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On soluble groups of finite rank with finitely many automorphism orbits

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Estudar automorfismos é decifrar como
algo muda sem deixar de ser o que é.

Livro das Funções

Resumo

Seja G um grupo. As órbitas da ação natural de $\mathrm{Aut}(G)$ em G são chamadas de "órbitas por automorfismos" de G, e o número de órbitas por automorfismos de G é denotado por $\omega(G)$. Se G é um grupo solúvel de posto finito com número finito de órbitas por automorfismos, então G possui um subgrupo característico nilpotente radicável e livre de torção K tal que $G = K \rtimes H$, onde H é um subgrupo finito (ver [4]). Provamos um teorema estrutural para grupos solúveis de posto finito com $\omega(G) = 4$ e $\omega(G) = 5$. Além disso, a decomposição $G = K \rtimes H$ nos levou a investigar grupos nilpotentes radicáveis e livres de torção. Neste contexto, provamos que o grupo das matrizes unitriangulares de dimensão n sobre o corpo dos números racionais $\mathbb{Q}$ possui infinitas órbitas por automorfismos quando $n > 5$, e um número finito de órbitas quando $n \leq 5$. Estes últimos resultados foram obtidos com o auxílio de métodos computacionais.

Título em português: Sobre grupos solúveis de posto finito com finitas órbitas por automorfismos.

Abstract

Let G be a group. The orbits of the natural action of $\mathrm{Aut}(G)$ on G are called "automorphism orbits" of G, and the number of automorphism orbits of G is denoted by $\omega(G)$. If G is a soluble group of finite rank with finitely many automorphism orbits, then G has a torsion-free radicable nilpotent characteristic subgroup K such that $G = K \rtimes H$, where H is a finite subgroup (see [4]). We prove a structure theorem about mixed order soluble groups of finite rank satisfying $\omega(G) = 4$ and $\omega(G) = 5$. Moreover, the decomposition $G = K \rtimes H$ led us to investigate torsion-free radicable nilpotent groups. In this topic, we prove that the unitriangular group of dimension n over the field of rational numbers $\mathbb{Q}$ has infinitely many orbits under the action of its automorphism groups when $n > 5$, and finitely many orbits when $n \leq 5$. These last result are obtained with the aid of computational methods.

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List of symbols

$\mathrm{Aut}(G)$	automorphism group of G
$\lvert G\rvert$	cardinality of G
$\omega(G)$	number of automorphism orbits of G
g^{α}	image of element g under automorphism α
g^{h}	conjugate of g by h: $h^{-1}gh$
$[g,h]$	commutator of g and h: $g^{-1}h^{-1}gh$
$Z(G)$	center of G
$C_G(H)$	centralizer of H in G
G'	commutator subgroup of G
$G^{(n)}$	nth term of derived series of G
$K \times H$	direct product of K and H
$K \rtimes H$	semidirect product of K and H
C_n	cyclic group of order n
$\mathbb{Q}$	group/set of rational numbers
$UT_n(\mathbb{Q})$	group of unitriangular matrices of dimension n over $\mathbb{Q}$
$D_n(\mathbb{Q})$	group of diagonal matrices of dimension n over $\mathbb{Q}$
$T_n(\mathbb{Q})$	group of triangular matrices of dimension n over $\mathbb{Q}$
$\det(A)$	determinant of matrix A
$\mathbb{F}_p[G]$	group algebra of G and $\mathbb{F}_p$
$\mathbb{F}_p$	field of order p
$\mathrm{char}(F)$	characteristic of F
$A \setminus B$	elements of A not in B
$G^{\#}$	$G \setminus 1$

Introduction

Let G be a group, and let $\mathrm{Aut}(G)$ denote its automorphism group. The group G is partitioned into orbits under the natural action of $\mathrm{Aut}(G)$, where two elements $g, h \in G$ are in the same orbit if there exists an automorphism $\alpha \in \mathrm{Aut}(G)$ such that $g^{\alpha} = h$. These orbits are referred to as *automorphism orbits* and the number of automorphism orbits is denoted by $\omega(G)$. A finite group is called a *k-orbit group* if it has exactly k automorphism orbits.

The identity group is the only example of an 1-orbit group. Furthermore, it is straightforward to verify that the finite 2-orbit groups are exactly the elementary abelian groups of prime-power order. Interestingly, an equivalent result for infinite groups remains unknown. Higman, Neumann, and Neumann constructed a non-abelian torsion-free simple group in which all nontrivial elements are conjugate ([23] (6.4.6)) showing the difference of infinite groups in this context.

The earliest works in the area of automorphism orbits focused on finite groups. The investigation into the relationships between groups and their automorphism orbits began with Higman [13] in 1963. Higman initially explored finite 2-groups in which the involutions constitute a single automorphism orbit. It is worth noting that a related problem had emerged earlier: the classification of groups with a prescribed number of conjugacy classes, which are the orbits under inner automorphisms. This problem was proposed by W. Burnside [8] in 1955 . Since then, the study of automorphism orbits has become a prominent area of interest among group theorists. Observe that automorphism orbits are unions of conjugacy classes and hence they give an example of *fusion* in the holomorph group $G \rtimes \mathrm{Aut}(G)$, a well-known concept established in the literature (see for instance [11, Chapter 7]).

Laffey and MacHale [16] in 1986 classified all finite soluble non-p-groups with $\omega(G) = 3$. We will explore this result in greater detail later on. Recently, finite 2-groups that are 3-orbit groups were classified by Bors and Glasby [7], while 3-orbit groups that are p-groups with odd prime p were classified by Li and Zhu [18], completing the classification of finite 3-orbit groups. Laffey and MacHale [16] in 1986 also described the structure for finite soluble non-p-groups with $\omega(G) = 4$. In the same work they proved that if G is a finite non-soluble group with $\omega(G) \leq 4$, then G is isomorphic to $PSL(2,4)$. These results represent one of the main approaches in the study of automorphism orbits: given a fixed number of orbits, the goal is to describe the structure of the group accordingly.

Later, Stroppel [27], showed that the only finite non-abelian simple groups G with $\omega(G) \leq 5$ are the groups $PSL(2,q)$ with $q \in \{4,7,8,9\}$. In [3], Bastos, Dantas and Garonzi proved that if G is a finite non-soluble group with $\omega(G) \leq 6$, then G is isomorphic to one of $PSL(2,q)$ with $q \in \{4,7,8,9\}$, $PSL(3,4)$ or $ASL(2,4)$. Studies of r-orbit groups for small values of $r > 3$ mainly focus on non-solvable groups. Finite simple r-orbit groups for $r \leq 100$ are determined in [15], and non-solvable r-orbit groups for $r \in \{4,5,6\}$ are classified (see [3], [16]).

Finite groups with certain special automorphism orbits have also received significant attention. Shult proved that a finite p-group for an odd prime p, in which the elements of order p form an automorphism orbit, is abelian [26]. In 1976, Gross [12] studied finite 2-groups whose involutions form an automorphism orbit. Finite groups in which elements of the same order lie in the same automorphism orbit, known as AT-groups, were studied by Zhang [30]. Many aspects of this topic have been explored, yet the area still presents numerous open questions that can serve as powerful motivation for further research.

These contributions are of significant interest, however, the main subject of this thesis is infinite groups, and we now turn our attention to their study. Schwachhöfer and Stroppel [25], in 1999, showed that if G is an abelian group with finitely many automorphism orbits, then

$$G = \mathrm{Tor}(G) \oplus D,$$

where D is a torsion-free radicable characteristic subgroup of G, and $\mathrm{Tor}(G)$ is the set of all torsion elements in G.

In 2017, Bastos and Dantas [2], proved that if G is an FC-group with finitely many automorphism orbits, then the derived subgroup G' is finite and G admits a decomposition

$$G = \mathrm{Tor}(G) \times A,$$

where A is a radicable characteristic subgroup of $Z(G)$. For more details concerning automorphism orbits of groups and infinite groups, see [27].

The main object of study in this thesis is soluble groups of finite rank. While the term ‘rank’ has many connotations in algebra, in soluble group theory it refers to the cardinality of a maximal linearly independent subset of some kind. In this thesis, we adopt the Prüfer rank.

Definition 0.1. A group is said to have finite *Prüfer rank* r if every finitely generated subgroup can be generated by r elements and r is the least such integer.

For example, $\mathbb{Q}^n$ has rank n, for any positive integer n. For simplicity, we will refer to it simply as *rank* throughout the text. Recall that a group G has *mixed order* if it contains non-trivial elements of finite order and also elements of infinite order. Note that a group G is said to be *radicable* if for every element g in G and every positive integer n, there exists an element h in G such that $h^n = g$.

In 2020, Bastos, Dantas and de Melo [4] proved

Theorem 0.1 ([4], Theorem A)**.** Let G be a soluble group of finite rank. If $\omega(G) < \infty$, then G has a torsion-free radicable nilpotent characteristic subgroup K such that

$$G = K \rtimes H,$$

where H is a finite subgroup.

The theorem above is essential to our work. It not only provides a key argument in the proof of a couple of our results, but also serves as a motivation to explore related classes of groups. In the same work they classified the mixed order soluble groups of finite rank such that $\omega(G) = 3$.

Theorem 0.2 ([4], Theorem B)**.** Let G be a mixed order soluble group of finite rank. We have $\omega(G) = 3$ if and only if $G = A \rtimes H$ where $|H| = p$ for some prime p, H acts fixed-point-freely on A and $A \cong \mathbb{Q}^n$ for some positive integer n.

Furthermore, Bastos, Dantas and de Melo in 2021 extended the results in [5] for virtually nilpotent groups such that $\omega(G) < \infty$.

Theorem 0.3 ([5], Theorem 1.1)**.** Let A be an abelian group and B a finite subgroup of $\mathrm{Aut}(A)$. Let $G = A \rtimes B$ be the semidirect product of A and B and assume that A is a characteristic subgroup of G. Then $\omega(G) < \infty$ if and only if A has finitely many automorphism orbits under the action of $C_{\mathrm{Aut}(A)}(B)$.

Theorem 0.4 ([5], Corollary 1.3)**.** Let G be a virtually nilpotent group with $\omega(G) < \infty$. Then G has a torsion-free radicable nilpotent subgroup K and a torsion subgroup H such that $G = K \rtimes H$. Moreover, the derived subgroup $G^{'} = D \times \mathrm{Tor}(G^{'})$, where D is a torsion-free nilpotent radicable characteristic subgroup.

Theorem 0.5 ([5], Corollary 1.4)**.** Let A be a finite dimensional vector space over $\mathbb{Q}$ and B a finite subgroup of $\mathrm{Aut}(A)$. Let $G = A \rtimes B$ be the semidirect product of A and B. Then $\omega(G) < \infty$ if and only if B is abelian.

Inpired by these works, we invetigated the structure of groups with 4 automorphism orbits. We obtained the following results. Throughout this paper, if K is a characteristic subgroup of a group G, we denote by $\omega_{\mathrm{Aut}(G)}(K)$ the number of automorphism orbits of K under the action of $\mathrm{Aut}(G)$.

Theorem A. Let G be a mixed order soluble group of finite rank with $\omega(G) = 4$. Then $G = K \rtimes H$, where K is a torsion free nilpotent characteristic radicable subgroup, H is a finite subgroup and one of the following three cases holds (q is a prime number):

1. H acts fixed-point-freely on K, $|H| = q$, and $\omega_{Aut(G)}(K) = 3$;
2. $K = \mathbb{Q}^n$, H is cyclic of order q^2 and the action of H is fixed-point-free;
3. $G = K \times H$, where $K = \mathbb{Q}^n$ and H is elementary abelian q-group.

We also establish a necessary and sufficient condition for a mixed order metabelian group to have exactly four automorphism orbits, as stated in the following theorem.

Theorem B. Let G be a mixed order metabelian group of finite rank. Then $\omega(G) = 4$ if and only if $G = K \rtimes H$ where one of the following holds:

1. $K = \mathbb{Q}^n$, H is cyclic of order q^2, the action of H is fixed-point-free and K decomposes as a $\mathbb{Q}[H]$-module into a direct sum of copies of an isomorphic irreducible $\mathbb{Q}[H]$-module;
2. $G = K \times H$, where $K = \mathbb{Q}^n$ and H is elementary abelian.

Now we consider infinite soluble groups of finite rank with five automorphism orbits.

Theorem C. Let G be a mixed order soluble group of finite rank with $\omega(G) = 5$. Then $G = K \rtimes H$, where K is a torsion-free nilpotent characteristic radicable subgroup, H is a finite subgroup and one of the following holds:

1. K is non-abelian, $G/Z(K)$ is non-abelian, and $\omega(G/Z(K)) \leq 4$;
2. $K = \mathbb{Q}^n$, $\omega_{Aut(G)}(K) = 3$, and $|H| = q^2$ for some prime q;
3. $K = \mathbb{Q}^n$, $\omega_{Aut(G)}(K) = 2$, $Z(G)$ is elementary abelian q-group and $\omega(G/Z(G)) = 3$.
4. $K = \mathbb{Q}^n$, $\omega_{Aut(G)}(K) = 2$, and H is a cyclic group with $\omega(H) = 4$.

Another approach we explored in our research, based on the decomposition provided by Theorem 0.1, is to consider torsion-free radicable nilpotent groups. Naturally, this leds us to investigate unitriangular matrices $UT_n(\mathbb{Q})$ of dimension $n \times n$ over the field of rational numbers $\mathbb{Q}$, as they present a natural example within this class of groups.

Definition 0.2. The *upper unitriangular matrix group* of dimension $n \times n$ over the field $\mathbb{Q}$, denoted by $UT_n(\mathbb{Q})$, is the group, under multiplication, with 1's on the diagonal, 0's below the diagonal, and arbitrary entries above the diagonal.

It is worth mentioning that $UT_n(\mathbb{Q})$ also belongs to the class of soluble groups of finite rank, for any positive integer n. This will be discussed when we consider the theoretical properties of this group.

The group $UT_n(\mathbb{Q})$ can be viewed as a subgroup of $GL_n(\mathbb{Q})$. In this context, it is known that its normalizer is $T_n(\mathbb{Q})$, the group of upper triangular matrices of dimension $n \times n$ over $\mathbb{Q}$ (for details see [1]). In particular, this subgroup decomposes as the semidirect product

$$T_n(\mathbb{Q}) = UT_n(\mathbb{Q}) \rtimes D_n(\mathbb{Q}),$$

were $D_n(\mathbb{Q}) \leq GL_n(\mathbb{Q})$ is the subgroup of $n \times n$ diagonal matrices over $\mathbb{Q}$. Note that $D_n(\mathbb{Q}) \cong (\mathbb{Q}^\times)^n$.

We now briefly discuss some related results. Borel and Steinberg stated the following problem: Is the number of the conjugacy classes of unipotent elements in a semisimple algebraic group finite? This was solved by Platonov for $p > 2$ for a field of characteristic p. Platonov also conjectured that in $UT_n(\mathbb{Q})$ the number of the unipotent conjugacy classes in general is infinite. Zalesskii [29], in 1968, showed that the number in question is infinite in the triangular linear group $T_n(\mathbb{Q})$.

Theorem 0.6 ([29], Proposition 1)**.** Unipotent elements of the triangular linear group $T_n(\mathbb{Q})$, $n \geq 6$, partition into infinitely many conjugacy classes.

This result is particularly relevant to our research, as conjugacy classes are contained within automorphism orbits.

As our study focus on automorphism orbits, we now turn to some results in the literature concerning the automorphism group of unitriangular matrices. The automorphism group of the group of unitriangular matrices over a field was studied by many authors [17, 20, 22, 28] . The first paper was in Russian, published by Pavlov in 1953. Pavlov studied the automorphism group of unitriangular matrices over a finite field of odd prime order. Weir [28] described the automorphism group of the group of unitriangular matrices over a finite field of odd characteristic. Maginnis [22] described it for the field of order two.

Finally, it was proved by Levchuk [17], in 1983, and reproved by Mahalanobis [20], in 2013, that the automorphism group of $UT_n(\mathbb{Q})$ is generated by certain automorphisms.

Theorem 0.7 ([20], Theorem 3.3)**.** The automorphism group of $UT_n(\mathbb{Q})$ is generated by extremal automorphisms, field automorphisms, diagonal automorphisms (conjugation by the diagonal matrices), inner automorphisms and central automorphisms (that is, the identity modulo the center).

Further details regarding these automorphisms will be provided in the Chapter 4. Inspired by these works we obtained the following results,

Theorem D. The unitriangular group $UT_n(\mathbb{Q})$ has finitely many automorphism orbits for $n < 5$. In particular, $\omega(UT_3(\mathbb{Q})) = 3$ and $\omega(UT_4(\mathbb{Q})) = 9$.

Later, we will see that as the dimension increases, the computations involved become significantly more complex. In the case of dimension 5, we were able to show that the number of automorphism orbits is finite, and we computed the number of orbits under the action of the group of upper triangular matrices $T_n(\mathbb{Q})$.

Theorem E. The unitriangular group $UT_5(\mathbb{Q})$ has finitely many automorphism orbits. Moreover, the number of orbits under the action of $T_n(\mathbb{Q})$ is 61.

Determining this number required a case-by-case analysis of elements in $UT_5(\mathbb{Q})$. Specifically, it was necessary to consider all possible combinations in which each entry is either zero or nonzero, resulting in a large number of distinct cases. To handle this complexity, we used the software SageMath [24] to assist in the calculations. These calculations are fully presented in the Appendix.

The next result introduces a class of groups that does not have finitely many orbits under automorphism, which stands in contrast to the previous results. The result mentioned by Zalesskii [29] shows that the classes of conjugate elements of the group $UT_n(\mathbb{Q})$ under the action of diagonal automorphisms and inner automorphisms are infinite. The strategy used in the proof of this theorem allowed us to obtain the following result,

Theorem F. The unitriangular group $UT_n(\mathbb{Q})$ has infinitely many automorphism orbits for $n > 5$.

The present work is divided into three chapters. In Chapter 1, we briefly introduce some basic topics, discuss previous results from the literature and present examples that illustrate and explore these results. The examples presented constitute an essential part of this work, we used the software GAP [9] to find some of them. In Chapter 2, we focus on proving Theorems A, B and C. In Chapter 3, we turn our attention to the group of unitriangular matrices, where we prove Theorems D, E, and F, we also present structural properties to this class of groups.

Chapter 1

Preliminaries

This chapter is divided into two parts. In the first part we will introduce some fundamental concepts and results of group theory that are applied in this thesis. In the second part, we analyze some theorems and provide illustrative examples. Although much of the discussion in this chapter focuses on finite groups, this is the only chapter where such emphasis occurs. In the remainder of the thesis, our attention will be directed primarily toward infinite groups.

