_\Gamma u_\lambda. \tag{5.2.40}$$

By combining (5.2.40) and (5.2.31), we deduce that $\sup_{\lambda \geq 0} \min_{\overline{(D_0)_R}} u_\lambda < +\infty$ and so by using (5.2.38), we deduce that there exists a large Λ such that

$$\inf_{\Gamma_R} u_\lambda > \sup_{\lambda \geq 0} \min_{\overline{(D_0)_R}} u_\lambda \;\forall\, \lambda \geq \Lambda. \tag{5.2.41}$$

By using (5.2.41), we have

$$\inf_{\Gamma_R} u_\lambda > \sup_{\lambda \geq 0} \min_{\overline{(D_0)_R}} u_\lambda \geq \min_{\overline{(D_0)_R}} u_\lambda = u_\lambda(x_\lambda) \;\forall\, \lambda \geq \Lambda. \tag{5.2.42}$$

So if $x_\lambda \in \Gamma_R$ for some $\lambda \geq \Lambda$, then we would have

$$u_\lambda(x_\lambda) \geq \inf_{\Gamma_R} u_\lambda > u_\lambda(x_\lambda),$$

due (5.2.42), which is an absurd. So $x_\lambda \in \Gamma$ for all $\lambda \geq \Lambda$. By (5.2.39), we imply that $\overline{B}_R(Y(x_\lambda)) \subset \overline{(D_0)_R}$ for all $\lambda \geq \Lambda$. Let $x \in \overline{B}_R(Y(x_\lambda))$. Then

$$u_\lambda(x) \geq \min_{\overline{B}_R(x_\lambda)} u_\lambda \geq \min_{\overline{(D_0)_R}} u_\lambda = u_\lambda(x_\lambda) \;\forall\, \lambda \geq \Lambda.$$

Since x is arbitrary, we have

$$u_\lambda(x) \geq u_\lambda(x_\lambda) \;\forall\, x \in \overline{B}_R(x_\lambda) \;\forall\, \lambda \geq \Lambda. \tag{5.2.43}$$

Let us define for each $\lambda \geq \Lambda$ the barrier function ψ_λ by

$$\psi_\lambda(x) := e^{-\alpha|x-Y(x_\lambda)|^2} - e^{-\alpha R^2},\ \forall\, x \in \overline{B}_R(Y(x_\lambda)).$$

Observe that for each $x \in B_R(Y(x_\lambda))$, we have

$$-\Delta\psi_\lambda(x) = \left(2\alpha N - 4\alpha^2\|x - Y(x_\lambda)\|^2\right) e^{-\alpha|x-Y(x_\lambda)|^2}.$$

Now, observe that

$$e^{-\alpha R^2} \leq e^{-\alpha\|x-Y(x_\lambda)\|^2}\ \forall\, x \in A_R := B_R(Y(x_\lambda)) \setminus \overline{B}_{R/2}(Y(x_\lambda))$$

and so

$$\begin{aligned}
(-\Delta - \mu(\lambda)m(x))\psi_\lambda(x) &= \left(2\alpha N - 4\alpha^2\|x - Y(x_\lambda)\|^2 - \mu(\lambda)m(x)\right) e^{-\alpha|x-Y(x_\lambda)|^2} + \\
&\quad + \mu(\lambda)m(x)e^{-\alpha R^2} = \\
&= \left(2\alpha N - 4\alpha^2\|x - Y(x_\lambda)\|^2\right) e^{-\alpha|x-Y(x_\lambda)|^2} + \mu(\lambda)m(x)\left(e^{-\alpha R^2} - e^{-\alpha|x-Y(x_\lambda)|^2}\right) \leq \\
&\leq \left(2\alpha N - 4\alpha^2\|x - Y(x_\lambda)\|^2\right) e^{-\alpha|x-Y(x_\lambda)|^2} \leq \\
&\leq \left(2\alpha N - 4\alpha^2\left(\frac{R}{2}\right)^2\right) e^{-\alpha|x-Y(x_\lambda)|^2} = \\
&= \alpha\left(2N - 4\alpha\left(\frac{R}{2}\right)^2\right) e^{-\alpha|x-Y(x_\lambda)|^2}.
\end{aligned}$$