1.1 Representation of abelian groups and complete reducibility

Let G be a group and consider a finite-dimensional vector space V over the field F.

Definition 1.1. A homomorphism ϕ from a group G to the group $GL_n(F)$ is called a *representation* of G over V of degree n, where n is the dimension of V over F.

Example 1.1. Let $G = C_4 = \langle a \mid a^4 = 1 \rangle$ be the cyclic group of order 4. Define the matrix $A \in GL_2(\mathbb{Q})$ by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and observe that

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The function $\rho : G \to GL_2(\mathbb{Q})$ given by $a^i \mapsto A^i$, where $0 \leq i \leq 3$, is a representation of C_4 over $\mathbb{Q}^2$.

Definition 1.2. A representation ϕ of a group G on a vector space V over a field F is said to be *faithful* if the map ϕ in injective. Also a representation ϕ is said to be *linear* if it has degree 1.

Definition 1.3. A representation ϕ of a group G on a vector space V over a field F is said to be *irreducible* if 0 and V are the only G^{ϕ}-invariant subspaces of V. Otherwise, ϕ is said to be *reducible*.

Definition 1.4. A representation ϕ of G on V over F is said to be *completely reducible* if there exists a decomposition

$$V = V_1 \oplus \cdots \oplus V_r,$$

where each V_i is a G^{ϕ}-invariant subspace of V and $\phi|_{V_i}$ is irreducible, for $1 \leq i \leq r$.

Now we will study some properties of representations of abelian groups. Such results will help in proving one of the main theorems regarding the classification of groups with a finite number of orbits under automorphisms.

For more details on the following results, we refer the reader to [11].

Theorem 1.1. ([11], Theorem 3.2.2) If G has a faithful irreducible representation, then $Z(G)$ is cyclic.

Theorem 1.2. ([11], Theorem 3.2.3) If ϕ is a irreducible representation of an abelian group G with kernel K, then G/K is cyclic. In particular, a non-cyclic abelian group has no faithful irreducible representation.

Theorem 1.3. ([11], Theorem 3.2.4) Let G be an abelian group of order n and F a field that contains a primitive n-th root of unity. Then every irreducible representation of G over F is linear.

Theorem 1.4. ([11], Theorem 3.2.5) If ϕ is a linear representation of G, then G/K is cyclic, where K is the kernel of ϕ. In particular, a non-cyclic group has no faithful representation of degree 1.

Now, we will establish sufficient criteria for a given representation to be completely reducible. We will refer to this result as Maschke's Theorem.

Theorem 1.5 (Maschke)**.** ([11], Theorem 3.3.1) Let φ be a representation of a finite group G on a finite-dimensional vector space V over a field F and assume that either F is of characteristic 0 or of characteristic relatively prime to $|G|$. Then φ is completely reducible.

Next, we present an application of Maschke's Theorem.

Theorem 1.6. Let V be a finite-dimensional vector space over a field F, and let Q be a non-cyclic abelian q-subgroup of $\mathrm{Aut}(V)$, with q a prime distinct from the characteristic of F. Then,

$$V = \prod_{x \in Q^{\#}} C_V(x),$$

where $C_V(x) = \{v \in V : v^x = v\}$. In particular, V is generated by its subgroups $C_V(x)$ with x in $Q^{\#}$.

We are interested in understanding finite rank solvable groups with finitely many orbits under the action of their automorphism group. Theorems 0.2 and 0.5, mentioned in the introduction, guide our investigation toward a semidirect product structure. In particular, let $G = K \rtimes H$, where $K = \mathbb{Q}^n$ and H is a finite abelian group, for some positive integer n.

First note that H acts on K, viewed as a vector space over $\mathbb{Q}$, so every element of H can be seen as linear operators of K. Therefore, in the context of linear algebra, it will be very useful to understand some polynomials. The first one is the *characteristic polynomial*, defined by $\det(x\mathbb{1} - A_h)$, where $\mathbb{1}$ is the identity matrix and A_h is the matrix associated to the linear operator $h \in H$. Other useful polynomials are the polynomials that annihilate the elements of H, in the sense that if $p(x)$ is a polynomial over F, that annihilate linear operator h, then $p(h) = 0$.

Definition 1.5. Let h be a linear operator on a finite-dimensional vector space V over the field F. The *minimal polynomial* for h is the (unique) monic generator of the ideal of polynomials over F which annihilates h.

In this context, we present a very useful theorem. For more details see [14].

Theorem 1.7 (Cayley-Hamilton)**.** ([14], Theorem 6.4) Let h be a linear operator on a finite-dimensional vector space V. If f is the characteristic polynomial of h, then $f(h) = 0$; in other words, the minimal polynomial divides the characteristic polynomial.

We now present some interesting results regarding automorphisms orbits.

Theorem 1.8. If $K = \mathbb{Q}^n$, then $\omega(K) = 2$.

Proof. The group K can be viewed as a vector space of dimension n over $\mathbb{Q}$. For any two non zero vectors $x, y \in K$, we can define basis $\mathcal{B}_1 = \{v_1, \dots, v_n\}$ and $\mathcal{B}_2 = \{u_1, \dots, v_n\}$ such that $x = v_1$ and $y = u_1$. It is known that there exists a unique bijective linear map extending the map $v_i \mapsto u_i$. Thus, every non trivial element of K is in the same automorphism orbit. □

Theorem 1.9. Let $G = K \rtimes H$, where $K = \mathbb{Q}^n$ and H is a finite abelian group, for some positive integer n. Then K is a characteristic subgroup of G.

Proof. Let $|H| = m$. We show that $G^m = K$. Since K is divisible, we have that $K = K^m \leq G^m$. Now, $G/K \cong H$, so $G^m \leq K$. Therefore, $G^m = K$, and we conclude that K is characteristic in G, as desired. □

Note that the above result holds for any divisible group. The next result concerns the case where the group decomposes as the direct product.

Theorem 1.10. Let $G = K \times H$, where $K = \mathbb{Q}^n$, for some positive integer n, and H is a finite abelian group. Then $\omega(G) = 2\omega(H)$.

Proof. First, note that G is an abelian group so the torsion elements form a characteristic subgroup $Tor(G)$ of G. Since $Tor(G) = H$, we have that $H \operatorname{char} G$ and $K \operatorname{char} G$. So if $\alpha \in Aut(G)$, then $(k_1, 1)^\alpha = (k_2, 1)$ and $(1, h_1)^\alpha = (1, h_2)$, for $h_i \in H$ and $k_i \in K$, $i = 1, 2$. So,

$$(k_1, h_1)^\alpha = ((k_1, 1)(1, h_1))^\alpha = (k_1, 1)^\alpha (1, h_1)^\alpha = (k_2, h_2).$$

We prove that $\mathrm{Aut}(G) \cong \mathrm{Aut}(K) \times \mathrm{Aut}(H)$. Define the map

$$f : \mathrm{Aut}(G) \to \mathrm{Aut}(K) \times \mathrm{Aut}(H)$$
$$\alpha \mapsto (\pi_1 \alpha \iota_1, \pi_2 \alpha \iota_2)$$

where $\pi_1 : G \to K$, $\pi_2 : G \to H$ are the projection homomorphisms, $\iota_1 : K \to G$, $\iota_2 : H \to G$ are the inclusion homomorphisms and product of homomorphism is given by $g^{\pi_1 \alpha} = (g^\alpha)^{\pi_1}$, for $g \in G$. Note that f is well-defined. Indeed, let $\alpha \in \mathrm{Aut}(G)$, and put $\alpha_i = \pi_i \alpha \iota_i$, for $i = 1, 2$. So $\alpha^f = (\alpha_1, \alpha_2)$. Note that α_1 is a group homomorphism, since projection and inclusion are homomorphism. Now we prove that $ker(\alpha_1) = 1$. Let $(k_1, 1)^\alpha = (x_1, 1)$ (recall that $K \operatorname{char} G$), so $(k_1)^{\alpha_1} = (k_1)^{\pi_1 \alpha \iota_1} = ((k_1, 1)^\alpha)^{\pi_1} = (x_1, 1)^{\pi_1} = x_1$. Then,

$$(x_1, 1)^\alpha = (x_1^{\alpha_1}, 1^{\alpha_2}) = (1, 1) = \alpha(1, 1)$$

and we conclude that $ker(\alpha_1) = 1$. As K is a finite dimension vector space, the Rank-Nullity Theorem shows that α_1 is also surjective. So $\alpha_1 \in \mathrm{Aut}(K)$. A similar argument holds to show that $\alpha_2 \in \mathrm{Aut}(H)$. Also, simple calculations show that f is an isomorphism.

Now we focus on automorphism orbits. From the map f, we know that it is possible to construct automorphisms of G that act as automorphisms of K and H on each respective component. Thus, to count the number of orbits, it suffices to multiply $\omega(K)$ and $\omega(H)$. Note that $\omega(K) = 2$ and $\omega(H)$ is finite. Therefore, we have that $\omega(G) = 2\omega(H)$. □

This result is interesting as it shows that, in this case, the number of orbits is always even. As an immediate consequence, we have the following corollary.

Corollary 1.1. Let $G = K \times H$, where $K = \mathbb{Q}^n$, for some positive integer n, and H is a finite elementary abelian p-group, for some prime p. Then $\omega(G) = 4$.

We observe that, in $G = K \times H$, where $K = \mathbb{Q}^n$, for some positive integer n, and H is a finite elementary abelian p-group the 4 automorphism orbits are $\{1\}$, $H \setminus \{1\}$, $K \setminus \{1\}$, $G \setminus (H \cup K)$.

Theorem 1.11. $G = K \rtimes H$, where $K = \mathbb{Q}^n$ and H is a finite abelian group, for some positive integer n. Then $K = C_K(H) \times [K,H]$.

Proof. Define the map $\theta : K \to K$ by $k^\theta = \frac{1}{|H|} \sum_{h \in H} k^h$, written additively. It can be easily verified that θ is a well-defined homomorphism. Note that θ is idempotent,

$$\theta^2 = \theta \left(\frac{1}{|H|} \sum_{h \in H} h \right) = \frac{1}{|H|} \sum_{h \in H} \theta h = \frac{1}{|H|} (|H|\theta) = \theta.$$

First we show that $C_K(H) = Im(\theta)$. Note that $Im(\theta) \leq C_K(H)$, once $\theta h = h\theta = \theta$ for all $h \in H$. Conversely, if $x \in C_K(H)$, then

$$x^\theta = \frac{1}{|H|} \sum_{h \in H} x^h = \frac{1}{|H|} \sum_{h \in H} x = \frac{1}{|H|} |H| x = x,$$

so $x \in Im(\theta)$.

Next set $A = [K,H]$ and $A_1 = \{k - k^\theta : k \in K\}$. As θ is an endomorphism and K is abelian, A_1 is a subgroup of K. Moreover, $k = k^\theta + (k - k^\theta)$ for $k \in K$, so $K = C_K(H) + A_1$. On the other hand, if $x \in C_K(H) \cap A_1$, we have $x = x^\theta$ and $x = y - y^\theta$ for some $y \in P$, whence $x = x^\theta = (y - y^\theta)^\theta = 0$, as θ is idempotent. It follows that $K = C_K(H) \oplus A_1$. In addition, our calculation shows that A_1 is the kernel of θ.

Finally, by definition, A is generated by the elements $-k + k^h$, $k \in K$, $h \in H$. But, $(-k + k^h)^\theta = -k^\theta + k^{h\theta} = -k^\theta + k^\theta = 0$, which implies that $A \subseteq A_1$. Conversely, for $k \in K$ we have

$$k - k^\theta = \frac{1}{|H|} \left[\sum_{h \in H} (k - k^h) \right] \in A$$

as each $k - k^h \in A$. Thus $A_1 \subseteq A$, whence $A_1 = A$ and the desired conclusion $K = C_K(H) \oplus A$ is established. And in multiplicative notation, we have $K = C_K(H) \times A$, so the theorem is proved.

□

Theorem 1.12. Let $G = K \rtimes H$, where $K = \mathbb{Q}^n$ and H is a finite abelian group, for some positive integer n. If $C_K(H) \neq 1$ and $C_K(H) < K$, then $\omega_{\mathrm{Aut}(G)}(K) \geq 4$.

Proof. First note that by Theorem 1.11, $K = C_K(H) \times [K,H]$. Let $\varphi \in Aut(G)$. We show that there are at least 4 different automorphism orbits of G in K. First note that $Z(G) = C_K(H)$. This means that $C_K(H)\,\mathrm{char}\,K$. Since $[H,K]\,\mathrm{char}\,K$, $C_K(H) \neq 1$ and $C_K(H) < K$, there is one orbit for 1, at least one orbit for $[H,K] \setminus \{1\}$, at least one orbit for $C_K(H) \setminus \{1\}$, and least one orbit for $K \setminus ([H,K] \cup C_K(H))$. So $\omega(K) \geq 4$, and we conclude that $\omega_{\mathrm{Aut}(G)}(K) \geq 4$. □

We now turn to the study of fixed-point-free automorphisms. The most important result on this subject is Thompson's Theorem on the nilpotency of groups admitting a fixed-point-free automorphism of prime order.

Definition 1.6. Denote

$$C_G(\phi) = \{x \in G \mid x^{\phi} = x\}.$$

An automorphism ϕ of a group G is said to be *fixed-point-free* if $C_G(\phi) = 1$, that is, if it fixes only the identity element. A group of automorphisms A of G is *fixed-point-free* if $C_G(A) = 1$, i.e., if it fixes only the identity element.

We now state Thompson's theorem. For more details on the following theorem, we refer the reader to [11].

Theorem 1.13 (Thompson)**.** ([11], Theorem 10.2.1) Let G be a finite group. If G admits a fixed-point-free automorphism of prime order, then G is nilpotent.

In this context, we prove,

Theorem 1.14. Let $G = K \rtimes H$, where $K = \mathbb{Q}^n$ and H is a finite abelian group, for some positive integer n. If every element in $G \setminus K$ has finite order, then H acts fixed-point-freely on K.

Proof. We prove the contrapositive. If a non-trivial element $h \in H$ is such that $k^h = k$, for some non trivial element $k \in K$, then the element $kh \in G \setminus K$ is such that $(kh)^n = k^n h^n$. Since K is torsion-free, it follows that kh has infinite order. □

1.2 Examples

This section presents some well-known results which will be used throughout the text. Beyond simply stating these theorems and propositions, we provide illustrative examples to clarify the conditions under which they apply. These examples not only demonstrate the practical utility of each result but also help the reader grasp their limitations and appropriate contexts for application.

In [16], T. J. Laffey and D. MacHale characterized finite groups G that are not p-groups and have the property that $\omega(G) = 3$ (see [16] Theorem 2.). This result is stated in the following theorem.

Theorem 1.15 ([16], Theorem 2.)**.** Let G be a finite group that is not of prime power order. The following are equivalent:

1. $\omega(G) = 3$;

2. $|G| = p^n q$, and G has a normal elementary abelian Sylow p-subgroup K, for some primes p, q, and for some integer $n \geq 1$. Furthermore, p is a primitive root mod q (i.e. $q-1$ is the least natural number e with $p^e \equiv 1$ *mod* q). Let H be a Sylow q-subgroup of G. Then K, regarded as a $\mathbb{F}_p[H]$-module, is a direct sum of $t \geq 1$ copies of the (unique) irreducible $\mathbb{F}_p[H]$-module of dimension $q-1$. In particular $|K| = p^{t(q-1)}$.

Now we provide some examples of finite 3-orbit groups which are not of prime power order and, in the notation of the above theorem, the subgroup K is a non-trivial direct sum of irreducible $\mathbb{F}_p[H]$-modules, ie, the number t varies.

Example 1.2. With the help of GAP system [9], we found that the following groups are 3-orbit groups. Note that 3 is a primitive root mod 2.

1. The group $G = \langle a,b \colon a^3 = b^2 = 1, bab = a^{-1} \rangle \cong C_3 \rtimes C_2$, where $|K| = 3$ and $t = 1$;

2. The group $G = \langle a,b,c \colon a^3 = b^3 = c^2 = 1, ab = ba, cac = a^{-1}, cbc = b^{-1} \rangle \cong (C_3 \times C_3) \rtimes C_2$, where $|K| = 3^2$ and $t = 2$;

3. The group $G = \langle a,b,c,d \colon a^3 = b^3 = c^3 = d^2 = 1, ab = ba, ac = ca, dad = a^{-1}, bc = cb, dbd = b^{-1}, dcd = c^{-1} \rangle \cong (C_3 \times C_3 \times C_3) \rtimes C_2$, where $|K| = 3^3$ and $t = 3$;

If we consider K as a vector space over $\mathbb{F}_p$, then saying that p is a primitive root modulo q means that the polynomial

$$x^q - 1 = (x-1)(x^{q-1} + \cdots + x + 1)$$

has no roots in $\mathbb{F}_p$ other than 1. The following example demonstrates the necessity of the assumptions in the above theorem.

Example 1.3. With the help of GAP system [9], we found that the group given by

$$G_1 := \langle a,b,c \colon a^7 = b^7 = c^3 = 1, ab = ba, cac^{-1} = a^4, cbc^{-1} = b^4 \rangle$$

has 4 automorphism orbits and the automorphism orbits are represented by elements of order 1,7,3,3. The group G_1 can be viewed as the semidirect product $(C_7 \times C_7) \rtimes C_3$. The characteristic subgroup $V := \langle a, b\rangle \cong C_7 \times C_7$ regarded as $\mathbb{F}_7[C_3]$-module is completely reducible as direct sum of two 1-dimension $\mathbb{F}_7[C_3]$-modules, say $V = V_1 \oplus V_2$, where $V_1 = \langle v\rangle$ and $V_2 = \langle w\rangle$. We will show that there is no automorphism that maps c to c^2. Assume by way of contradiction that there is an automorphism θ that maps c to c^2. The elements c and c^2 can be viewed as the matrix

$$c = \begin{pmatrix} \bar{4} & \bar{0} \\ \bar{0} & \bar{4} \end{pmatrix}, c^2 = \begin{pmatrix} \bar{2} & \bar{0} \\ \bar{0} & \bar{2} \end{pmatrix} \in GL_2(V).$$

Then we would have

$$\bar{4}(v+w)^\theta = ((v+w)^c)^\theta = ((v+w)^\theta)^{c^\theta} = ((v+w)^\theta)^{c^2} = \bar{2}(v+w)^\theta.$$

A contradiction. So there is no automorphism that maps c to c^2.

Thus the assumption that K, regarded as a $\mathbb{F}_p[H]$-module, is a direct sum of $t \geq 1$ copies of the (unique) irreducible $\mathbb{F}_p[H]$-module of dimension $q-1$ is essential.

Example 1.4. With the help of GAP system [9], we found that the group given by

$$G_2 := \langle a, b, c \colon a^7 = b^7 = c^3 = 1, ab = ba, cac^{-1} = a^4, cbc^{-1} = b^2\rangle$$

has 4 automorphism orbits, and the automorphism orbits are represented by elements of order 1,7,7,3. The characteristic subgroup $V := \langle a, b\rangle \cong C_7 \times C_7$ regarded as $\mathbb{F}_7[C_3]$-module is completely reducible as direct sum of two 1-dimension $\mathbb{F}_7[C_3]$-modules, say $V = V_1 \oplus V_2$, where $V_1 = \langle v\rangle$ and $V_2 = \langle w\rangle$. And c can be viewed as the matrix

$$c = \begin{pmatrix} \bar{4} & \bar{0} \\ \bar{0} & \bar{2} \end{pmatrix} \in GL_2(\mathbb{F}_3).$$

If v and $v+w$ are in the same automorphism orbit, then there is an automorphism θ such that

$$\bar{4}(v+w) = \bar{4}v^\theta = (\bar{4}v)^\theta = (v^c)^\theta = (v^\theta)^{c^\theta} = (v+w)^{c^\theta},$$

this is $\bar{4}v + \bar{2}w$ or $\bar{2}v + \bar{4}w$, which can't happen. Hence there is no group automorphism of G that permutes the elements v and $v+w$ and V is divided into three automorphism orbits. It is also interesting to note that $\langle v\rangle$ and $\langle w\rangle$ are not isomorphic modules, once they have different eigenvalues. Indeed, the existence of a module isomorphism $f:\langle v\rangle \to \langle w\rangle$ would imply

$$\bar{2}w = w^c = f(v)^c = f(v^c) = f(\bar{4}v) = \bar{4}f(v) = \bar{4}w.$$

A contradiction.

Therefore, two crucial assumptions are that the normal Sylow subgroup, viewed as a $\mathbb{F}_p[H]$-module, decomposes as a direct sum of isomorphic $\mathbb{F}_p[H]$-submodules and that p is a primitive root modulo q. In [4] (Theorem B) it was proved that if G is mixed order soluble group of finite rank, then $\omega(G) = 3$ if and only if $G = K \rtimes H$ where $K = \mathbb{Q}^n$ and H is of order q acting fixed-point-freely on K. In particular, note that in this case the polynomial $x^q - 1 = (x-1)(x^{q-1} + \cdots + x + 1)$ always has no roots in $\mathbb{Q}$ other than 1. Also, as mentioned in the introduction, Examples 1.3 and 1.4 shows that in some sense the upper bound 3^n in Theorem C does not hold in the finite case.

We now consider 4-orbit groups. The following structure theorem for finite soluble groups G that are not of prime power order and with $\omega(G) = 4$ was proven by Laffey and MacHale in [16] (see Theorem 4).

Theorem 1.16 ([16], Theorem 4.)**.** Let G be a finite soluble group which is not of prime power order such that $\omega(G) = 4$. Then $|G| = p^a q^b$, and G has a normal Sylow p-subgroup P for some primes p, q. Let Q be a Sylow q-subgroup of G. Then one of the following holds:

1. Q acts fixed-point-freely on P, $|Q| = q$, and P is a 2-orbit or 3-orbit group;

2. P is elementary abelian, and Q is cyclic of order q^2;

3. P is elementary abelian, and Q is the quartenion group of order 8;

4. $G = P \times Q$, where P, Q are elementary abelian.

We provide some examples of finite soluble groups with $\omega(G) = 4$ that are not p-groups to better illustrate the preceding theorem.

Example 1.5. 1. With the help of GAP system [9], we found that

$$G := D_{18} = \langle a, b \colon a^9 = b^2 = 1, a^b = a^{-1}\rangle,$$

has 4 automorphism orbits. Let $P = \langle a\rangle$ and $Q = \langle b\rangle$. Note that $G = P \rtimes Q \cong C_9 \rtimes C_2$, Q acts fixed-point-freely on P, $|Q| = 2$, and P is 3-orbit group. So this exemplifies item 1. of Theorem 1.16.

2. With the help of GAP system [9], we found that

$$G := \langle a,b \colon a^7 = b^3 = 1, a^b = a^4 \rangle,$$

has 4 automorphism orbits. Let $P = \langle a \rangle$ and $Q = \langle b \rangle$. Note that $G = P \rtimes Q \cong C_7 \rtimes C_3$, Q acts fixed-point-freely on P, $|Q| = 3$, and P is 2-orbit group. The automorphism orbits are represented by elements of order 1,7,3,3. So this also exemplifies item 1. We note that this example also applies to demonstrate the necessity of the assumptions in Theorem 1.15.

3. With the help of GAP system [9], we found that the group

$$\begin{aligned} G := \langle a,b,c,d \colon a^3 = b^3 = c^4 = 1, d^2 = c^2, b^d = ab = ba, \\ a^c = b^{-1}, a^d = a^{-1}b, b^c = a, c^d = c^{-1} \rangle, \end{aligned}$$

has 4 automorphism orbits. Let $P = \langle a,b \rangle \cong C_3 \times C_3$ and $Q = \langle c \rangle \cong C_4$. Note that $G = P \rtimes Q \cong \mathrm{PSU}(2,7)$, Q acts fixed-point-freely on P, $|Q| = 4$, and P is elementary abelian. So this exemplifies item 2.

4. The group

$$G := \langle a \colon a^6 = 1 \rangle \cong C_2 \times C_3,$$

has 4 automorphism orbits. We note that there is only one element of order 2, also the inversion automorphism is sufficient to prove that all elements of order 3 are in the same automorphism orbit as well as all elements of order 6. So this exemplifies item 4.

In [5], it was proved that for $G = A \rtimes B$, where A is a finite-dimensional $\mathbb{Q}$-vector space and $B \leq \mathrm{Aut}(A)$, we have $\omega(G) < \infty$ if and only if, B is abelian. It shows that case (3) does not occur in the infinite case. The following example shows that the converse of item (3) does not hold. We denote the quaternion group of order 8 by Q_8.

Example 1.6. Using GAP system [9], we verified that for primes $p \in \{3,5,7,11\}$ the groups $G = (C_p \times C_p) \rtimes Q_8$, where Q_8 acts faithfully on $C_p \times C_p$ have exactly four automorphism orbits and $(C_{13} \times C_{13}) \rtimes Q_8$ has exactly five.

Therefore, Theorem 1.16 does not fully characterize all finite soluble groups with exactly four automorphism orbits that are not p-groups.

We present some examples of finite soluble groups, that are not p-groups, with 5 automorphism orbits. These examples will be relevant in the next chapter.

Example 1.7. 1. With the help of GAP system [9], we found that

$$G := \langle a, b \colon a^6 = 1, b^2 = a^3, bab^{-1} = a^{-1} \rangle,$$

has 5 automorphism orbits. Let $P = \langle a \rangle$ and $Q = \langle b \rangle$. Note that $G = P \rtimes Q \cong C_3 \rtimes C_4$, Q acts fixed-point-freely on P, $|Q| = 4$. The automorphism orbits are represented by elements of order 1,2,3,4,6.

2. With the help of GAP system [9], we found that

$$G := \langle a, b \colon a^{10} = 1, b^2 = a^5, bab^{-1} = a^{-1} \rangle,$$

has 5 automorphism orbits. Let $P = \langle a \rangle$ and $Q = \langle b \rangle$. Note that $G = P \rtimes Q \cong C_5 \rtimes C_4$, Q acts fixed-point-freely on P, $|Q| = 4$. The automorphism orbits are represented by elements of order 1,2,5,4,10.

3. With the help of GAP system [9], we found that

$$G := \langle a, b \colon a^6 = b^2 = 1, bab = a^{-1} \rangle,$$

has 5 automorphism orbits. Note that $G \cong (C_3 \rtimes C_2) \times C_2$, $Z(G) = \langle a^3 \rangle$ has order 2, the quotient $G/Z(G)$ is a 3-orbit group of composite order and the automorphism orbits are represented by elements of order 1,2,2,3,6.

4. With the help of GAP system [9], we found that the group

$$G := \langle a, b, c \colon a^5 = b^5 = c^6 = 1, ab = ba, cac^{-1} = a^2b^3, cbc^{-1} = a^{-1}b^{-1} \rangle,$$

has 5 automorphism orbits. Let $P = \langle a, b \rangle \cong C_5 \times C_5$ and $Q = \langle c \rangle \cong C_6$. Note that $G = P \rtimes Q \cong (C_5 \times C_5) \rtimes C_6$, Q acts fixed-point-freely on P, the automorphism orbits are represented by elements of order 1,2,3,5,6.

Now we present an example with 6 automorphism orbits.

Example 1.8. Let $G = S_3 \times S_3$, we show that $\omega(G) = 6$.

We begin by proving that $\mathrm{Aut}(S_3 \times S_3) \cong (S_3 \times S_3) \rtimes C_2$, where the cyclic group of order 2 acts by permutation on the entries of $S_3 \times S_3$. Let $g = (g_1, g_2) \in G$ and $\phi \in \mathrm{Aut}(G)$. First, we show that there exist homomorphisms $\alpha, \beta, \gamma, \delta : S_3 \to S_3$ such that $\phi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Consider the projection homomorphisms $\pi_1, \pi_2 : S_3 \times S_3 \to S_3$ and the inclusion maps

$\iota_1, \iota_2 : S_3 \to S_3 \times S_3$ defined as follows:

$$(g_1, g_2)^{\pi_1} = g_1 \qquad (g_1, g_2)^{\pi_2} = g_2$$
$$(g)^{\iota_1} = (g, 1) \qquad (g)^{\iota_2} = (1, g)$$

Define

$$\alpha = \pi_1 \phi \iota_1 \qquad \beta = \pi_1 \phi \iota_2$$
$$\gamma = \pi_2 \phi \iota_1 \qquad \delta = \pi_2 \phi \iota_2.$$

Then the maps $\alpha, \beta, \gamma, \delta$ are homomorphisms from S_3 to itself. Moreover, observe that using notation of [6], we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} g_1^{\alpha} g_2^{\beta} \\ g_1^{\gamma} g_2^{\delta} \end{pmatrix} = \begin{pmatrix} g_1^{\pi_1 \phi \iota_1} g_2^{\iota_1 \phi \pi_2} \\ g_1^{\iota_2 \phi \pi_1} g_2^{\pi_2 \phi \iota_2} \end{pmatrix} \tag{1.1}$$

$$= \begin{pmatrix} (g_1, 1)^{\pi_1 \phi} (1, g_2)^{\pi_1 \phi} \\ (g_1, 1)^{\pi_2 \phi} (1, g_2)^{\pi_2 \phi} \end{pmatrix} = \begin{pmatrix} (g_1, g_2)^{\pi_1 \phi} \\ (g_1, g_2)^{\pi_2 \phi} \end{pmatrix} = g^{\phi}. \tag{1.2}$$

Since ϕ is an automorphism, we can deduce some properties of the images $(S_3)^{\alpha}, (S_3)^{\beta} \leq S_3$. From (1.2) we obtain that $[(S_3)^{\alpha}, (S_3)^{\beta}] = 1$ and $(S_3)^{\alpha}(S_3)^{\beta} = S_3$. Hence, we conclude that $(S_3)^{\alpha}$ and $(S_3)^{\beta}$ are normal subgroups of S_3. The normal subgroups of S_3 are 1, $\langle (1\ 2\ 3) \rangle$, and S_3. But given the properties above, the only possible cases are $(S_3)^{\alpha} = 1$, $(S_3)^{\beta} = S_3$ or $(S_3)^{\alpha} = S_3$, $(S_3)^{\beta} = 1$. This implies that the only nontrivial homomorphisms are such that either $\alpha \in \mathrm{Aut}(G)$ and β is the trivial map, or $\beta \in \mathrm{Aut}(G)$ and α is trivial. The same holds for γ and δ. Therefore, ϕ must be of the form

$$\phi = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \quad \text{or} \quad \phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix},$$

where 0 is the zero homomorphism. Note, however, that

$$\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can then define the function

$$f : \mathrm{Aut}(S_3 \times S_3) \to (S_3 \times S_3) \rtimes C_2$$
$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \mapsto (\alpha, \delta)$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto \sigma.$$

where σ has order 2 and permutes the entries in $S_3 \times S_3$. It is easy to verify that f is an isomorphism. Noting that $\mathrm{Aut}(S_3) \cong S_3$, we conclude that $\mathrm{Aut}(S_3 \times S_3) \cong (S_3 \times S_3) \rtimes C_2$.

Now consider the action of $\mathrm{Aut}(G)$ on G, where the orbit of an element $g \in G$ is the set

$$Orb(g) = \{g^{\phi} : \phi \in \mathrm{Aut}(G)\}.$$

Since we have already described the elements of $\mathrm{Aut}(G)$, we can compute all the orbits. They are as follows:

$$Orb\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$Orb\begin{pmatrix} (12) \\ 1 \end{pmatrix} = \left\{ \begin{pmatrix} (12) \\ 1 \end{pmatrix}, \begin{pmatrix} (13) \\ 1 \end{pmatrix}, \begin{pmatrix} (23) \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ (12) \end{pmatrix}, \begin{pmatrix} 1 \\ (13) \end{pmatrix}, \begin{pmatrix} 1 \\ (23) \end{pmatrix} \right\}$$

$$Orb\begin{pmatrix} (12) \\ (12) \end{pmatrix} = \left\{ \begin{pmatrix} (12) \\ (12) \end{pmatrix}, \begin{pmatrix} (13) \\ (12) \end{pmatrix}, \begin{pmatrix} (23) \\ (12) \end{pmatrix}, \begin{pmatrix} (12) \\ (13) \end{pmatrix}, \begin{pmatrix} (13) \\ (13) \end{pmatrix}, \begin{pmatrix} (23) \\ (13) \end{pmatrix}, \begin{pmatrix} (12) \\ (23) \end{pmatrix}, \begin{pmatrix} (13) \\ (23) \end{pmatrix}, \begin{pmatrix} (23) \\ (23) \end{pmatrix}, \right\}$$

$$Orb\begin{pmatrix} (123) \\ 1 \end{pmatrix} = \left\{ \begin{pmatrix} (123) \\ 1 \end{pmatrix}, \begin{pmatrix} (132) \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ (123) \end{pmatrix}, \begin{pmatrix} 1 \\ (132) \end{pmatrix} \right\}$$

$$Orb\begin{pmatrix} (123) \\ (123) \end{pmatrix} = \left\{ \begin{pmatrix} (123) \\ (123) \end{pmatrix}, \begin{pmatrix} (132) \\ (132) \end{pmatrix}, \begin{pmatrix} (123) \\ (132) \end{pmatrix}, \begin{pmatrix} (132) \\ (123) \end{pmatrix}, \right\}$$

$$\begin{aligned} Orb\begin{pmatrix}(12)\\(123)\end{pmatrix} = \Bigg\{ & \begin{pmatrix}(12)\\(123)\end{pmatrix}, \begin{pmatrix}(13)\\(123)\end{pmatrix}, \begin{pmatrix}(23)\\(123)\end{pmatrix}, \begin{pmatrix}(123)\\(12)\end{pmatrix}, \begin{pmatrix}(123)\\(13)\end{pmatrix}, \begin{pmatrix}(123)\\(23)\end{pmatrix}, \\ & \begin{pmatrix}(12)\\(132)\end{pmatrix}, \begin{pmatrix}(13)\\(132)\end{pmatrix}, \begin{pmatrix}(23)\\(132)\end{pmatrix}, \begin{pmatrix}(132)\\(12)\end{pmatrix}, \begin{pmatrix}(132)\\(13)\end{pmatrix}, \begin{pmatrix}(132)\\(23)\end{pmatrix} \Bigg\} \end{aligned}$$

So $\omega(G) = 6$.

This concludes the preliminary chapter. In what follows, we turn our attention to soluble groups of finite rank with finitely many automorphism orbits, which form the main focus of this work.

Chapter 2

Mixed order soluble groups of finite rank

This chapter is devoted to the study of 4-orbit and 5-orbit groups that are mixed order soluble groups of finite rank. We prove two structural theorems and provide some examples of such groups.

2.1 Soluble groups with 4 automorphism orbits

In this section we prove Theorem A.

Theorem A. Let G be a mixed order soluble group of finite rank with $\omega(G) = 4$. Then $G = K \rtimes H$, where K is a torsion free nilpotent characteristic radicable subgroup, H is a finite subgroup and one of the following three cases holds (q is a prime number):

1. H acts fixed-point-freely on K, $|H| = q$, and $\omega_{Aut(G)}(K) = 3$;
2. $K = \mathbb{Q}^n$, H is cyclic of order q^2 and the action of H is fixed-point-free;
3. $G = K \times H$, where $K = \mathbb{Q}^n$ and H is elementary abelian q-group.

Before proving Theorem A, we present some related results. The authors of [5] proved

Theorem 2.1 ([5], Corollary 1.4)**.** Let A be a finite dimensional vector space over $\mathbb{Q}$ and B a finite subgroup of $\mathrm{Aut}(A)$. Let $G = A \rtimes B$ be the semidirect product of A and B. Then $\omega(G) < \infty$ if and only if B is abelian.

In a different work [4], the same authors proved

Theorem 2.2 ([4], Theorem A)**.** Let G be a soluble group of finite rank. If $\omega(G) < \infty$, then G has a torsion-free radicable nilpotent characteristic subgroup K such that

$$G = K \rtimes H,$$

where H is a finite subgroup.

Theorem 2.3 ([4], Theorem B)**.** Let G be a mixed order soluble group of finite rank. We have $\omega(G)=3$ if and only if $G=A\rtimes H$ where $|H|=p$ for some prime p, H acts fixed-point-freely on A and $A=\mathbb{Q}^n$ for some positive integer n.

Now we establish an auxiliary lemma.

Lemma 2.1. Let $G=A\rtimes B$, where $A=\mathbb{Q}^m$, $B\leq Aut(A)$, $|B|=q^n$ is an elementary abelian q-group and $Z(G)=1$. Then $\omega_{Aut(G)}(A)\geq n+1$.

Proof. If $n=1$, then H acts fixed-point-freely on A, and consequently $\omega(G)=3$, by Theorem 2.3. So $\omega_{\mathrm{Aut}(G)}(A)=2$, as desired. Thus, we can assume that $n\geq 2$. We consider A as a $\mathbb{Q}[B]$-module. By Maschke's Theorem, A is a completely reducible $\mathbb{Q}[B]$-module. Hence

$$A=A_1\oplus\ldots\oplus A_k,$$

where A_i is an irreducible $\mathbb{Q}[B]$-submodule of A, $1\leq i\leq k$. Since B is abelian, B/B_i is cyclic, where B_i denotes the kernel of the representation of B on A_i. Moreover, since B is elementary abelian and non-cyclic, we have that B_i is a maximal subgroup of B. Therefore, for each A_j there exists a maximal subgroup B_i such that $A_j\leq C_A(B_i)$.