Take $\alpha > 2N/(R^2)$, whence the constant between the parenthesis is negative and so

$$\begin{aligned}
(-\Delta - \mu(\lambda)m(x))\psi_\lambda(x) &\leq \alpha\left(2N - 4\alpha\left(\frac{R}{2}\right)^2\right) e^{-\alpha|x-Y(x_\lambda)|^2} \leq \\
&\leq \alpha\left(2N - 4\alpha\left(\frac{R}{2}\right)^2\right) e^{-\alpha|x-Y(x_\lambda)|^2} \leq \\
&\leq \alpha\left(2N - 4\alpha\left(\frac{R}{2}\right)^2\right) e^{-\alpha R^2}\ \forall\, x \in A_R,
\end{aligned}$$

that is,

$$(-\Delta - \mu(\lambda)m(x))\psi_\lambda(x) \leq -\omega < 0\ \forall\, x \in A_R, \tag{5.2.44}$$

where

$$\omega = -\alpha\left(2N - 4\alpha\left(\frac{R}{2}\right)^2\right)e^{-\alpha R^2}.$$

Since $\overline{B}_{R/2}(Y(x_\lambda))$ is a compact subset of D_0, then

$$\lim_{\lambda\to+\infty}\min_{\overline{B}_{R/2}(Y(x_\lambda)))} u_\lambda(x) = +\infty. \tag{5.2.45}$$

Let us define

$$c_\lambda := \frac{\min_{\overline{B}_{R/2}(Y(x_\lambda)))} u_\lambda(x) - u_\lambda(x_\lambda)}{e^{-\alpha R^2/4} - e^{-\alpha R^2}}.$$

By combining (5.2.31) and (5.2.40), we deduce that

$$\lim_{\lambda\to+\infty} u_\lambda(x_\lambda) < +\infty. \tag{5.2.46}$$

Since (5.2.45) and (5.2.46), then

$$\lim_{\lambda\to+\infty} c_\lambda = +\infty. \tag{5.2.47}$$

By the definition of c_λ, we have that

$$u_\lambda(x) \geq u_\lambda(x_\lambda) + c_\lambda\left(e^{-\alpha R^2/4} - e^{-\alpha R^2}\right)\ \forall\, x \in \overline{B}_{R/2}(Y(x_\lambda)). \tag{5.2.48}$$

For every $\lambda \geq \Lambda$, let us define the auxiliary function

$$v_\lambda := u_\lambda - u_\lambda(x_\lambda) - c_\lambda \psi_\lambda \text{ in } A_R.$$

By (5.2.48), we have

$$v_\lambda \geq 0 \text{ on } \partial B_{R/2}(Y(x_\lambda)).$$

Since $\psi_\lambda = 0$ on $\partial B_R(Y(x_\lambda))$ and (5.2.43), we have

$$v_\lambda = u_\lambda - u_\lambda(x_\lambda) \geq 0 \text{ on } \partial B_R(Y(x_\lambda)).$$

Thus

$$v_\lambda \geq 0 \text{ on } \partial A_R\ \forall\, \lambda \geq \Lambda. \tag{5.2.49}$$

Now observe that

$$\begin{aligned}(-\Delta-\mu(\lambda)m(x))v_\lambda &= \mu(\lambda)m(x)u_\lambda(x_\lambda)-c_\lambda(-\Delta-\mu(\lambda)m(x))\psi_\lambda \\ &\geq \mu(\lambda)m(x)u_\lambda(x_\lambda)+\omega c_\lambda \;\forall\, x\in A_R,\end{aligned}$$

where we used the definition of u_λ and (5.2.44).