Observe that the components A_i are contained in at least n distinct centralizers of maximal subgroups of B, since the intersection of any $n-1$ maximal subgroups of B is non-trivial and $Z(G)=1$. Thus, we may select n distinct components, say $A_1,\ldots,A_n$, with each $A_i\leq C_A(B_i)$ for distinct maximal subgroups $B_1,\ldots,B_n$. Let $v_i\in A_i$ be a non-trivial element of A_i for each i. Now we show that $v_1,v_1v_2,\ldots,v_1v_2\cdots v_n$ are in distinct orbits. Since

$$C_G(x)^{\alpha}=C_G(x^{\alpha}),$$

for all $x\in G$ and $\alpha\in Aut(G)$, it is sufficient to show that these elements have non-isomorphic centralizers. It is a straightforward calculation that $C_G(v_1)=AB_1$, $C_G(v_1v_2)=A(B_1\cap B_2)$, $C_G(v_1v_2\cdots v_i)=A(B_1\cap B_2\cap\cdots\cap B_i),\ldots,C_G(v_1v_2\cdots v_n)=A(B_1\cap B_2\cap\cdots\cap B_n)$, as desired. □

We proceed with the proof of Theorem A.

Proof of Theorem A. By Theorem 2.2, $G=K\rtimes H$ where K is a torsion-free radicable nilpotent characteristic subgroup and H is a finite group.

First, suppose that $\omega_{Aut(G)}(K)=3$. Then either K is nilpotent of class 2 with its center $Z(K)$ occupying two automorphism orbits in G, or K is abelian. In both cases, every element of $G \setminus K$ has prime order q and acts fixed-point-freely on K. However, the case where K is abelian contradicts Theorem 2.3. Therefore, K is nilpotent of class 2.

Assume now that $\omega_{Aut(G)}(K) = 2$. Since K is nilpotent, we have $K = \mathbb{Q}^n$. If $C_H(K) \neq 1$, then $C_G(K) = K \times C_H(K)$ is an abelian characteristic subgroup with at least four automorphism orbits, by Lemma 2.1. Thus, in this case $G = K \times H$.

Then H acts faithfully on K. Note that $C_K(H) = Z(G)$, then $C_K(H) = 1$ since $\omega_{Aut(G)}(K) = 2$. This implies, via Theorem 2.3, that the quotient group $G/K \cong H$ is a finite abelian group with at most three automorphism orbits. By Theorem 1.15, we conclude that $G/K \cong H$ have a prime power order.

If H is not a cyclic group, then H contains an elementary abelian q-group B of rank at least two for some prime q such that KB is characteristic. By Lemma 2.1 this leads to a contradiction, and therefore H must be cyclic of order q^2. □

Therefore if G is a mixed order soluble group of finite rank with $\omega(G) = 4$, then $G = K \rtimes H$, where K is either isomorphic to a finite direct product of $\mathbb{Q}$ or class-2 nilpotent. We now provide an example illustrating the latter case.

Example 2.1. Define $\lambda^3 = 1$ for some $1 \neq \lambda \in \mathbb{C}$ and consider the splitting field $F = \mathbb{Q}[\lambda] = \{x + y\lambda : x, y \in \mathbb{Q}\}$ over $\mathbb{Q}$ of the separable polynomial $X^3 - 1$ and the group $K = UT_3(F)$ of upper 3 by 3 unitriangular matrices over the field F. Define a map $\alpha : K \to K$ by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \lambda a & \lambda^2 c \\ 0 & 1 & \lambda b \\ 0 & 0 & 1 \end{pmatrix}.$$

The map α extends to a fixed-point-free automorphism of K of order 3. By denoting $H = \langle \alpha \rangle$ we construct the semidirect product $G = K \rtimes H$.

We will argue that the group G in Example 2.1 has 4 automorphism orbits. First we will show that every non-trivial element in K' is in the same automorphism orbit. Recall that the commutator subgroup K' of K consists of unitriangular matrices $(a_{ij}) \in K$ such that $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq 3$, $j \neq 1$, $i \neq j$. We now show that all non-trivial elements of K' lie in the same automorphism orbit. For any two elements $k_1, k_2 \in K$, conjugation by

$$\begin{pmatrix} k_1^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k_2 \end{pmatrix} \in T_5(F)$$

provides an automorphism that connects two non-trivial elements in K'.

Now we show that $K \setminus K'$ form a single automorphism orbit. By Theorem 3 of [17], for every $A = (a_{ij}) \in GL_2(F)$ the map θ_A given by

$$k \mapsto \begin{pmatrix} 1 & aa_{1,1}+ba_{2,1} & \left(c-\frac{ab}{2}\right)\det(A)+\frac{(aa_{1,1}+ba_{2,1})(aa_{1,2}+ba_{2,2})}{2} \\ 0 & 1 & aa_{1,2}+ba_{2,2} \\ 0 & 0 & 1 \end{pmatrix},$$

extends to an automorphism of K. The existence of these automorphisms implies that any two elements in $K \setminus K'$ are connected, and thus $UT_3(F)$ has exactly 3 automorphism orbits. Further examples of groups with 3 automorphism orbits can be found in [21]. Now since α commutes with θ_A for every $A \in GL_2(F)$, we can define the extension of θ_A to G by fixing $\alpha \mapsto \alpha$. This extension yields an automorphism of G.

Now we will show that every element in $G \setminus K$ is in a single automorphism orbit. We will show that α and α^{-1} are in the same automorphism orbit. Observe that $\mathbb{Q}[\lambda]/\mathbb{Q}$ is a Galois extension. Hence, we can define a field automorphism $\overline{\varphi} : \lambda \mapsto \lambda^2$ that can be extended to an automorphism of K by the action $\varphi : x+y\lambda \mapsto x+y\lambda^2$ on each matrix entry of K (since it is compatible with the operations of matrix multiplication). In fact, the map φ is complex conjugation. Define the map $\varphi : G \to G$ by

$$k \mapsto \begin{pmatrix} 1 & \varphi(a) & \varphi(c) \\ 0 & 1 & \varphi(b) \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha \mapsto \alpha^{-1}.$$

Basic calculations show that this map extends to an automorphism of G.

Now it remains to show that the element $z\alpha$ for $z \in Z(K)$ is in the same orbit as the element $k\alpha$ for $k \notin Z(K)$. We will present an element $x \in K$ such that $(k\alpha)^x = z\alpha$. Let

$$k = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in K.$$

Note that $(k\alpha)^x = x^{-1}k\alpha x = x^{-1}k\alpha x\alpha^{-1}\alpha = x^{-1}kx^{\alpha^{-1}}\alpha$. And this last product corresponds to

$$\begin{pmatrix} 1 & a+(\lambda^2-1)x_1 & b+c\lambda^2 x_1+(\lambda-1)x_2-ax_3-(\lambda^2+1)x_1x_3 \\ 0 & 1 & c+(\lambda^2-1)x_3 \\ 0 & 0 & 1 \end{pmatrix}\alpha.$$

Note that we can easily find the entries for matrix x by simply equating the above matrix with $z\alpha$. Hence every element in $G\setminus K$ is in a single automorphism orbit. Consequently, the group presented in Example 2.1 has 4 orbits under automorphism.

We now turn to the proof of Theorem B which we state again for convenience.

Theorem B. Let G be a mixed order metabelian group of finite rank. Then $\omega(G)=4$ if and only if $G=K\rtimes H$ where one of the following holds:

1. $K=\mathbb{Q}^n$, H is cyclic of order q^2, the action of H is fixed-point-free and K decomposes as a $\mathbb{Q}[H]$-module into a direct sum of copies of isomorphic irreducible $\mathbb{Q}[H]$-module;

2. $G=K\times H$, where $K=\mathbb{Q}^n$ and H is elementary abelian.

We already proved that item 2 is a sufficient condition for a group to have 4 automorphism orbits (see Theorem 1.10). Now we prove that item 1 is a sufficient condition for a group to have 4 automorphism orbits. We begin with an auxiliary lemma.

Lemma 2.2. Let $G=K\rtimes H$ where $K=\mathbb{Q}^n$ and H is a cyclic group of order q^2. If $\omega(G)=4$, then H acts fixed-point-freely on K, and K decomposes as a $\mathbb{Q}[H]$-module into a direct sum of copies of the same irreducible $\mathbb{Q}[H]$-module.

Proof. Using Theorem 2.1, we obtain that H acts fixed-point-freely on K. By Maschke's Theorem $K=K_1\oplus K_2\oplus\cdots\oplus K_r$, where K_i is an irreducible $\mathbb{Q}[H]$-module, for $1\le i\le r$. If $H=\langle h\colon h^{q^2}=1\rangle$, we choose a basis for K_i so that h is represented by the companion matrix of its minimal polynomial $m_i(X)$ on K_i. Hence K_i is determined up to $\mathbb{Q}[H]$-isomorphism by the minimal polynomial $m_i(X)$ of h on K_i.

Since $h^{q^2}=1$ we conclude that the minimal polynomial $m_h(x)$ of $T_h: K\to K$, given by $k\mapsto k^h$ divides

$$x^{q^2}-1=(x-1)(1+x+x^2+\cdots+x^q+\cdots+x^{q^2-1}).$$

Since h acts fixed-point-freely,

$$m_h(x) \text{ divides } (1+x+x^2+\cdots+x^q+\cdots+x^{q^2-1}).$$

Assume, by way of contradiction, that K_1 is not isomorphic to K_2. Hence $m_1(X)\neq m_2(X)$. Let $0\neq u_i\in K_i$, for $i=1,2$. Since G is a 4 orbit group, the characteristic subgroup K occupies

2 automorphism orbits, hence there exists $\sigma \in \mathrm{Aut}(G)$ with $u_1^\sigma = u_1 + u_2$. Now $h^{\sigma^{-1}} = h^k w$ for some $w \in K$ and let $g(X)$ be the minimal polynomial of h^k on K_1. Consider

$$u_1^\sigma g(h) = u_1^\sigma (g(h^{\sigma^{-1}}))^\sigma = u_1^\sigma (g(h^k))^\sigma = [u_1 g(h^k)]^\sigma = 0.$$

Hence $(u_1 + u_2)g(h) = 0$. But the minimal polynomial of h on $u_1 + u_2$ is $m_1(X)m_2(X)$. Since $\deg(g) = \deg(m_1)$, we have a contradiction. □

Theorem 2.4. Let $G = K \rtimes H$ where $K = \mathbb{Q}^n$ and H is a cyclic group of order q^2. If H acts fixed-point-freely on K and K, regarded as a $\mathbb{Q}[H]$-module, is a direct sum of copies of the irreducible $\mathbb{Q}[H]$-module, then $\omega(G) = 4$.

Proof. First, we determine the orders of elements in $G \setminus K$, demonstrating that they must be either q or q^2. Set $H = \langle h \rangle$. The minimal polynomial $m_h(x)$ divides

$$1 + x + x^2 + \cdots + x^q + \cdots + x^{q-1}.$$

Using the identity $(xy)^n = x^n (y^{x^{n-1}}) \ldots y^x y$, it is clear that for every $k \in K$, $(h^j k)^{q^2} = 1$.

Now we show that any two elements of K are conjugate by some automorphism. Let $u, v \in K$ be non-trivial elements. Since K is the direct sum of isomorphic $\mathbb{Q}[H]$-modules, we have that $K_0 = \{vf(h) \colon f(X) \in \mathbb{Q}[X]\}$ and $K_1 = \{uf(h) \colon f(X) \in \mathbb{Q}[X]\}$ are irreducible $\mathbb{Q}[H]$-submodule of K. If $K_1 = K_0$, then $K = K_0 \oplus K_2$, where K_2 is a complement of K_1, and the map σ given by $u^\sigma = v$, $h^\sigma = h$ and $w^\sigma = w$ extends to an automorphism of G, for $w \in K_2$. Not that K_1 and K_2 are fixed. If $K_1 \neq K_0$, then $K = K_0 \oplus K_1 \oplus K_2$, where K_2 is a complement of K_1, and the map σ given by $u^\sigma = v$, $v^\sigma = u$ and $h^\sigma = h$ extends to an automorphism of G. Note that K_1 and K_2 are switched and K_2 is fixed.

Now we must show that elements of order q^2 form a single orbit under $\mathrm{Aut}(G)$. Note that for any $h \in H$ the map $k \to k^{-1}k^h$ is an automorphism of K since h acts fixed-point-freely. Therefore, the elements of the coset hK are in the same orbit by conjugation. So it suffices to show that there exists $\sigma \in \mathrm{Aut}(G)$ such that $h^\sigma = h^i$ for every positive integer i coprime to q. Let $K = K_1 \oplus \cdots \oplus K_t$ be the sum of irreducible $\mathbb{Q}[H]$-modules and let $0 \neq k_j \in K_j$. Each $k \in K_j$ can be written in the form $k = k_j f(h)$ for some $f(X) \in \mathbb{Q}[X]$ with $\deg(f) < q^2 - 1$. Define a map $h^\sigma = h^i$, $k_j^\sigma = k_j$ and $(k_j f(h))^\sigma = k_j f(h^\sigma)$. Note that if $0 \neq k \in K$ is such that $kg(h) = 0$ for some $g(X) \in \mathbb{Q}[X]$, then $\Phi_{q^2-1}(X)$ (the cyclotomic polynomial for $q^2 - 1$) divides $g(X)$ and hence it also divides $g(X^i)$. So $kg(h^i) = 0$ and the extension of the map σ is well defined, hence it extends to an automorphism of G. Finally, observe that as i ranges over all possible values coprime with q, it simultaneously covers all powers of h^q. Thus, the elements of order q constitute a single orbit. □

Combining this result with Theorem A, we establish Theorem B.

2.2 Soluble groups with 5 automorphism orbits

Following the result presented in the previous section, it is natural to ask whether similar structural conclusions can be drawn for groups with five automorphism orbits. This question arises as a natural continuation of the investigation. Now we explore this case in more detail.

We start by providing a proof of Theorem C and later we provide some examples.

Theorem C. Let G be a mixed order soluble group of finite rank with $\omega(G) = 5$. Then $G = K \rtimes H$, where K is a torsion free nilpotent characteristic radicable subgroup, H is a finite subgroup and one of the following holds:

1. K is non-abelian, $G/Z(K)$ is non-abelian, and $\omega(G/Z(K)) \leq 4$;
2. $K = \mathbb{Q}^n$, $\omega_{Aut(G)}(K) = 3$, and $|H| = q^2$ for some prime q;
3. $K = \mathbb{Q}^n$, $\omega_{Aut(G)}(K) = 2$, $Z(G)$ is elementary abelian q-group and $\omega(G/Z(G)) = 3$.
4. $K = \mathbb{Q}^n$, $\omega_{Aut(G)}(K) = 2$, and H is a cyclic group with $\omega(H) = 4$.

Proof of Theorem B. By Theorem 2.2, $G = K \rtimes H$ where K is a torsion-free radicable nilpotent characteristic subgroup and H is a finite group.

- **Claim 1.** If K is nilpotent of class 2 or 3, then $G/Z(K)$ is non-abelian, and $\omega(G/Z(K)) \leq$ 4.

Suppose K is nilpotent of class 2. Then $\omega_{Aut(G)}(K) \geq 3$. If $[K,H] \leq K'$, we will show that $G = K \times H$, and consequently $\omega(G) \geq 6$. To see this, set $h \in H$ and define the mapping $\alpha_h : k \mapsto k^{-1}k^h$ from K to K'. Since $K' = Z(K)$, for $x, y \in K$ we have

$$(xy)^{\alpha_h} = (xy)^{-1}(xy)^h = y^{-1}x^{-1}x^h y^h = y^{-1}x^{\alpha_h}y^h = x^{\alpha_h}y^{\alpha_h}.$$

So α_h is an homomorphism of K to K' for any $h \in H$. Note that α_h maps K into an abelian group so K' is contained in the kernel of α_h and so $[K',h] = 1$. For any $k \in K$ we have $k^h = k[k,h]$, $k^{h^i} = k[k,h]^i$ and then $k = k[k,h]^{|h|}$, i some positive integer. Since K is torsion-free we obtain that $[k,h] = 1$ and $G/Z(K)$ is non-abelian, as desired.

If K is nilpotent of class 3, we can use the same argument to see that if

$$[K/\gamma_3(K), H\gamma_3(K)/\gamma_3(K)] \leq K'/\gamma_3(K),$$

then $[K/\gamma_3(K), H\gamma_3(K)/\gamma_3(K)] = 1$. Thus $\omega(G/\gamma_3(K)) \geq 6$, and so $\omega(G) \geq 6$.

- **Claim 2.** If K is abelian with $\omega_{\mathrm{Aut}(G)}(K) = 3$, then $|H| = q^2$ for some prime q.

Suppose that K is abelian with $\omega_{\mathrm{Aut}(G)}(K) = 3$. In this case, H is an abelian group with at most three automorphism orbits, while $G \setminus K$ has exactly two automorphism orbits. By Theorem 1.15, this implies H cannot have composite order. Moreover, G must be non-abelian, since in the direct product $K \times H$ of abelian groups, K would have exactly two automorphism orbits.

Note that if $C_K(H) \neq 1$, then $K = C_K(H) \times [K,H]$, where both factors are characteristic subgroups, which would force $\omega_{\mathrm{Aut}(G)}(K) \geq 4$, contradicting our hypothesis. Thus $C_K(H) = 1$.

When H is not cyclic, Lemma 2.1 shows that $G \setminus K$ contains both elements of prime order q and elements of infinite order, with $|H| = q^2$ for some prime q. If H is cyclic, we immediately conclude $|H| = q^2$.

- **Claim 3.** If K is abelian with $\omega_{\mathrm{Aut}(G)}(K) = 2$ and $Z(G) \neq 1$, then $Z(G)$ is elementary abelian q-group and $\omega(G/Z(G)) = 3$.

Suppose that K is abelian and $\omega_{Aut(G)}(K) = 2$. In this case, $Z(G) = C_H(K)$. If $C_H(K) \neq 1$, the subgroup $K \times C_H(K)$ must have four automorphism orbits, and consequently $C_H(K)$ must be elementary abelian. Thus, all elements of $G \setminus C_G(K)$ have order q for some prime q. Note that $Z(G) = C_K(H) \times C_H(K)$, therefore H acts fixed-point-freely on $KZ(G)/Z(G)$ and $\omega(G/Z(G)) = 3$.

- **Claim 4.** If K is abelian with $\omega_{\mathrm{Aut}(G)}(K) = 2$ and $Z(G) = 1$, then H is a cyclic group with $\omega(H) = 4$.

If $C_H(K) = 1$, we may apply Lemma 2.1 to conclude that H is cyclic with $\omega(H) = 4$, as desired. □

Now we present some examples. The first two demonstrate the existence of groups with exactly five automorphism orbits and elementary abelian centers. We point out that these cases are analogous to the finite ones in Example 1.7.

Example 2.2.

1. $G = \mathbb{Q} \rtimes C_4$, where $C_4 = \langle b \mid b^4 = 1 \rangle$ is cyclic of order 4 and the action is given by $q^\sigma = -q$. We observe that $Z(G) = \langle b^2 \rangle$.

2. $G = (\mathbb{Q} \rtimes C_2) \times C_2$. The first C_2 factor acts on $\mathbb{Q}$ via canonical inversion. We observe that $Z(G) \cong C_2$.

3. $G = \mathbb{Q}^2 \rtimes C_6$ where $C_6 = \langle b \mid b^6 = 1 \rangle$. And b acts on $\mathbb{Q}^2$ by

$$b = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

   We note that $\omega_{\mathrm{Aut}(G)}(\mathbb{Q}^2) = 2$, the elements of order 2 are in the same automorphism orbit, the elements of order 3 are in the same automorphism orbit and the elements of order 6 are in the same automorphism orbit.

Following this result, it is natural to ask what can be said about mixed order soluble groups of finite rank with six or more automorphism orbits. For instance, if G is a mixed order soluble groups of finite rank with $\omega(G) = 6$, the decomposition of Theorem 0.1

$$G = K \rtimes H,$$

holds, where K is a torsion-free nilpotent characteristic radicable subgroup and H is a finite subgroup. In the present work, we do not pursue this case in depth, leaving it as a potential direction for future investigation. However, we present an example of a finite rank mixed-order soluble group with 6 automorphism orbits to compare to Example 1.8, where we considered $S_3 \times S_3$.

Example 2.3. Let $G = (\mathbb{Q} \rtimes C_2) \times (\mathbb{Q} \rtimes C_2)$. We prove that $\omega(G) = 6$. We begin by describing their automorphism group. Note that this group can be viewed as $G = \mathbb{Q}^2 \rtimes (\langle b_1 \rangle \times \langle b_2 \rangle)$, where

$$b_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The group G can also be viewed as $G = ((\{0\} \times \mathbb{Q}) \rtimes \langle b_1 \rangle) \times ((\mathbb{Q} \times \{0\}) \rtimes \langle b_2 \rangle)$ and also $G \cong N \times N$, where $N = \mathbb{Q} \rtimes C_2$. We prove that $\omega(G) = 6$. Let $g = (g_1, g_2) \in G = N \times N$ and $\phi \in \mathrm{Aut}(G)$. First, we show that there exist homomorphisms $\alpha, \beta, \gamma, \delta : N \to N$ such that $\phi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Consider the projection homomorphisms $\pi_1, \pi_2 : N \times N \to N$ and the inclusion maps $\iota_1, \iota_2 : N \to N \times N$ defined as follows:

$$(g_1, g_2)^{\pi_1} = g_1 \qquad (g_1, g_2)^{\pi_2} = g_2$$
$$(g)^{\iota_1} = (g, 1) \qquad (g)^{\iota_2} = (1, g)$$

Define:

$$\alpha = \pi_1 \phi \iota_1 \qquad \beta = \pi_1 \phi \iota_2$$
$$\gamma = \pi_2 \phi \iota_1 \qquad \delta = \pi_2 \phi \iota_2.$$

Thus, the maps $\alpha, \beta, \gamma, \delta$ are homomorphisms from N to itself. Moreover, observe that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} g_1^\alpha g_2^\beta \\ g_1^\gamma g_2^\delta \end{pmatrix} = \begin{pmatrix} g_1^{\pi_1 \phi \iota_1} g_2^{\iota_1 \phi \pi_2} \\ g_1^{\iota_2 \phi \pi_1} g_2^{\pi_2 \phi \iota_2} \end{pmatrix} \tag{2.1}$$

$$= \begin{pmatrix} (g_1, 1)^{\pi_1 \phi} (1, g_2)^{\pi_1 \phi} \\ (g_1, 1)^{\pi_2 \phi} (1, g_2)^{\pi_2 \phi} \end{pmatrix} = \begin{pmatrix} (g_1, g_2)^{\pi_1 \phi} \\ (g_1, g_2)^{\pi_2 \phi} \end{pmatrix} = g^\phi. \tag{2.2}$$

Now, since ϕ is an automorphism, we can determine some properties of the images $(N)^\alpha, (N)^\beta \leq N$. From (2.1) we obtain that $[(N)^\alpha, (N)^\beta] = 1$ and $(N)^\alpha (N)^\beta = N$. Thus, we conclude that $(N)^\alpha, (N)^\beta \lhd N$. Given the properties above, the only possibilities are $(N)^\alpha = 1, (N)^\beta = N$ or $(N)^\alpha = N, (N)^\beta = 1$. This shows that the homomorphisms must be such that either $\alpha \in \mathrm{Aut}(N)$ and β is the trivial homomorphism, or $\beta \in \mathrm{Aut}(N)$ and α is trivial. The same applies to γ and δ. Hence, we can write either $\phi = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ or $\phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$. But notice that

$$\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This determines all automorphisms of G. Furthermore, we can define the map

$$f : \mathrm{Aut}(N \times N) \to (\mathrm{Aut}(N) \times \mathrm{Aut}(N)) \rtimes S_2$$
$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \mapsto (\alpha, \delta)$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto \sigma.$$

It is easy to see that f is an isomorphism. We know that $\omega(N) = 3$, where one orbit is for e, one orbit is for torsion-free elements and one orbit is for elements of order 2. Now we count the number of orbits of G. By the automorphism $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ we know that there are at most $\omega(N)^2 = 9$ orbits in G. The orders of the elements occurring in each possible orbit are of the following types

$$(1,1), (1, \text{torsion-free}), (1,2),$$

$$(\text{torsion-free}, 1), (\text{torsion-free}, \text{torsion-free}), (\text{torsion-free}, 2),$$

$$(2,1), (2, \text{torsion-free}), (2,2).$$

Now, the automorphism $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ shows that the following types of elements are in the same automorphism orbit

$$(\text{torsion-free}, 1) \quad \text{and} \quad (1, \text{torsion-free}),$$

$$(1,2) \quad \text{and} \quad (2,1),$$

$$(2, \text{torsion-free}) \quad \text{and} \quad (\text{torsion-free}, 2).$$

As all automorphisms have been taken into account, there are exactly $\omega(N)^2 - 3 = 6$ automorphism orbits in G.

This concludes the chapter about mixed order soluble groups of finite rank with 4 and 5 automorphism orbits.

Chapter 3

Unitriangular matrices over $\mathbb{Q}$

In the beginning of this work we mentioned Theorem 0.1, which says that if G is a soluble group of finite rank with $\omega(G) < \infty$, then G contains a torsion-free characteristic nilpotent subgroup K such that $G = K \rtimes H$, where H is a finite group.

The group of unitriangular matrices plays a significant role within the class of nilpotent groups, serving as a fundamental example. We investigate certain properties of these groups in the context of our study. In particular, we considered the group of upper triangular matrices of dimension n over the field of rational numbers, $UT_n(\mathbb{Q})$.

3.1 General overview

3.1.1 Group theoretic properties

Definition 3.1. The *upper unitriangular matrix group* of dimension $n \times n$ over the field $\mathbb{Q}$, denoted $UT_n(\mathbb{Q})$, is the group, under multiplication, with 1's on the diagonal, 0's below the diagonal, and arbitrary entries above the diagonal.

Explicitly,

$$UT_n(\mathbb{Q}) = \left\{ \begin{pmatrix} 1 & * & * & \dots & * \\ 0 & 1 & * & \dots & * \\ 0 & 0 & 1 & \dots & * \\ \dots & \dots & \dots & \dots & * \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}_{n\times n} : \text{all star-marked entries vary arbitrarily over } \mathbb{Q} \right\}.$$

This subsection is based on the paper [20] by Mahalanobis, in which he finds a set of generators for the automorphism group of the group of unitriangular matrices over a field. . We begin by defining some well known properties about $UT_n(\mathbb{Q})$.

We define the elementary matrix $xe_{i,j}$ to be the $n \times n$ matrix with x in the (i, j) position, 1's in the diagonal and 0 elsewhere, where $x \in \mathbb{Q}$. In what follows, we shall discuss the central series of $UT_n(\mathbb{Q})$.

Definition 3.2. Define

$$\gamma_k = \{M = (m_{i,j}) \in UT_n(\mathbb{Q})\colon m_{i,j} = 0,\ i < j,\ j - i < k\}.$$

In other words, the $\gamma_1 = UT_n(\mathbb{Q})$. The subgroup γ_2 is the commutator of $UT_n(\mathbb{Q})$. It consists of all upper unitriangular matrices with the first superdiagonal entries zero. The first superdiagonal can be specified by all entries (i, j) with $j - i = 1$. Similarly γ_3 consists of all matrices with the first two superdiagonals zero and so on. It follows that $\gamma_n = 1$. We denote the identity matrix by $\mathbb{1}$.

It is known that

Proposition 3.1 ([20], Proposition 1.1)**.** In $UT_n(\mathbb{Q})$, the lower central series and the upper central series are identical and it is of the form

$$UT_n(\mathbb{Q}) = \gamma_1 > \gamma_2 > \ldots > \gamma_{n-1} > \gamma_n = \mathbb{1}.$$

With this discussion, it becomes clear that $UT_n(\mathbb{Q})$ has nilpotency class $n - 1$. This is a very basic property, but we will present an example to illustrate the idea clearly.

Example 3.1. The group $UT_4(\mathbb{Q})$ has nilpotency class 3 and the lower central series is

$$\gamma_1 = \left\{ \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & 1 & a_{2,3} & a_{2,4} \\ 0 & 0 & 1 & a_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} : a_{i,j} \in \mathbb{Q} \right\},$$

$$\gamma_2 = \left\{ \begin{pmatrix} 1 & 0 & a_{1,3} & a_{1,4} \\ 0 & 1 & 0 & a_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : a_{i,j} \in \mathbb{Q} \right\},$$

$$\gamma_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & a_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : a_{i,j} \in \mathbb{Q} \right\},$$

$$\gamma_4 = \{\mathbb{1}\}.$$

We now proceed to analyze the derived series. In [10] it was discussed that the kth term in the derived series for $UT_n(\mathbb{Q})$ is $UT_n(\mathbb{Q})^{(k)} = \gamma_{2^k}$. See the following example.

Example 3.2. The group $UT_4(\mathbb{Q})$ has derived lenght 3 and the derived series is

$$UT_4(\mathbb{Q})^{(1)} = \left\{ \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & 1 & a_{2,3} & a_{2,4} \\ 0 & 0 & 1 & a_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} : a_{i,j} \in \mathbb{Q} \right\},$$

$$UT_4(\mathbb{Q})^{(2)} = \left\{ \begin{pmatrix} 1 & 0 & a_{1,3} & a_{1,4} \\ 0 & 1 & 0 & a_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : a_{i,j} \in \mathbb{Q} \right\},$$

$$UT_4(\mathbb{Q})^{(3)} = \{\mathbb{1}\}.$$

Note that there is a bijection between the commutator subgroup $UT_4(\mathbb{Q})^{(2)}$ and the group $UT_3(\mathbb{Q})$. However, there is no isomorphism between them, since one is abelian while the other has nilpotency class 2.

Proposition 3.2. The group of unitriangular matrices $UT_n(\mathbb{Q})$ is a soluble group of finite rank, for any positive integer n.

Proof. The solubility is clear. To see that the group has finite rank, observe that, based on the description of γ_i and the discussion above, the quotients γ_i/γ_{i+1} are direct sums of a finite number of copies of $\mathbb{Q}$, for $1 \leq i \leq n-1$. And the property of having finite rank is extension closed. In fact, if N and G/N have rank r and r' respectively, then G has rank at most $r+r'$. Therefore, we can conclude that $UT_n(\mathbb{Q})$ is a soluble group of finite rank for any positive integer n. □

We now proceed to define a class of maximal abelian subgroups, which will play an important role in our subsequent analysis.

Definition 3.3. For $j > i$ let us define $N_{i,j}$ to be the subset of $UT_n(\mathbb{Q})$ all of whose matrices have all rows greater than the i^{th} row zero and all columns less than the j^{th} column zero, except from the diagonal entries.

It is straightforward to see that $N_{i,j}$ is an abelian normal subgroup of $UT_n(\mathbb{Q})$. We now present some examples.

Example 3.3. Let $G = UT_4(\mathbb{Q})$. The subgroup $N_{1,2}$ and $N_{2,3}$ has elements of the form

$$N_{1,2} = \left\{ \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : a_{i,j} \in \mathbb{Q} \right\},$$

$$N_{2,3} = \left\{ \begin{pmatrix} 1 & 0 & a_{1,3} & a_{1,4} \\ 0 & 1 & a_{2,3} & a_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : a_{i,j} \in \mathbb{Q} \right\}.$$

Levchuk proved

Proposition 3.3 ([17], Lemma 5.)**.** The centralizer of $N_{i,j}$ is

$$C_{UT_n(\mathbb{Q})}(N_{i,j}) = N_{j-1,i+1}.$$

Levchuk also mentioned that the subgroups $N_{i,i+1}$ are maximal abelian normal subgroups of $UT_n(\mathbb{Q})$, for $i = 1, 2, \ldots n-1$. Knowing this property is particularly useful, because the image of maximal abelian normal subgroups under automorphism are maximal abelian normal subgroups.

Example 3.4. Let $G = UT_4(\mathbb{Q})$. The subgroup $N_{2,3}$ has elements of the form

$$N_{2,3} = \left\{ \begin{pmatrix} 1 & 0 & a_{1,3} & a_{1,4} \\ 0 & 1 & a_{2,3} & a_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : a_{i,j} \in \mathbb{Q} \right\}.$$

It is a maximal abelian subgroup isomorphic to $\mathbb{Q}^4$. This subgroup is also a characteristic subgroup of G, proof of this fact can be found in [20] or [17].

3.1.2 Ring of niltriangular matrices and automorphism group

The unitriangular group $UT_n(\mathbb{Q})$ is isomorphic to the associated group (with multiplication $a \circ b = a + b + ab$) of the ring $NT_n(\mathbb{Q})$ of upper niltriangular matrices of degree n over $\mathbb{Q}$, ie, all entries on and below the main diagonal are zero. See [20] and [17].

$$NT_n(\mathbb{Q}) = \left\{ \begin{pmatrix} 0 & * & * & \dots & * \\ 0 & 0 & * & \dots & * \\ 0 & 0 & 0 & \dots & * \\ \dots & \dots & \dots & \dots & * \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{n\times n} : \text{all star-marked entries vary arbitrarily over } \mathbb{Q} \right\}.$$

The notation for this group is $(NT_n(\mathbb{Q}), \circ)$ and the isomorphism is given by $x \mapsto \mathbb{1}_n + x$, for $x \in NT_n(\mathbb{Q})$ and $\mathbb{1}_n$ the identity matrix of size n. In the literature, authors typically prefer to work within the ring $NT_n(\mathbb{Q})$ (for instance [20],[17]). Accordingly, we will adopt this notation when stating the theorems that will be used in our analysis and whenever the group $UT_n(\mathbb{Q})$ is under consideration, it will be specified.

In [17] Levchuk provided a complete description of the automorphisms of the associated group of $NT_n(K)$, where K is an associative ring with identity, and also specified the structure in the case where the underlying ring is commutative in which the element 2 is invertible. We will state this particular case and then focus on working over the field $\mathbb{Q}$.

Let K be a commutative ring in which the element 2 is invertible. In order to state the characterization theorem for the automorphism group of the associated group of $NT_n(K)$ we begin by describing some automorphisms of this group.

For $i < j$ and $x \in K$ we define the matrix unit $x\varepsilon_{i,j}$ to be the $n \times n$ matrix with x in the (ij) position and 0 elsewhere. The relations on the associated group are

$$(x\varepsilon_{i,j}) \circ (y\varepsilon_{i,j}) = (x+y)\varepsilon_{i,j}, \tag{3.1}$$

$$[x\varepsilon_{i,j}, y\varepsilon_{k,l}] = \begin{cases} xy\varepsilon_{i,l} & \text{whenever } j = k, \\ -xy\varepsilon_{k,j} & \text{whenever } i = l, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

from the relations above it follows, that a set of generators for $NT_n(K)$ is of the form $x\varepsilon_{i,i+1}$, $x \in K$ and $i = 1, 2, \ldots, n-1$.

Transformations of the elementary matrices in $NT_n(K)$ can be extended to an automorphism of the associated group, provided that the transformation preserves relations (3.1) and (3.2), for details see [17] Lemma 3.

Now we will present some examples of automorphisms of the associated group of the ring. We present them as action on the generators $x\varepsilon_{i,i+1}$ and they can be extended using relation (3.2), for $i = 1, 2, \ldots, n-1$. The automorphisms are as follows:

1. **Inner automorphisms:** We will denote by $\mathcal{J}$ the subgroup of inner automorphisms of the associated group. For an invertible matrix $A = (a_{i,j}) \in NT_n(K)$, the inner automorphism induced by A is defined by

$$X \mapsto A^{-1} \circ X \circ A,$$

   where $X \in NT_n(K)$;

2. **Diagonal automorphisms:** We will denote by D the subgroup of diagonal automorphisms. For a diagonal matrix $\operatorname{diag}[d_1, \ldots, d_n]$, where each d_i is invertible, $i = 1, \ldots, n$, the diagonal automorphism induced by $\operatorname{diag}[d_1, \ldots, d_n]$ is given by

$$\varepsilon_{i,j} \mapsto d_i^{-1}\varepsilon_{i,j}d_j;$$

3. **Central automorphisms:** The subgroup of central automorphisms will be denoted by $\mathcal{Z}$, it is generated by the automorphisms

$$x\varepsilon_{i,i+1} \mapsto x\varepsilon_{i,i+1} + x^{\lambda}\varepsilon_{1,n},$$

where λ is a linear map of K^{+} to itself;

4. **Extremal Automorphisms:** We will denote by $\mathcal{U}$ the subgroup of extremal automorphisms, these automorphisms are given by the rule,

$$x\varepsilon_{1,2} \to x(\varepsilon_{1,2} + \lambda\varepsilon_{2,n}) + \frac{\lambda x^2}{2}\varepsilon_{1,n},$$
$$x\varepsilon_{n-1,n} \to x(\varepsilon_{n-1,n} + \mu\varepsilon_{1,n-1}) + \frac{\mu x^2}{2}\varepsilon_{1,n}, \quad x \in K,$$

for λ, μ running through K. All other generators remain fixed;

5. **Field Automorphisms:** We will denote by $\overline{\mathrm{Aut}}(K)$ the subgroup formed by induced field automorphisms. It is generated by the automorphisms

$$x\varepsilon_{i,i+1} \mapsto x^{\mu}\varepsilon_{i,i+1},$$

where μ is a field automorphism and $i = 1, \dots, n-1$;

6. **Flip Automorphisms:** Let W denote the subgroup generated by the flip automorphism which is given by

$$x\varepsilon_{i,j} \mapsto (-1)^{i-j-1}x\varepsilon_{n-j+1,n-i+1}.$$

This automorphism is given by flipping the matrix by the anti-diagonal and changing the sign of some entries. This is an automorphism of order 2.