Once that $\mu(\lambda)u_\lambda(x_\lambda)$ is bounded and $\lim\limits_{\lambda\to+\infty} c_\lambda=+\infty$, it follows that

$$(-\Delta-\mu(\lambda)m(x))v_\lambda>0\;\forall\, x\in A_R\;\forall\,\lambda\geq\Lambda_1, \tag{5.2.50}$$

for some large $\Lambda_1\geq\Lambda$. Now, observe that

$$\mu(\lambda)<\sigma_1^{D_0}[-\Delta;m(x)]<\sigma_1^{A_R}[-\Delta;m(x)],$$

by whence

$$-\mu(\lambda)m(x)>-\sigma_1^{A_R}[-\Delta;m(x)]m(x)\;\forall\, x\in A_R$$

and so

$$\sigma_1^{A_R}[-\Delta-\mu(\lambda)m(x)]>\sigma_1^{A_R}[-\Delta-\sigma_1^{A_R}[-\Delta;m(x)]m(x)]=0. \tag{5.2.51}$$

So by the Strong Maximum Principle Caracterization, we (5.2.50) and (5.2.51) that

$$v_\lambda>0\;\forall\, x\in A_R.$$

By the definition of v_λ, it follows that

$$u_\lambda(x)\geq u_\lambda(x_\lambda)+c_\lambda\psi_\lambda(x_\lambda)\;\forall\, x\in A_R\;\forall\,\lambda\geq\Lambda_1. \tag{5.2.52}$$

Now let us define

$$n_\lambda:=\frac{Y(x_\lambda)-x_\lambda}{R}.$$

By definition,

$$\frac{\partial u_\lambda}{\partial n_\lambda}(x_\lambda)=\lim_{t\to0^+}\frac{u_\lambda(x_\lambda+tn_\lambda)-u_\lambda(x_\lambda)}{t}.$$

Since $x_\lambda + tn_\lambda \in A_R$, whenever $0 < t < R/2$, then we can use (5.2.52) in order to get

$$\begin{aligned} \frac{u_\lambda(x_\lambda + tn_\lambda) - u_\lambda(x_\lambda)}{t} &\geq \frac{c_\lambda \psi_\lambda(x_\lambda + tn_\lambda)}{t} \\ &= \frac{c_\lambda \left(e^{-\alpha(R-t)^2} - e^{-\alpha R^2} \right)}{t} \end{aligned}$$

Since

$$\lim_{t \to 0^+} \frac{e^{-\alpha(R-t)^2} - e^{-\alpha R^2}}{t} = 2\alpha R e^{-\alpha R^2},$$

we get that

$$\frac{\partial u_\lambda}{\partial n_\lambda}(x_\lambda) \geq 2\alpha R e^{-\alpha R^2} c_\lambda,$$

by whence

$$\lim_{\lambda \to +\infty} \frac{\partial u_\lambda}{\partial n_\lambda}(x_\lambda) = \infty. \tag{5.2.53}$$

By H_{2d}), it follows that there exists $\varepsilon > 0$ such that

$$\emptyset \neq \mathcal{V} := \{x \in \Omega \setminus \overline{D_0}; \mathrm{dist}(x, \Gamma) < \varepsilon\} \subset V_0$$

(see Figure 10).

Note that $\partial\Gamma = \Gamma \cup \Gamma_1$. Now consider the problem

$$\begin{cases} -\Delta w = \mu(\lambda) \inf_{\mathcal{V}} m w - b(x) w^p & \text{in } \mathcal{V} \\ w = u_\lambda(x_\lambda) & \text{on } \Gamma \\ w = 0 & \text{on } \Gamma_1, \\ w > 0 & \text{in } \mathcal{V}. \end{cases} \tag{5.2.54}$$