Now we state the theorem of characterization of the automorphism group of $NT_n(K)$.

Theorem 3.1 ([17], Corollary 5)**.** Let K be a commutative ring in which the element 2 is invertible. Then the group of automorphism of the associated group of $NT_n(K)$ coincides with the product

$$(\mathcal{Z} \rtimes \mathrm{GL}_2(K)) \rtimes \overline{\mathrm{Aut}}(K),$$

for $n = 3$, and for $n > 3$, coincides with the product

$$((((\mathcal{Z}\mathcal{J}) \rtimes \mathcal{U}) \rtimes W) \rtimes D) \rtimes \overline{\mathrm{Aut}}(K).$$

We consider the particular case where the field is the rational numbers $\mathbb{Q}$. This allows to give a conciser decomposition of its automorphism group. It is well known that the only field automorphism in $\mathbb{Q}$ is the trivial one. So as a corollary of the above discussion and the mentioned results we have.

Corollary 3.1. The group of automorphism of $\mathrm{Aut}((NT_n(\mathbb{Q}),\circ))$ coincides with the product

$$((((\mathcal{Z}\mathcal{J})\rtimes\mathcal{U})\rtimes W)\rtimes D,$$

for $n>3$ and coincides with the product

$$\mathcal{Z}\rtimes \mathrm{GL}_2(\mathbb{Q}),$$

for $n=3$.

The corollary above describes the automorphism group of the associated group $(NT_n(\mathbb{Q}),\circ)$. As previously mentioned, authors in the literature generally prefer to work in the associated group $(NT_n(\mathbb{Q}),\circ)$. For this reason, we stated the existing theorems in terms of the associated group. However, it is straightforward to verify that the analogous results hold for the case of $UT_n(\mathbb{Q})$. We now proceed to argue that the automorphism group of the unitriangular group $UT_n(\mathbb{Q})$ can also be described analogously.

It was mentioned that the map from the associated group $(NT_n(\mathbb{Q}),\circ)$ to the group $(UT_n(\mathbb{Q}),\cdot)$ with operation $\cdot$ matrix multiplication, defined by $x\mapsto \mathbb{1}+x$ is an isomorphism of groups. Let us denote this isomorphism by

$$\begin{aligned}\theta : (NT_n(\mathbb{Q}),\circ) &\to (UT_n(\mathbb{Q}),\cdot)\\ x &\mapsto \mathbb{1}+x.\end{aligned}$$

Since $x\varepsilon_{i,i+1}$, $x\in\mathbb{Q}$ and $i=1,2,\dots,n-1$, is a set of generators for $(NT_n(\mathbb{Q}),\circ)$, their image by θ is also a set of generators of $UT_n(\mathbb{Q})$. Observe that in our notation, $(x\varepsilon_{i,i+1})^{\theta}=xe_{i,i+1}$. Define a map

$$\begin{aligned}\Phi : \mathrm{Aut}((NT_n(\mathbb{Q}),\circ)) &\to \mathrm{Aut}((UT_n(\mathbb{Q}),\cdot))\\ \alpha &\mapsto \theta\alpha\theta^{-1}.\end{aligned}$$

The map Φ is easily seen to be a homomorphism. It has the inverse $\theta^{-1}\alpha\theta$, so it follows that Φ is an isomorphism. In this way, we obtain a decomposition for the unitriangular group analogous to the one in Corollary 3.1. Moreover, to describe its automorphisms, it suffices to apply Φ. We illustrate the case for central automorphisms, the others automorphisms behave in the same way.

Example 3.5. Let $z\in\mathcal{Z}$ be a central automorphism of $(NT_n(\mathbb{Q}),\circ)$ generated by the automorphisms $x\varepsilon_{i,i+1}\mapsto x\varepsilon_{i,i+1}+x^{\lambda}\varepsilon_{1,n}$, where λ is a linear map of $\mathbb{Q}^{+}$ to itself. In $(UT_n(\mathbb{Q}),\cdot)$, the central automorphism z^Φ has the following action on generators,

$$\begin{aligned}(xe_{i,i+1})^{z^\Phi} &= (xe_{i,i+1})^{\theta z \theta^{-1}} = ((-\mathbb{1}) + xe_{i,i+1})^{\theta z} = (x\varepsilon_{i,i+1})^{\theta z} = \\ &= (x\varepsilon_{i,i+1} + x^\lambda \varepsilon_{1,n})^\theta = \mathbb{1} + (x\varepsilon_{i,i+1} + x^\lambda \varepsilon_{1,n}),\end{aligned}$$

for $x \in \mathbb{Q}$. So the central automorphism z^Φ of $(UT_n(\mathbb{Q}), \cdot)$ is defined by

$$xe_{i,i+1} \mapsto \mathbb{1} + (x\varepsilon_{i,i+1} + x^\lambda \varepsilon_{1,n}).$$

In this way, the passage of automorphisms between $(NT_n(\mathbb{Q}), \circ)$ and $(UT_n(\mathbb{Q}), \cdot)$ is clear. In the remainder of the text, the notation becomes increasingly heavy, and to improve readability, we will omit explicit references to the application Φ.

We present an example containing the image of central, extremal and flip automorphisms for a specific element.

Example 3.6. Let $x \in UT_4(\mathbb{Q})$, such that

$$x = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In term of $e_{i,i+1}$ notation, we have $x = [e_{1,2}, e_{2,3}]e_{2,3}e_{1,2}$. This is the decomposition of x in terms of the generators $e_{i,i+1}$.

1. We apply the central automorphism z given by

$$xe_{i,i+1} \mapsto \mathbb{1} + (x\varepsilon_{i,i+1} + x\varepsilon_{1,n}),$$

so the elements of the decomposition of x are mapped to

$$e_{1,2}^z = \mathbb{1} + (\varepsilon_{1,2} + \varepsilon_{1,4}),$$

$$e_{2,3}^z = \mathbb{1} + (\varepsilon_{2,3} + \varepsilon_{1,4}),$$

$$\begin{aligned}[e_{1,2}, e_{2,3}]^z &= \mathbb{1} + ([\varepsilon_{1,2} + \varepsilon_{1,4}, \varepsilon_{2,3} + \varepsilon_{1,4}]) = \\ &= \mathbb{1} + (([\varepsilon_{1,2}, \varepsilon_{2,3}]) + ([\varepsilon_{1,2}, \varepsilon_{1,4}]) + ([\varepsilon_{1,4}, \varepsilon_{2,3}]) + ([\varepsilon_{1,4}, \varepsilon_{1,4}])) = \\ &= \mathbb{1} + \varepsilon_{1,3}.\end{aligned}$$

We can now compute the image of x by z.

$$x^z = \begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&1&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = \begin{pmatrix}1&1&1&2\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

2. We apply the flip automorphism w given by

$$x\varepsilon_{i,j} \mapsto (-1)^{i-j-1}x\varepsilon_{n-j+1,n-i+1}.$$

so the elements of the decomposition of x are mapped to

$$e_{1,2}^w = \mathbb{1} + \varepsilon_{3,4},$$

$$e_{2,3}^w = \mathbb{1} + \varepsilon_{2,3},$$

$$e_{1,3}^w = \mathbb{1} + (-1)\varepsilon_{2,4}.$$

We can now compute the image of x by z.

$$x^w = \begin{pmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}.$$

3. We apply the extremal automorphism u given by

$$x\varepsilon_{1,2} \to x(\varepsilon_{1,2} + \varepsilon_{2,4}) + \frac{x^2}{2}\varepsilon_{1,4}.$$

All other generators remain fixed. So the elements of the decomposition of x are mapped to

$$e_{1,2}^w = \mathbb{1} + (\varepsilon_{1,2} + \varepsilon_{2,4} + \frac{1}{2}\varepsilon_{1,4}),$$

$$e_{2,3}^w = \mathbb{1} + \varepsilon_{2,3},$$

$$[e_{1,2},e_{2,3}]^z = \mathbb{1}+([\varepsilon_{1,2}+\varepsilon_{2,4}+\frac{1}{2}\varepsilon_{1,4}),\varepsilon_{2,3}]) =$$
$$= \mathbb{1}+(([\varepsilon_{1,2},\varepsilon_{2,3}])+([\varepsilon_{2,4},\varepsilon_{2,3}])+([\frac{1}{2}\varepsilon_{1,4},\varepsilon_{2,3}])) =$$
$$= \mathbb{1}+\varepsilon_{1,3}.$$

We can now compute the image of x by z.

$$x^u = \begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&1&0&\frac{1}{2}\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix} = \begin{pmatrix}1&1&1&\frac{1}{2}\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Understanding the automorphism group will be extremely useful for studying the orbits. Recall that if $A \leq \mathrm{Aut}(G)$ acts in G, the action partitions G into orbits. In this sense, it will be convenient to select an element from each orbit and write T for the resulting set representatives. Then G is the disjoint union

$$G = \bigcup_{t\in T} Orb_G(t).$$

This motivates the definition,

Definition 3.4. The set of elements of T are called *orbit representatives* to the action of A in G.

We conclude the general overview section with a useful property.

Proposition 3.4. Let $X \in UT_n(\mathbb{Q})$. If an entry on the first superdiagonal of X is nonzero, it cannot be transformed to zero via conjugation by a diagonal or unitriangular matrix.

Proof. The group $UT_n(\mathbb{Q})$ can be viewed as a subgroup of $GL_n(\mathbb{Q})$. In this context, it is known that its normalizer is $T_n(\mathbb{Q})$, the group of upper triangular matrices of dimension $n \times n$ over $\mathbb{Q}$ (for details see [1]). In particular, this subgroup decomposes as the semidirect product

$$T_n(\mathbb{Q}) = UT_n(\mathbb{Q}) \rtimes D_n(\mathbb{Q}),$$

were $D_n(\mathbb{Q})$ is the subgroup of $n \times n$ diagonal matrices over $\mathbb{Q}$. We will analyze the action of the normalizer on $UT_n(\mathbb{Q})$ and on its corresponding abelianization.

Let $T \in T_n(\mathbb{Q})$, then $T = UD$, for $U \in UT_n(\mathbb{Q})$ and $D \in D_n(\mathbb{Q})$. Moreover $U = \mathbb{1} + N$, where $\mathbb{1}$ is the identity matrix and N is a nilpotent matrix, such that $N^n = 0$. Therefore,

$$U^{-1} = (\mathbb{1}+N)^{-1} = \mathbb{1} + \sum_{k=1}^{n-1} (-1)^k N^k = \mathbb{1} - N + N^2 - \cdots + (-1)^{n-1} N^{n-1}.$$

Thus $T^{-1} = D^{-1}U^{-1}$ can be computed and

$$T^{-1}XT = D^{-1}(U^{-1}XU)D,$$

can also be computed.

Set $X = (x_{i,j}) \in UT_n(\mathbb{Q})$, $U = (a_{i,j}) \in UT_n(\mathbb{Q})$, $D = \operatorname{diag}[d_1, \ldots, d_n]$ and $T^{-1}XT = (b_{ij})$. The formula above allows us to compute all entries of the matrix $T^{-1}XT$. However, as the entries move further from the diagonal, the expression becomes increasingly complex. Nonetheless, the first and second superdiagonals can be calculated with relative ease. By applying the matrix multiplication algorithm, we obtain the following values in the entries of the first and second superdiagonals of the resulting matrix repectively,

$$b_{i,i+1} = \frac{d_{i+1}}{d_i} x_{i,i+1}, \tag{3.3}$$

$$b_{ii+2} = \frac{a_{i+1,i+2}x_{i,i+1}d_{i+1} + d_{i+1}d_{i+2}x_{i,i+2} - a_{i,i+1}x_{i+1,i+2}d_{i+2}}{d_i d_{i+1}}. \tag{3.4}$$

Note that the entries on the first superdiagonal are not affected by the matrix U, they only depend on elements of D, which are nonzero. This means that if an entry of X on the first superdiagonal is nonzero, it cannot be transformed to zero via conjugation by a diagonal or unitriangular matrix. □

3.2 The unitriangular group $UT_3(\mathbb{Q})$

With the previous section, we now have a description of the automorphism group of $UT_n(\mathbb{Q})$. To compute the number of orbits, it remains to analyze the action on suitable elements and determine how they relate under automorphism. In this section we prove,

Theorem 3.2. The unitriangular group $UT_3(\mathbb{Q})$ has 3 automorphism orbits.

Proof. Let $G := UT_3(\mathbb{Q})$. We show that elements in $G' \setminus \{\mathbb{1}\}$ and $G \setminus G'$ coincide with the two non trivial automorphism orbits. If $x, y \in G' \setminus \{\mathbb{1}\}$, for

$$x = \begin{pmatrix} 1 & 0 & x_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 & y_{1,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

it is clear that there is a diagonal matrix A given by

$$A = \begin{pmatrix} x_{1,3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y_{1,3} \end{pmatrix},$$

such that $x^A = y$. Since G' is a characteristic subgroup, its non trivial elements form an automorphism orbit. Now we analyze elements in $G \setminus G'$. Set

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix} \in G \setminus G'.$$

We now proceed by considering three distinct cases. First assume $x_{1,2} \neq 0$ and $x_{2,3} = 0$. Conjugation by matrix A yields

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_{1,2}^{-1} & -x_{1,3}x_{1,2}^{-1} \\ 0 & 0 & 1 \end{pmatrix}, \quad x^A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B_1.$$

Similarly assume $x_{1,2} = 0$ and $x_{2,3} \neq 0$. Conjugation by matrix A yields

$$A = \begin{pmatrix} 1 & x_{1,3}x_{2,3}^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_{2,3}^{-1} \end{pmatrix}, \quad x^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = B_2.$$

Now assume $x_{1,2} \neq 0$ and $x_{2,3} \neq 0$. Conjugation by matrix A yields

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_{1,2}^{-1} & x_{1,3}(x_{1,2}^2 x_{2,3})^{-1} \\ 0 & 0 & (x_{2,3}x_{1,2})^{-1} \end{pmatrix}, \quad x^A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = B_3.$$

Now we construct automorphisms that connects the three matrices B_1, B_2, B_3 above. In [17] Levchuk states that in $\mathbb{Q}$ automorphism, up to multiplication by a central automorphism, has the form

$$\mathbb{1}+x\varepsilon_{1,2} \to \mathbb{1}+\left(x(a_{1,1}\varepsilon_{1,2}+a_{1,2}\varepsilon_{2,3})+\frac{a_{1,1}a_{1,2}x^2}{2}\varepsilon_{1,3}\right),$$
$$\mathbb{1}+x\varepsilon_{2,3} \to \mathbb{1}+\left(x(a_{2,1}\varepsilon_{1,2}+a_{2,2}\varepsilon_{2,3})+\frac{a_{2,1}a_{2,2}x^2}{2}\varepsilon_{1,3}\right), \quad x \in \mathbb{Q},$$

where $(a_{i,j}) \in \mathrm{GL}_2(\mathbb{Q})$. We can choose two following matrices that serve, up to conjugation by triangular matrix, to connect these elements

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}).$$

In other words, A_1 sends B_2 to B_3 and A_2 sends B_2 to B_1. This proves that $G \setminus G'$ forms an automorphism orbit. Thus $\omega(G) = 3$. □

3.3 The unitriangular group $UT_4(\mathbb{Q})$

In this section we prove,

Theorem 3.3. The unitriangular group $UT_4(\mathbb{Q})$ has 9 automorphism orbits and 16 orbits under the action of $T_4(\mathbb{Q})$. Furthermore the number of orbits in the following characteristic subgroups are

	$\{\mathbb{1}\}$	γ_3	γ_2	$N_{2,3}$	G
$\mathrm{Aut}(G)$	1	2	4	5	9
$T_4(\mathbb{Q})$	1	2	5	7	16

Table 3.1 Number of orbits for $n = 4$.

In the table each row corresponds to a specific group of automorphisms acting on the group, while each column represents a characteristic subgroup. For instance, under the action of $T_4(\mathbb{Q})$, the characteristic subgroup γ_2 has 5 distinct orbits.

Proof. First, we outline the strategy used. We establish a partition for $UT_4(\mathbb{Q})$ and for each subset of the partition, we will present a finite number of orbit representatives. The approach involves taking an arbitrary element from the subset and constructing an automorphism that maps this element to the representative. It is worth noting that another interesting problem is to count the number of orbits under the action of conjugation by triangular matrices, since finite number of orbits under the action of conjugation by triangular matrices implies finite automorphism orbits. So we will first consider the action of $T_4(\mathbb{Q})$.

Consider the following partition of $UT_4(\mathbb{Q})$

$$UT_4(\mathbb{Q}) = Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6 \cup Y_7 \cup Y_8,$$

where

$$Y_1 = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q} \right\},$$

$$Y_2 = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, x_{1,2}, x_{2,3}, x_{3,4} \neq 0 \right\},$$

$$Y_3 = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, x_{1,2}, x_{2,3} \neq 0 \right\},$$

$$Y_4 = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, x_{1,2}, x_{3,4} \neq 0 \right\},$$

$$Y_5 = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, x_{1,2} \neq 0 \right\},$$

$$Y_6 = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, x_{2,3} \neq 0 \right\},$$

$$Y_7 = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, x_{2,3}, x_{3,4} \neq 0 \right\},$$

$$Y_8 = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, x_{3,4} \neq 0 \right\}.$$

This partition was motivated by the analysis in Propposition 3.4, since conjugation by triangular matrices modifies the entries in the first superdiagonal, but does not send a non zero entry to zero. Note that each subset of the partition corresponds to a union of cosets in the quotient group G/G'. So the conjugacy classes of elements in one subset Y_i remains in Y_i, for $1 \leq i \leq 8$.

•**Claim 1.** The number of orbits under the action of $T_4(\mathbb{Q})$ is finite.