Due to Theorem A.3.1, the problem (5.2.54) admits a unique positive solution w_λ. Note that u_λ is a supersolution of (5.2.54) and, for any $0 \leq k \leq 1$, kw_λ is a subsolution of (5.2.54). Assuming that k is small enough such that $kw_\lambda \leq u_\lambda$, we deduce that $kw_\lambda \leq w_\lambda \leq u_\lambda$ by the method of sub and supersolution. Moreover, by combining $w_\lambda \leq u_\lambda$ with $w_\lambda(x_\lambda) = u_\lambda(x_\lambda)$, it follows that

$$\frac{\partial w_\lambda}{\partial n_\lambda}(x_\lambda) \geq \frac{\partial u_\lambda}{\partial n_\lambda}(x_\lambda) \; \forall \, \lambda \geq \Lambda_1.$$

Since (5.2.53), then

$$\lim_{\lambda\to+\infty}\frac{\partial w_\lambda}{\partial n_\lambda}(x_\lambda)=\infty. \tag{5.2.55}$$

Let $\overline{w}$ be the positive solution of

$$\begin{cases} -\Delta w=\sup\limits_{\lambda\geq 0}\mu(\lambda)\inf\limits_{\mathcal{V}} mw-b(x)w^p & \text{in } \mathcal{V},\\ w=\sup\limits_{\lambda\geq 0}u_\lambda(x_\lambda) & \text{on } \Gamma,\\ w=0 & \text{on } \Gamma_1,\\ w>0 & \text{in } \mathcal{V}. \end{cases} \tag{5.2.56}$$

Given $\lambda\geq 0$, consider $0\leq k(\lambda)\leq 1$ such that $k(\lambda)w_\lambda\leq\overline{w}$. Thus $k(\lambda)w_\lambda\leq w_\lambda\leq\overline{w}$ by the method of sub and supersolution. Consequently,

$$\sup_{\lambda\geq 0}\|w_\lambda\|_0<+\infty. \tag{5.2.57}$$

Now let us apply Theorem A.2.2. Let us define $B_\lambda(x,w)=\mu(\lambda)\left(\inf\limits_{R} m\right)w-b(x)w^p$. Observe that since B_λ does not depend on ∇w, it follows that the B_λ satisfies the conditions of Theorem A.2.2, where the constants of (A.2.1) does not depend on λ. Moreover, there exists $M_0>0$ such that $w_\lambda\leq M_0$ for all $\lambda\geq\Lambda_1$ due to (5.2.57). Then we can conclude that

$$\sup_{\lambda\geq\Lambda_1}\|w_\lambda\|_{1,\alpha}<+\infty \tag{5.2.58}$$

due to Theorem 1 of [40], which contradicts (5.2.55).

□

In the following, we will proof the items of Theorem 0.0.6 in ascending order.

Proof of item 1i) of Theorem 0.0.5: It follows directly by applying Theorem 4.2.2.

Proof of item 1ii) of Theorem 0.0.5: The blow-up (0.0.22) follows directly by Corollary 5.2.1.

The convergence (0.0.24) follows from Theorem 5.2.2.

Now fix $\lambda \in [0,\Lambda)$, let us denote $m_+ = \max\{0,m\}$ and let U_λ be the unique positive solution of the problem

$$\begin{cases} -\Delta u + \Lambda V(x)u = \|\mu\|_\infty \|m_+\|_\infty u - b(x)u^p \text{ in } \Omega\setminus\bar{B_0}, \\ u = \sup\limits_{x\in\partial B_0} u_\lambda \text{ on } \partial B_0, \\ u = 0 \text{ on } \partial\Omega\setminus\partial B_0, \\ u > 0 \text{ in } \Omega\setminus\bar{B_0}, \end{cases} \tag{5.2.59}$$

whose existence and uniqueness are guaranteed by Corollary 4.4 of [45]. Note that if $K \geq 1$, then KU_λ is a supersolution of (5.2.59). Moreover, $u_\lambda|_{\overline{\Omega}\setminus B_0}$ is a subsolution. Take $K = K(\lambda)$ sufficiently large such that $u_\lambda|_{\overline{\Omega}\setminus B_0} \leq KU_\lambda$. By using the subsupersolution method and the uniqueness of positive solution of (5.2.59), we deduce that

$$u_\lambda|_{\overline{\Omega}\setminus B_0} \leq U_\lambda \leq KU_\lambda.$$

and so the proof of (0.0.25) proceeds by following the same steps of Theorem 1.1 of [39].

Proof of items 2i) and 2ii) of Theorem 0.0.5: It follows directly from Theorem 4.2.3.

Proof of item 2iii) of Theorem 0.0.5: (0.0.26) follows from Corollary 5.2.1. The convergence (0.0.27) follows from Theorem 5.2.3.

Now assume that $\bar{B}_0 \subset V_0 \subset \bar{V}_0 \subset \Omega$. Fix $\lambda \in [0,+\infty)$ and consider the unique positive solution U_λ of

$$\begin{cases} -\Delta u + \lambda V(x) = \sigma_1^{D_0}[-\Delta; m]\|m_+\|_\infty u - b(x)u^p & \text{in } \Omega \\ u = \sup\limits_{x\in\partial B_0} u_\lambda & \text{on } \partial B_0, \\ u = 0 & \text{on } \partial\Omega\setminus\partial B_0, \\ u > 0 & \text{in } \Omega \end{cases} \tag{5.2.60}$$

whose existence and uniqueness is guaranteed by Corollary 4.4 of [45]. Note that if $K \geq 1$, then KU_λ is a supersolution of (5.2.60). Moreover, $u_\lambda|_{\overline{\Omega}\setminus B_0}$ is a subsolution. Take $K = K(\lambda)$ sufficiently large such that $u_\lambda|_{\overline{\Omega}\setminus B_0} \leq KU_\lambda$. By applying the subsupersolution method and the uniqueness of positive solution of (5.2.60), we deduce that

$$u_\lambda|_{\overline{\Omega}\setminus B_0} \leq U_\lambda \leq KU_\lambda$$

and so the proof of (0.0.25) proceeds by following the same steps of Theorem 1.1 of [39]. Consider any compact subset $K \subset \Omega \setminus \bar{V}_0$. By testing against a positive $\varphi \in C_0^\infty(K)$ in the definition of U_λ, we deduce that

$$\liminf_{\lambda \to +\infty} \int_K V(x) U_\lambda^2 < \infty.$$

Consequently $\lim_{\lambda \to +\infty} U_\lambda(x) = 0$ for all $x \in \Omega \setminus \bar{V}_0$. So the proof of (0.0.28) proceeds by following the same steps of Theorem 1.1 of [39].

Proof of item 3) of Theorem 0.0.5: Let λ_n be any sequence as in the hypothesis of item 3). Consider a fixed $n \in \mathbb{N}$. According to Theorem 5.2.1, the following convergences hold.

$$\lim_{\mu \downarrow \sigma_1^\Omega[-\Delta + \lambda_n V; m]} \|u_{\lambda_n,\mu}\|_\infty = 0 \tag{5.2.61}$$

and

$$\lim_{\mu \uparrow \sigma_1^{B_0}[-\Delta + \lambda_n V; m]} u_{\lambda_n,\mu} = +\infty \text{ uniformly in compact subsets of } B_0. \tag{5.2.62}$$

Let $\sigma_1^\Omega[-\Delta + \lambda_n V; m] < \underline{\mu}_n(\lambda_n) < \sigma_1^{B_0}[-\Delta + \lambda_n V; m]$. By (5.2.61), we imply that we can assume that $\underline{\mu}_n(\lambda_n)$ is sufficiently close to $\sigma_1^\Omega[-\Delta + \lambda_n V; m]$ such that the positive solution $\underline{u}_n$ of ($S_{\lambda,\mu}$) associated with $(\lambda_n, \underline{\mu}_n(\lambda_n))$ satisfy

$$\|\underline{u}_n\|_\infty < \frac{1}{n}.$$

On the other hand, by (5.2.62) we deduce that there exists $\overline{\mu}_n(\lambda_n)$ and a positive solution $\overline{u}_n$ of ($S_{\lambda,\mu}$) associated to $(\lambda_n, \overline{\mu}_n(\lambda_n))$ and $x_n \in B_0$ such that

$$\overline{u}_n(x_n) > n.$$

In order to conclude the proof of item 3) of Theorem 0.0.6, we must show that

$$\lim_{n \to +\infty} \underline{\mu}(\lambda_n) = \lim_{n \to +\infty} \overline{\mu}(\lambda_n) = \sigma_1^{D_0}[-\Delta; m]. \tag{5.2.63}$$