Let $T_4(\mathbb{Q})$ act on G, this action partitions G into orbits $\{O_i\}_{i\in I}$. To prove this claim, we proceed as follows: we take an arbitrary element from one of the subsets Y_i in the partition and find a triangular matrix $A \in T_4(\mathbb{Q})$ such that, upon conjugation by A, we obtain a finite set of elements of $UT_4(\mathbb{Q})$, where

$$A = \begin{pmatrix} d_1 & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & d_2 & a_{2,3} & a_{2,4} \\ 0 & 0 & d_3 & a_{3,4} \\ 0 & 0 & 0 & d_4 \end{pmatrix}.$$

This means that we consider a finite partition in which each subset is infinite, and for each subset, we find a finite set of elements such that any element in the subset can be mapped to one of these representatives by conjugation.

Since Y_1 corresponds to the derived subgroup, it is expected to contain several other characteristic subgroups. For this reason, we will treat Y_1 last. In what follows we provide an extended explanation for the first case and the other cases are analogous.

Subset Y_2:

The subset Y_2 has elements of the form

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_{1,2}, x_{2,3}, x_{3,4}$ are non zero. We want to find entries of matrix A such that

$$x^A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We computed $x^A = (b_{ij}) \in UT_4(\mathbb{Q})$ and obtained the following entries

$$\begin{aligned}
&b_{1,1} = 1, b_{2,2} = 1, b_{3,3} = 1, b_{4,4} = 1, \\
&b_{1,2} = \frac{d_2 x_{1,2}}{d_1}, b_{1,3} = \frac{a_{2,3} d_2 x_{1,2} + d_2 d_3 x_{1,3} - a_{1,2} d_3 x_{2,3}}{d_1 d_2}, \\
&b_{1,4} = \frac{a_{2,4} d_2 d_3 x_{1,2} + a_{3,4} d_2 d_3 x_{1,3} + d_2 d_3 d_4 x_{1,4} - a_{1,2} a_{3,4} d_3 x_{2,3} - a_{1,2} d_3 d_4 x_{2,4} + a_{1,2} a_{2,3} d_4 x_{3,4} - a_{1,3} d_2 d_4 x_{3,4}}{d_1 d_2 d_3}, \\
&b_{2,3} = \frac{d_3 x_{2,3}}{d_2}, b_{2,4} = \frac{a_{3,4} d_3 x_{2,3} + d_3 d_4 x_{2,4} - a_{2,3} d_4 x_{3,4}}{d_2 d_3}, b_{3,4} = \frac{d_4 x_{3,4}}{d_3}.
\end{aligned}$$

To find the matrix A, we equate the corresponding entries b_{ij} with those of the candidate representative and set up a system of equations. Solving this system of equations we found that matrix A can defined by

$$\begin{aligned}
&d_1 = 1, d_2 = \frac{1}{x_{1,2}}, d_3 = \frac{1}{x_{1,2} x_{2,3}}, d_4 = \frac{1}{x_{1,2} x_{2,3} x_{3,4}}, \\
&a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = \frac{-x_{1,3}}{x_{1,2}^2 x_{2,3}}, \\
&a_{2,4} = -\frac{x_{1,2} x_{1,4} x_{2,3} - x_{1,2} x_{1,3} x_{2,4} - x_{1,3}^2 x_{3,4}}{x_{1,2}^3 x_{2,3}^2 x_{3,4}}, a_{3,4} = -\frac{x_{1,2} x_{2,4} + x_{1,3} x_{3,4}}{x_{1,2}^2 x_{2,3}^2 x_{3,4}}.
\end{aligned}$$

This means that all elements of Y_2 can be mapped to the matrix above via conjugation by an appropriate matrix. Most of the subsets Y_i can be treated in the same manner, following the general procedure outlined above.

Subset Y_3:

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where matrix A is defined by

$$d_1 = 1, d_2 = \frac{1}{x_{1,2}}, d_3 = \frac{1}{x_{1,2}x_{2,3}}, d_4 = 1,$$
$$a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = \frac{-x_{1,3}}{x_{1,2}^2 x_{2,3}}, a_{2,4} = \frac{x_{2,4}x_{1,3} - x_{1,4}x_{2,3}}{x_{1,2}x_{2,3}}, a_{3,4} = \frac{-x_{2,4}}{x_{2,3}}.$$

Subset Y_4:

A few specific cases require a more detailed analysis. The initial idea for Y_4 was to proceed as in the previous case, however we were unable to identify a single representative to which all elements could be conjugated. As a result, we divided the analysis into a few separate cases to account for the distinct behaviors observed. First assume $x_{1,2}x_{2,4} + x_{1,3}x_{3,4} \neq 0$ and $x_{1,3} \neq 0$.

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & x_{35} \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = \frac{1}{x_{1,2}}, d_3 = \frac{x_{3,4}}{x_{1,2}x_{2,4} + x_{1,3}x_{3,4}}, d_4 = \frac{1}{x_{1,2}x_{2,4} + x_{1,3}x_{3,4}},$$
$$a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = \frac{x_{2,4}}{x_{1,2}x_{2,4} + x_{1,3}x_{3,4}}, a_{2,4} = 0, a_{3,4} = -\frac{x_{1,4}}{x_{1,2}x_{1,3}x_{2,4} + x_{1,3}^2 x_{3,4}}.$$

Now assume $x_{1,2}x_{2,4} + x_{1,3}x_{3,4} \neq 0$ and $x_{1,3} = 0$.

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & x_{3,5} \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = \frac{1}{x_{1,2}}, d_3 = \frac{x_{3,4}}{x_{1,2}x_{2,4}}, d_4 = \frac{1}{x_{1,2}x_{2,4}},$$
$$a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = \frac{1}{x_{1,2}}, a_{2,4} = \frac{-x_{1,4}}{x_{1,2}^2 x_{2,4}}, a_{3,4} = 0.$$

Now assume $x_{1,2}x_{2,4} + x_{1,3}x_{3,4} = 0$ and $x_{1,3} \neq 0$.

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = \frac{1}{x_{1,2}}, d_3 = 1, d_4 = \frac{1}{x_{3,4}},$$
$$a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = -\frac{x_{1,3}}{x_{1,2}}, a_{2,4} = 0, a_{3,4} = -\frac{x_{1,4}}{x_{1,3}x_{3,4}}.$$

Now assume $x_{1,2}x_{2,4} + x_{1,3}x_{3,4} = 0$ and $x_{1,3} = 0$.

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = \frac{1}{x_{1,2}}, d_3 = 1, d_4 = \frac{1}{x_{3,4}},$$
$$a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = 0, a_{2,4} = -\frac{x_{1,4}}{x_{1,2}x_{3,4}}, a_{3,4} = 0.$$

Subset Y_5:

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = \frac{1}{x_{1,2}}, d_3 = 1, d_4 = \frac{1}{x_{1,2}x_{2,4}},$$
$$a_{1,2} = \frac{x_{1,4}}{x_{1,2}x_{2,4}}, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = \frac{-x_{1,3}}{x_{1,2}}, a_{2,4} = 0, a_{3,4} = 0.$$

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = \frac{1}{x_{1,2}}, d_3 = 1, d_4 = 1,$$
$$a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = \frac{-x_{1,3}}{x_{1,2}}, a_{2,4} = 0, a_{3,4} = \frac{-x_{1,4}}{x_{1,3}}.$$

Subset Y_6:

First assume $x_{1,4}x_{2,3} - x_{1,3}x_{2,4} \neq 0$.

$$x = \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = 1, d_3 = \frac{1}{x_{2,3}}, d_4 = -\frac{x_{2,3}}{x_{1,4}x_{2,3} - x_{1,3}x_{2,4}},$$
$$a_{1,2} = \frac{x_{1,3} - x_{2,3}}{x_{2,3}}, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = 0, a_{2,4} = 0, a_{3,4} = \frac{x_{1,4}x_{2,3} - (x_{1,3} - x_{2,3})x_{2,4}}{x_{1,4}x_{2,3}^2 - x_{1,3}x_{2,3}x_{2,4}}.$$

Now assume $x_{1,4}x_{2,3} - x_{1,3}x_{2,4} = 0$.

$$x = \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = 1, d_3 = \frac{1}{x_{2,3}}, d_4 = 1,$$
$$a_{1,2} = \frac{x_{1,3}}{x_{2,3}}, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = 0, a_{2,4} = 0, a_{3,4} = -\frac{x_{2,4}}{x_{2,3}}.$$

Subset Y_7:

First assume $x_{1,3} \neq 0$.

$$x = \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = 1, d_3 = \frac{1}{x_{2,3}}, d_4 = \frac{1}{x_{2,3}x_{3,4}},$$
$$a_{1,2} = \frac{x_{1,3}}{x_{2,3}}, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = \frac{-x_{1,4}x_{2,3} + x_{1,3}x_{2,4}}{x_{1,3}x_{2,3}x_{3,4}},$$
$$a_{2,4} = 0, a_{3,4} = \frac{-x_{1,4}}{x_{1,3}x_{2,3}x_{3,4}}.$$

Now assume $x_{1,3} = 0$.

$$x = \begin{pmatrix} 1 & 0 & 0 & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = 1, d_3 = \frac{1}{x_{2,3}}, d_4 = \frac{1}{x_{2,3}x_{3,4}},$$
$$a_{1,2} = 0, a_{1,3} = \frac{x_{1,4}}{x_{2,3}x_{3,4}}, a_{1,4} = 0, a_{2,3} = 0, a_{2,4} = 0,$$
$$a_{3,4} = \frac{-x_{2,4}}{x_{2,3}^2 x_{3,4}}.$$

Subset Y_8:

First assume $x_{1,3} \neq 0$.

$$x = \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$\begin{aligned} &d_1 = 1, d_2 = 1, d_3 = \frac{1}{x_{1,3}}, d_4 = \frac{1}{x_{1,3}x_{3,4}}, \\ &a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = \frac{x_{2,4}}{x_{1,3}x_{3,4}}, \\ &a_{2,4} = 0, a_{3,4} = \frac{-x_{1,4}}{x_{1,3}^2 x_{3,4}}. \end{aligned}$$

Now assume $x_{1,3} = 0$.

$$x = \begin{pmatrix} 1 & 0 & 0 & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$\begin{aligned} &d_1 = 1, d_2 = 1, d_3 = 1, d_4 = \frac{1}{x_{3,4}}, \\ &a_{1,2} = 0, a_{1,3} = \frac{x_{1,4} - x_{3,4}}{x_{3,4}}, a_{1,4} = 0, \\ &a_{2,3} = \frac{x_{2,4}}{x_{3,4}}, a_{2,4} = 0, a_{3,4} = 0. \end{aligned}$$

Subset Y_1:

In this case, the set corresponds to the derived subgroup. Therefore, we will divide the analysis into cases according to the form of the matrix. First consider elements $x \in Y_1$ of the form, where $x_{1,4} \neq 0$

$$x = \begin{pmatrix} 1 & 0 & 0 & x_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = x_{1,4}, d_2 = 1, d_3 = 1, d_4 = 1,$$
$$a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = 0, a_{2,4} = 0, a_{3,4} = 0.$$

Now consider elements $x \in Y_1$ of the form, where $x_{1,3} \neq 0$.

$$x = \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = 1, d_3 = \frac{1}{x_{1,3}}, d_4 = 1,$$
$$a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = 0, a_{2,4} = 0, a_{3,4} = \frac{-x_{1,4}}{x_{1,3}}.$$

Now consider elements $x \in Y_1$ of the form, where $x_{1,3}, x_{2,4} \neq 0$.

$$x = \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = 1, d_3 = \frac{1}{x_{1,3}}, d_4 = \frac{1}{x_{2,4}},$$
$$a_{1,2} = 0, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = 0, a_{2,4} = 0, a_{3,4} = \frac{-x_{1,4}}{x_{1,3}x_{2,4}}.$$

Finally consider elements $x \in Y_1$ of the form, where $x_{2,4} \neq 0$.

$$x = \begin{pmatrix} 1 & 0 & 0 & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x^A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Where matrix A is defined by

$$d_1 = 1, d_2 = x_{2,4}, d_3 = 1, d_4 = 1,$$
$$a_{1,2} = x_{1,4}, a_{1,3} = 0, a_{1,4} = 0, a_{2,3} = 0, a_{2,4} = 0, a_{3,4} = 0.$$

All cases have been computed and we found that every element of $UT_4(\mathbb{Q})$ can be conjugated to an element of the set $\mathcal{S}$, where

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\left. \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \cup \{\mathbb{1}\}.$$

This proves Claim 1.

•**Claim 2.** There are 16 orbits under the action of $T_4(\mathbb{Q})$.

It has already been mentioned that the orbits under the action of $T_4(\mathbb{Q})$ remain within each Y_i. Therefore, to verify that the elements in the set $\mathcal{S}$ form a set of orbit representatives, it suffices to check that the conjugacy class under the action of $T_4(\mathbb{Q})$ of a specific element does not contain any other element of the set $\mathcal{S}$. We will present a specific case, as the others follow in an analogous manner.

The conjugacy class of the candidate for orbit representative in Y_5

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

has the form

$$\begin{pmatrix} 1 & \frac{d_2}{d_1} & \frac{a_{2,3}}{d_1} & \frac{a_{2,4}d_2 - a_{1,2}d_4}{d_1d_2} \\ 0 & 1 & 0 & \frac{d_4}{d_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for $d_1, d_2, d_4 \in \mathbb{Q} \setminus \{0\}$ and $a_{1,2}, a_{2,3}, a_{2,4} \in \mathbb{Q}$. Notice that this conjugacy class does not contain

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

once d_4 is nonzero. So both elements belong to different orbits under the action of $T_4(\mathbb{Q})$.

Upon doing all calculations, this proves that the set $\mathcal{S}$ is a set of orbit representatives for the action of $T_4(\mathbb{Q})$, so Claim 2 is proved. In particular, this also shows that the number of automorphism orbits for $n = 4$ is finite.

•**Claim 3.** $\omega(UT_4(\mathbb{Q})) \leq 9$.

Now to prove that the number of automorphism orbits is at least 9 we will use our understanding of the automorphisms and attempt to connect the representatives through their action. For instance, using the central automorphism $xe_{i,i+1} \mapsto xe_{i,i+1} + xe_{1n}$, and conjugation we were able to connect $e_{2,3} \mapsto e_{2,4}e_{2,3}e_{1,2}$.

$$e_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{cent}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{conjug}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e_{2,4}e_{2,3}e_{1,2}$$

Using extremal and conjugation automorphism we were able to connect $e_{3,4}e_{1,2}$ and $e_{3,4}e_{1,3}^{-1}e_{1,2}$.

$$e_{3,4}e_{1,2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{extr}} \begin{pmatrix} 1 & 1 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{conjug}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e_{3,4}e_{1,3}^{-1}e_{1,2}$$

$$e_{1,2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{extr}} \begin{pmatrix} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{conjug}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e_{2,4}e_{1,2}$$

Using flip automorphism we were able to connect $e_{1,3} \mapsto e_{2,4}$ and $e_{1,2} \mapsto e_{2,4}e_{1,2}$.

$$e_{1,3} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{flip}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e_{2,4}$$

Using flip and conjugation automorphism we were able to connect $e_{3,4}e_{1,2} \mapsto e_{3,4}e_{1,3}^{-1}e_{1,2}$ and $e_{3,4} \mapsto e_{1,2}$.

$$e_{2,3}e_{1,2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{flip}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{conjug}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e_{3,4}e_{2,3}$$

$$e_{1,2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{flip}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e_{3,4}$$

$$e_{2,4}e_{1,2} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{flip}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto^{\text{conjug}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e_{3,4}e_{1,3}$$

We obtain the following connections between elements.

$$\begin{aligned} &\{e_{1,4}\}, \{e_{1,3}, e_{2,4}\}, \{e_{2,4}e_{1,3}\}, \\ &\{e_{3,4}e_{2,3}e_{1,2}\}, \{e_{2,3}e_{1,2}, e_{3,4}e_{2,3}\}, \\ &\{e_{3,4}e_{1,3}^{-1}e_{1,2}, e_{3,4}e_{1,2}\}, \\ &\{e_{3,4}, e_{1,2}, e_{2,4}e_{1,2}, e_{3,4}e_{1,3}\}, \\ &\{e_{2,4}e_{2,3}e_{1,2}, e_{2,3}\}. \end{aligned}$$

This implies that we have at most 9 orbits automorphism orbits. And the possible orbit representatives are:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

•**Claim 4.** $\omega(UT_4(\mathbb{Q})) = 9$.

It is not possible to connect the remaining representatives. Consider the following elements: $e_{1,4}, e_{1,3}, e_{2,4}e_{1,3}, e_{2,3}, e_{2,3}e_{1,2}, e_{1,2}, e_{3,4}e_{1,2}, e_{3,4}e_{2,3}e_{1,2}$, they are respectively:

$$\begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \in \gamma_3 \setminus \{e\},$$

$$\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix} \in \gamma_2 \setminus \gamma_3,$$

$$\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \in N_{2,3} \setminus \gamma_2,$$

$$\begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix} \in G \setminus \mathrm{N}_{2,3}\,.$$

Let us show that each element is in a distinct orbit under automorphisms. We can restrict the analysis to check whether $e_{1,3}, e_{2,4}e_{1,3}$ share the same orbit and whether $e_{2,3}e_{1,2}, e_{1,2}, e_{3,4}e_{1,2}, e_{3,4}e_{2,3}e_{1,2}$ share the same orbit.

Note that $e_{1,3}$ belongs to the subgroup $\gamma_2 \cap N_{1,2}$. The subgroup $N_{1,2}$ can only be mapped to itself or to the other maximal abelian subgroup $N_{3,4}$. Thus, given an automorphism $\alpha \in \mathrm{Aut}(G)$, we have that $e_{1,3}^{\alpha} \in \gamma_2 \cap N_{1,2}$ or $e_{1,3}^{\alpha} \in \gamma_2 \cap N_{3,4}$. On the other hand, the element $e_{2,4}e_{1,3}$ does not belong to any of these subgroups, so they cannot be in the same orbit under automorphisms.