Note that since $(\lambda_n, \underline{\mu}(\lambda_n)), (\lambda_n, \overline{\mu}(\lambda_n)) \in \mathcal{S}$, then

$$\sigma_1^\Omega[-\Delta + \lambda_n V; m] < \underline{\mu}_n(\lambda_n) < \sigma_1^{B_0}[-\Delta + \lambda_n V; m]$$

and

$$\sigma_1^{\Omega}[-\Delta+\lambda_n V;m] < \overline{\mu}_n(\lambda_n) < \sigma_1^{B_0}[-\Delta+\lambda_n V;m].$$

By passing to the limit in the inequalities above, using Lemma 1.2.1 and the fact that $B_0 \cap V_0 = V_0$, we deduce (5.2.63).

Conclusion of Chapter 5

As noted in the introduction of the thesis, the information provided by Theorems 0.0.5 and 0.0.6 imply some interesting inferences about the models $(R_{\lambda,\mu})$ and $(S_{\lambda,\mu})$, in the point of view of population dynamics.

In the case of Theorem 0.0.5, it is worth mention some open problems. The behavior of the positive solutions u_λ of $(R_{\lambda,\mu})$ when $\lambda \to \infty$ in the case $\sigma_1^{\Omega}[-\Delta;\chi_{A_0}] = \sigma_1^{A_+\cup B_0}[-\Delta;\chi_{A_0}] = \mu$. The behavior of u_λ in the boundary of ∂C_i when $\lambda \to \infty$. Due to the lack of continuity in the convergence $1/(1+\lambda a(x))$ to χ_{A_0} when $\lambda \to \infty$, we were not capable of applying the ideas of Section 1.3. On the other hand, since the set ∂C_i is not necessarily contained in $\{x \in \Omega; \chi_{A_0}(x) > 0\}$, then ∂C_i does not satisfy the conditions to apply the same arguments used in the proof of Theorem 5.2.3.

Theorem 0.0.6 also leave some open problems. For example, the behavior of the positive solutions u_λ in the boundary of B_0 when $\lim_{\lambda\uparrow\Lambda}\mu_0(\lambda) = \sigma_1^{B_0}[-\Delta+\Lambda V;m]$, without requiring the transversality condition (0.0.23). Also the hypothesis about the analiticity of m, required in order to obtain the blow-up in the boundary of u_λ, excludes functions m such that

$$\operatorname{int}\{x \in \Omega; m(x) = 0\} \neq \emptyset.$$

Finally the relaxation of the hypothesis $\Gamma \subset V_0$ in item $2iii)$, which is is intimately related with the theory of metasolutions of [45]. Indeed, this hypothesis implies that $\partial\mathcal{V}$ is composed by two connected components Γ and Γ_1. This condition is crucial in the above proof, in order to guarantee (5.2.58). In fact, the a priori boundedness (5.2.58) is a consequence of the boundary conditions of the problem (5.2.54) to be $\mathcal{C}^1$ bounded uniformly with respect to λ, once that they are constants. Without the aforementioned condition in Γ, $\partial\mathcal{V}$ would possibly be a unique component and the technique involved in the proof would force us to define the boundary conditions of the problem (5.2.54) that would possibly not be $\mathcal{C}^1$ bounded uniformly with respect to λ. This technical obstacle is an open problem.

Conclusion

Proposing Theorem 0.0.1 of continuation of solutions for operators satisfying a notion of compactness in an open subset, we were able to obtain a connected of positive solutions of ($P_{\lambda,\mu}$) and ($Q_{\lambda,\mu}$). The asset of the operator to be defined on a open subset, allowed us to find solutions very close to a certain singularity region of the working parameter (see 0.0.4 and 0.0.7). The same asset, also gave us explicit values of the parameter λ for what we have positive solutions of ($P_{\lambda,\mu}$) (see 0.0.5). The abstract formulation of Theorem 0.0.1 makes it possible to applying it for different families of PDEs that include terms that can present technical difficulties. For example, the nonlinearity $u\Delta(u^2)$ in ($P_{\lambda,\mu}$) and the nonlocal term $|u|_r$ in ($Q_{\lambda,\mu}$).

The refinement of properties for positive eigenfunctions and their first eigenvalues with respect to a varying parameter λ (as seen in Lemma 1.1.1, Theorem 1.2.1, and Theorem 1.3.2), coupled with the a priori boundedness from Theorem 4.2.1 and the subsolution provided by Theorem 4.3.1, enabled us to prove fine qualitative information about the behavior of positive solutions for the logistic models ($R_{\lambda,\mu}$) and ($S_{\lambda,\mu}$).

Some questions about the problems ($P_{\lambda,\mu}$), ($Q_{\lambda,\mu}$), ($R_{\lambda,\mu}$) and ($S_{\lambda,\mu}$) remain open as noted in the conclusions of Chapters 2, 3 and 5. For the sake of completeness of this conclusion, we will point out three of what we consider the most challenging of them. First, the existence of solutions of ($P_{\lambda,\mu}$) (respectively, ($Q_{\lambda,\mu}$)) outside $\mathcal{U}$ (respectively, $\mathcal{V}$). Second, whether the blow-up of the positive solutions u_λ of ($S_{\lambda,\mu}$), in ∂B_0, when $\lambda \to \Lambda$ still occurs without the transversality condition. Lastly, whether the solutions u_λ of ($S_{\lambda,\mu}$) blows-up in a component Γ of $\partial(V_0 \cap B_0)$ without requiring $\Gamma \subset V_0$.

Appendix A

Some results on strong maximum principle, regularity and metasolutions

To ensure completeness, we have included in this appendix specific results from the literature that we considered pertinent to the preceding text.

A.1 Strong Maximum Principle

Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$ and $c \in L^\infty(\Omega)$. We will denote $L = -\Delta + c$. In line with [44], we define the following.

Definition A.1.1. *A function $u \in W^{2,q}(\Omega)$, with $q > N$, is said to be a supersolution of L if*

$$\begin{cases} -\Delta u + cu \geq 0 \text{ in } \Omega, \\ \qquad\quad u \geq 0 \text{ on } \partial\Omega. \end{cases}$$

If one of the above inequalities is strict in some subset of Ω with positive measure, then u is said to be a strict supersolution of L.

Theorem A.1.1 (Krein-Rutman)**.** *The operator L admits a real eigenvalue $\sigma_1[L]$, called the first eigenvalue of L, which is simple and is associated with a unique eigenfunction, up to a multiplicative constant, and it can be assumed to be positive. Moreover, $\sigma_1[L]$ is the only eigenvalue associated with a positive eigenfunction. Any other eigenvalue $\sigma \in \mathbb{R}$ of L must satisfy $\sigma_1[L] \leq \sigma$.*

Theorem A.1.2. *$\sigma_1[L] > 0$ if and only if L admits a strict supersolution. Moreover, if $\sigma_1[L] > 0$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a supersolution of L, then $u > 0$ in Ω.*

For more details about the above results see [4] and [44].

A.2 Regularity

Theorem A.2.1. *Let $0 \leq c \in L^\infty(\Omega)$. Then there is a constant $C > 0$ independent of u such that*

$$\|u\|_{W^{2,q}(\Omega)} \leq C|(-\Delta + c)u|_{L^q(\Omega)}$$

for all $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, $1 < q < \infty$.

See Lemma 9.17 of [34].

Theorem A.2.2. *Let* α, λ, Λ *and* M_0 *be positive constants with* $\alpha \leq 1$ *and* $\Lambda \geq \lambda$. *Let* κ *and* Φ *be nonnegative constants, and let* Ω *be a bounded domain in* $\mathbb{R}^N$ *with* $C^{1,\alpha}$ *boundary. Assume that A and B satisfy the following conditions*

$$a^{ij}(x,z,p)\xi_i\xi_j \geq \lambda(\kappa + |p|)^m|\xi|^2,$$

$$|a^{ij}(x,z,p)| \leq \Lambda(\kappa + |p|)^m,$$

$$|A(x,z,p) - A(y,w,p)| \leq \Lambda(1+|p|)^{m+1}(|x-y|^{\alpha} + |z-w|^{\alpha}),$$

$$|B(x,z,p)| \leq \Lambda(1+|p|)^{m+2} \tag{A.2.1}$$

for all $(x,z,p) \in \partial\Omega \times [-M_0, M_0] \times \mathbb{R}^N$, *all* $(y,w) \in \Omega \times [-M_0, M_0]$, *and all* $\xi \in \mathbb{R}^N$. *If* ϕ *is in* $C^{1,\alpha}(\partial\Omega)$ *with* $|\varphi|_{1+\alpha} \leq \Phi$ *and if u is a bounded weak solution of the Dirichlet problem*

$$\operatorname{div} A(x,u,\nabla u) + B(x,u,\nabla u) = 0 \textit{ in } \Omega, u = \phi \textit{ on } \partial\Omega$$

with $|u| \leq M_0$ *in* Ω, *then there is a positive constant* $\beta = \beta(\alpha, \Lambda/\lambda, m, n)$ *such that u is in* $C^{1,\beta}(\Omega)$. *Moreover,*

$$|u|_{1+\beta} \leq C(\alpha, \Lambda/\lambda, m, M_0, n, \Phi, \Omega).$$

The interested reader may see [40].

Theorem A.2.3. *Let* $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *be a continuous function such that* $\partial f/\partial\xi$ *and* $\partial f/\partial\eta$ *exist and are continuous where* (x,ξ,η) *denotes a generic point of* $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$. *Assume also that there is an increasing function* $c : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|f(x,\xi,\eta)| \leq c(|\xi|)(1+|\eta|^2) \; \forall \; (x,\xi,\eta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N.$$

Then there is an increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *such that if* $u \in W^{2,q}(\Omega)$, $q > N$, *is a solution of*

$$\begin{cases} -\Delta u = f(x,u,\nabla u) \textit{ in } \Omega, \\ \quad u = 0 \textit{ on } \partial\Omega, \end{cases}$$

then

$$\|u\|_{W^{2,q}(\Omega)} \leq \gamma(\|u\|_{C(\overline{\Omega})}).$$

See [5], for more details.

A.3 Metasolutions

When dealing with diffusive logistic models with refuge, the concept of metasolution arises from the presence of a subregion of the habitat where the carrying capacity degenerates to infinity (see [45]). The following result from [45] will be used in this work.

Let $0 \leq b \in \mathcal{C}(\overline{\Omega})$ and denote $B_0 = \operatorname{int}\{x \in \overline{\Omega}; b(x) = 0\}$. Assume that $\partial(\Omega \setminus \overline{B}_0) = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$ and that if $\Gamma_i \cap \overline{B}_0 \neq \emptyset$, for some $i \in \{1,2\}$, then $\Gamma_i \subset \overline{B}_0$.

Theorem A.3.1. *Given* $\lambda \in \mathbb{R}$, *the problem*

$$\begin{cases} -\Delta u = \lambda u - b(x)u^p & \text{in } \Omega \setminus \overline{B}_0 \\ \quad u = M & \text{on } \Gamma_1, \\ \quad u = 0 & \text{on } \Gamma_2 \\ \quad u > 0 & \text{in } \Omega \setminus \overline{B}_0 \end{cases}$$

possesses a unique positive solution.

The above result is a particular case of Corollary 4.4 of [45].

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