Now, note that the element $e_{1,2} \in N_{1,2}$, and since $N_{1,2}$ is a maximal abelian subgroup, its image under an automorphism belongs to $N_{1,2}$ or $N_{3,4}$. The other elements $e_{2,3}e_{1,2}, e_{3,4}e_{1,2}, e_{3,4}e_{2,3}e_{1,2}$ do not belong to these subgroups, so $e_{1,2}$ does not share its orbit under automorphisms with the other elements. Moreover, since $e_{2,3}e_{1,2} \in N_{2,3}N_{1,2} \setminus N_{2,3}$, applying $\alpha \in \mathrm{Aut}(G)$ we have that

$$(e_{2,3}e_{1,2})^{\alpha} = e_{2,3}^{\alpha} e_{1,2}^{\alpha}.$$

We have already seen that $e_{2,3}^{\alpha}e_{1,2}^{\alpha}$ belongs to $N_{2,3}N_{1,2}$ or $N_{2,3}N_{3,4}$. Finally, note that $e_{3,4}e_{1,2} \in N_{3,4}N_{1,2}$ and, under an automorphism, it can only be mapped to elements in $N_{3,4}N_{1,2}$. Therefore, $e_{2,3}e_{1,2}$, $e_{3,4}e_{1,2}$, and $e_{3,4}e_{2,3}e_{1,2}$ are in distinct orbits. □

This proves,

Theorem D. The unitriangular group $UT_n(\mathbb{Q})$ has finitely many automorphism orbits for $n < 5$. In particular, $\omega(UT_3(\mathbb{Q})) = 3$ and $\omega(UT_4(\mathbb{Q})) = 9$.

3.4 The unitriangular group $UT_5(\mathbb{Q})$

In this section we prove,

Theorem E. The unitriangular group $UT_5(\mathbb{Q})$ has finitely many automorphism orbits. Moreover, the number of orbits under the action of $T_5(\mathbb{Q})$ is 61.

Proof. We already have an algorithm to find representatives of orbits under the action of conjugation by $T_5(\mathbb{Q})$, so we will restrict our analysis to this action. Note that if the action of $T_5(\mathbb{Q})$ yields finitely many orbits, then the action of $\mathrm{Aut}(UT_5(\mathbb{Q}))$ also produces finitely many orbits.

We will first proceed in a manner analogous to the previous case. Consider the following partition of $UT_5(\mathbb{Q})$

$$\begin{aligned} UT_5(\mathbb{Q}) =& Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6 \cup Y_7 \cup Y_8 \cup Y_9 \cup Y_{10} \cup Y_{11} \cup \\ & \cup Y_{12} \cup Y_{13} \cup Y_{14} \cup Y_{15} \cup Y_{16}, \end{aligned}$$

where

$$Y_1 = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & 0 & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & 0 & x_{3,5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q} \right\},$$

$$Y_2 = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & 0 & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & 0 & x_{3,5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{1,2} \neq 0 \right\},$$

$$Y_3 = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & 0 & x_{3,5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{2,3} \neq 0 \right\},$$

$$Y_4 = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & 0 & x_{3,5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{1,2}, x_{2,3} \neq 0 \right\},$$

$$Y_5 = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & 0 & x_{3,5} \\ 0 & 0 & 0 & 1 & x_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{2,3}, x_{4,5} \neq 0 \right\},$$

$$Y_6 = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{2,3}, x_{3,4} \neq 0 \right\},$$

$$Y_7 = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{1,2}, x_{2,3}, x_{3,4} \neq 0 \right\},$$

$$Y_8 = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & 0 & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & 0 & x_{3,5} \\ 0 & 0 & 0 & 1 & x_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{1,2}, x_{4,5} \neq 0 \right\},$$

$$Y_9 = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & 0 & x_{3,5} \\ 0 & 0 & 0 & 1 & x_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{1,2}, x_{2,3}, x_{4,5} \neq 0 \right\},$$

$$Y_{10} = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} \\ 0 & 0 & 0 & 1 & x_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{1,2}, x_{2,3}, x_{3,4}, x_{4,5} \neq 0 \right\},$$

$$Y_{11} = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & 0 & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & 0 & x_{3,5} \\ 0 & 0 & 0 & 1 & x_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{4,5} \neq 0 \right\},$$

$$Y_{12} = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & 0 & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{3,4} \neq 0 \right\},$$

$$Y_{13} = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & 0 & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} \\ 0 & 0 & 0 & 1 & x_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{3,4}, x_{4,5} \neq 0 \right\},$$

$$Y_{14} = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & 0 & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{1,2}, x_{3,4} \neq 0 \right\},$$

$$Y_{15} = \left\{ \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & x_{2,3} & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} \\ 0 & 0 & 0 & 1 & x_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{2,3}, x_{3,4}, x_{4,5} \neq 0 \right\},$$

$$Y_{16} = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ 0 & 1 & 0 & x_{2,4} & x_{2,5} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} \\ 0 & 0 & 0 & 1 & x_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : x_{i,j} \in \mathbb{Q}, \quad x_{1,2}, x_{3,4}, x_{4,5} \neq 0 \right\}.$$

•**Claim 1.** The number of orbits under the action of $T_5(\mathbb{Q})$ is finite.

We proceeded in an analogous manner of $UT_4(\mathbb{Q})$: for each subset of the partition, we identified a finite set of orbit representatives under the action of $T_5(\mathbb{Q})$. This was done by selecting an element $x \in Y_i$, $1 \leq i \leq 16$, and finding a matrix $A \in T_n(\mathbb{Q})$ that conjugates it to a chosen representative.

$$A = \begin{pmatrix} d_1 & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ 0 & d_2 & a_{2,3} & a_{2,4} & a_{2,5} \\ 0 & 0 & d_3 & a_{3,4} & a_{3,5} \\ 0 & 0 & 0 & d_4 & a_{4,5} \\ 0 & 0 & 0 & 0 & d_5 \end{pmatrix}$$

However, determining these representatives required a case-by-case analysis. We had to consider the cases where each entry is either zero or nonzero in order to solve the system of equations, this leads to a large number of cases. Moreover, there are more equations than in the case for dimension 4. Therefore, we used software SageMath [24] to solve these systems of equations. The `solve`[1] function solves equations. To use it, first we specify some variables; then the arguments to solve are a system of equations, together with the variables for which to solve.

For instance, we want to find matrix A that conjugates element

$$x = \begin{pmatrix} 1 & x_{1,2} & 0 & 0 & 0 \\ 0 & 1 & x_{2,3} & 0 & 0 \\ 0 & 0 & 1 & x_{3,4} & 0 \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in Y_{10} \quad \text{to} \quad y = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The entries of the matrix x^A are functions of the variables d_k and $a_{i,j}$, for $1 \leq k \leq 5$ and $1 \leq i < j \leq 5$. We equate these entries with the corresponding entries of y. This defines a system of 10 equations, which we denote by `eq1`, `eq2`, `eq3`, `eq4`, `eq5`, `eq6`, `eq7`, `eq8`, `eq9`, and `eq10`. We use the `solve` function with this set of equations as **input** to solve for the variables d_k and $a_{i,j}$. The code used to call the function is:

```
solve(  [eq1,eq2,eq3,eq4,eq5,eq6,eq7,eq8,eq9,eq10],
        d1,d2,d3,d4,d5,a12,a13,a14,a15,a23,a24,a25,a34,a35,a45)
```

[1]https://doc.sagemath.org/html/en/tutorial/tour_algebra.html

The **output** is

$$
\begin{aligned}
& d_1 = 1, d_2 = \frac{1}{x_{1,2}}, d_3 = \frac{1}{x_{1,2}x_{2,3}}, d_4 = \frac{1}{x_{1,2}x_{2,3}x_{3,4}}, d_5 = \frac{1}{x_{1,2}x_{2,3}x_{3,4}x_{45}}, \\
& a_{1,2} = 0, a_{1,3} = 1, a_{1,4} = 1, a_{1,5} = 1, \\
& a_{2,3} = 0, a_{2,4} = \frac{1}{x_{1,2}}, a_{2,5} = \frac{1}{x_{1,2}}, \\
& a_{3,4} = 0, a_{3,5} = \frac{1}{x_{1,2}x_{2,3}}, \\
& a_{4,5} = 0.
\end{aligned}
$$

It can be easily verified that the matrix A with those entries satisfy $x^A = y$.

The computations involved for all the cases are quite extensive, and for the sake of readability, they are presented in full in the Appendix. Our work can be reproduced, we provide the code with all systems of equations on the GitHub repository[2].

Based on the computations presented in the Appendix, we established a finite list of candidates for orbit representatives in $UT_5(\mathbb{Q})$ for the action of $T_5(\mathbb{Q})$. This proves Claim 1.

•**Claim 2.** The number of orbits under the action of $T_5(\mathbb{Q})$ is 61.

To determine a valid set of representatives, further argumentation is required. Consider the characteristic series,

$$\{\mathbb{1}\} - \gamma_4 - \gamma_3 - N_{24} - \gamma_2 - G.$$

The elements of Y_1 are precisely the elements of γ_2. To determine a set of orbit representatives in γ_2 for the action of $T_5(\mathbb{Q})$, we take all the candidates and verify whether there exists a matrix that conjugates one into another. If no such matrix exists, it means they belong to different orbits. All the subgroups mentioned above are normal, so they do not share orbits under $T_5(\mathbb{Q})$. Therefore, it suffices to verify whether the candidate representatives are connected to the others within the same subset. This can be done computing the conjugacy class of a specific element and observing that it doesn't contain any other element of the list. We will now explicit the representatives.

The orbits representatives in γ_4 are

[2]https://github.com/juhmit/unitriangular_n5.git

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $\gamma_3 \setminus \gamma_4$ are

$$\begin{pmatrix}1&0&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0&0\\0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1&0\\0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $N_{24} \setminus \gamma_3$ are

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1&0\\0&1&0&1&1\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $\gamma_2 \setminus N_{24}$ are

$$\begin{pmatrix}1&0&1&0&0\\0&1&0&1&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&0&0\\0&1&0&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$
$$\begin{pmatrix}1&0&1&0&0\\0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0&0\\0&1&0&1&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1&0\\0&1&0&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The final case $G \setminus \gamma_2$ is more intricate as it involves all elements from Y_i, for $i = 2, \ldots, 16$. Therefore, it becomes necessary to organize the group in an appropriate way, we will focus on certain subgroups, mainly the maximal abelian normal subgroups $N_{1,2}, N_{2,3}, N_{3,4}, N_{4,5}$, the characteristic subgroup γ_2, and products between them. Denote

$$\begin{aligned} N_{1,2} &:= H_1, \\ N_{2,3} &:= H_2, \\ N_{3,4} &:= H_3, \\ N_{4,5} &:= H_4, \\ \gamma_2 &:= Z_3. \end{aligned}$$

In the appendix, we present the calculations and list the candidates for orbit representatives. Based on these candidates, we verify that they are indeed representatives by proceeding as follows: All the subgroups mentioned above are normal, so they do not share orbits under $T_5(\mathbb{Q})$. Therefore, it suffices to verify whether the candidate representatives are connected to the others within the same subset. This can be done with a simple matrix conjugation computation. We will now explicit the representatives.

The orbits representatives in $H_1Z_3 \setminus Z_3$ are

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The orbits representatives in $H_2Z_3 \setminus Z_3$ are

$$\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0&1\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&1&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$
$$\begin{pmatrix}1&0&0&1&0\\0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_3Z_3 \setminus Z_3$ are

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0&1\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&1\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$
$$\begin{pmatrix}1&0&1&0&0\\0&1&0&0&1\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_4Z_3 \setminus Z_3$ are

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&1&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},$$
$$\begin{pmatrix}1&0&0&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_1H_2Z_3 \setminus (H_1Z_3 \cup H_2Z_3)$ are

$$\begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_1H_3Z_3 \setminus (H_1Z_3 \cup H_3Z_3)$ are

$$\begin{pmatrix}1&1&1&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0&0\\0&1&0&0&1\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&1&0&0\\0&1&0&0&1\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_1H_4Z_3 \setminus (H_1Z_3 \cup H_4Z_3)$ are

$$\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0&0\\0&1&0&1&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_2H_3Z_3 \setminus (H_2Z_3 \cup H_2Z_3)$ are

$$\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0&1\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_2H_4Z_3 \setminus (H_2Z_3 \cup H_3Z_3)$ are

$$\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1&0\\0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_3H_4Z_3 \setminus (H_3Z_3 \cup H_4Z_3)$ are

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_1H_2H_3Z_3 \setminus (H_1H_2Z_3 \cup H_1H_3Z_3 \cup H_2H_3Z_3)$ are

$$\begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_1H_3H_4Z_3 \setminus (H_1H_4Z_3 \cup H_1H_3Z_3 \cup H_3H_4Z_3)$ are

$$\begin{pmatrix}1&1&0&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&1&1&0&0\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_1H_2H_4Z_3 \setminus (H_1H_2Z_3 \cup H_1H_4Z_3 \cup H_2H_4Z_3)$ are

$$\begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_2H_3H_4Z_3 \setminus (H_2H_3Z_3 \cup H_2H_4Z_3 \cup H_3H_4Z_3)$ are

$$\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}.$$

The orbits representatives in $H_1H_2H_3H_4Z_3 \setminus (H_1H_2H_3Z_3 \cup H_1H_3H_4Z_3 \cup H_1H_2H_4Z_3 \cup H_2H_3H_4Z_3)$ are

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The number of orbit representatives we obtained for each of the sets is presented in the table below.

Subsets	number of orbit rep.
Z_3	15
$H_1Z_3 \setminus Z_3$	6
$H_2Z_3 \setminus Z_3$	5
$H_3Z_3 \setminus Z_3$	5
$H_4Z_3 \setminus Z_3$	6
$H_1H_2Z_3 \setminus (H_1Z_3 \cup H_2Z_3)$	2
$H_1H_3Z_3 \setminus (H_1Z_3 \cup H_3Z_3)$	4
$H_1H_4Z_3 \setminus (H_1Z_3 \cup H_4Z_3)$	3
$H_2H_3Z_3 \setminus (H_2Z_3 \cup H_2Z_3)$	2
$H_2H_4Z_3 \setminus (H_2Z_3 \cup H_3Z_3)$	4
$H_3H_4Z_3 \setminus (H_3Z_3 \cup H_4Z_3)$	2
$H_1H_2H_3Z_3 \setminus (H_1H_2Z_3 \cup H_1H_3Z_3 \cup H_2H_3Z_3)$	1
$H_1H_3H_4Z_3 \setminus (H_1H_4Z_3 \cup H_1H_3Z_3 \cup H_3H_4Z_3)$	2
$H_1H_2H_4Z_3 \setminus (H_1H_2Z_3 \cup H_1H_4Z_3 \cup H_2H_4Z_3)$	2
$H_2H_3H_4Z_3 \setminus (H_2H_3Z_3 \cup H_2H_4Z_3 \cup H_3H_4Z_3)$	1
$H_1H_2H_3H_4Z_3 \setminus (H_1H_2H_3Z_3 \cup H_1H_3H_4Z_3 \cup H_1H_2H_4Z_3 \cup H_2H_3H_4Z_3)$	1
total	61

Table 3.2 Number of orbits representatives under action of $T_5(\mathbb{Q})$.

This proves Theorem E. □

Due to the large number of possible orbite representatives, we do not compute the number of automorphism orbits of the entire group $UT_5(\mathbb{Q})$.

3.5 The unitriangular group $UT_n(\mathbb{Q})$, for $n \geq 6$

In this section we prove Theorem F. In the previous sections, we presented a method for computing automorphism orbits and finding orbit representatives. It is then natural to attempt to extend this method to higher dimensions. However, a very interesting phenomenon occurs when we increase to dimension 6, the number of automorphism orbits becomes infinite. We will prove this in the present section.

We begin by recalling that the group $UT_n(\mathbb{Q})$ is a normal subgroup of $T_n(\mathbb{Q})$ and that the normalizer of $UT_n(\mathbb{Q})$ in $GL_n(\mathbb{Q})$ is exactly the group $T_n(\mathbb{Q})$. In connection with this, we have the following result due to Zalesskii.

Theorem 3.4 ([29], Proposition 1)**.** Unipotent elements of group $T_n(\mathbb{Q})$, $n \geq 6$, partition into infinitely many conjugacy classes.

For the proof see Proposition 1 of [29]. We are now in a position to prove Theorem F.

Theorem F. The unitriangular group $UT_n(\mathbb{Q})$ has infinitely many automorphism orbits for $n > 5$.

Proof. Assume $n = 6$. By Corollary 3.1 and the comments following it, we may consider the action of $((((\mathcal{Z}\mathcal{J}) \rtimes \mathcal{U}) \rtimes W) \rtimes D$ over $UT_n(\mathbb{Q})$. Define a subset S of $UT_n(\mathbb{Q})$, such that $(x_{i,j}) \in S$ if and only if $(x_{i,j})$ has zeros elsewhere, 1's at the diagonal and at positions $(5,6)$, $(4,6)$, $(3,4)$, $(2,5)$, $(1,3)$ and any rational number at position $(1,2)$. That is, matrices of the form

$$\begin{pmatrix} 1 & x_{1,2} & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{6\times 6},$$

where the $(1,2)$ entry vary arbitrarily over $\mathbb{Q}$. We prove that if elements

$$(x_{i,j}) = \begin{pmatrix} 1 & x_{1,2} & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (y_{i,j}) = \begin{pmatrix} 1 & y_{1,2} & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in S,$$

are in the same automorphism orbit, then $x_{1,2} = y_{1,2}$.

Let φ be an automorphism of $UT_n(\mathbb{Q})$, such that $\varphi = dwukz$ for $z \in \mathcal{Z}$, $k \in \mathcal{J}$, $u \in \mathcal{U}$, $w \in W$, $d \in D$ and choose $(x_{i,j}) \in S$. If $(x_{i,j})^{\varphi} \in S$, then $(x_{i,j})^{dwuk} \in S$, so z can be ignored since this type of automorphism only modifies the $(1,6)$-entry. Denote $(x_{i,j})^{dwuk} = (y_{i,j}) \in S$.

Consider d conjugation by diagonal matrix $\mathrm{diag}[d_1,d_2,d_3,d_4,d_5,d_6]$. Note that $(x_{i,j})^d$ has zeros elsewhere except at the positions $(5,6)$, $(4,6)$, $(3,4)$, $(2,5)$, $(1,3)$, $(1,2)$, since the diagonal automorphism maps $x_{i,j} \mapsto d_i^{-1} x_{i,j} d_j$. Now, the flip automorphism has order 2 and is given by flipping the matrix by the anti-diagonal. If w is trivial or not, we also have that $(x_{i,j})^{dw}$ has zeros elsewhere except at the positions $(5,6)$, $(4,6)$, $(3,4)$, $(2,5)$, $(1,3)$, $(1,2)$. The matrices are

$$(x_{i,j})^d = \begin{pmatrix} 1 & \frac{d_2}{d_1}x_{1,2} & \frac{-d_3}{d_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{d_5}{d_2} & 0 \\ 0 & 0 & 1 & \frac{d_4}{d_3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{d_6}{d_4} \\ 0 & 0 & 0 & 0 & 1 & \frac{d_6}{d_5} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(x_{i,j})^{dw} = \begin{pmatrix} 1 & \frac{d_6}{d_5} & \frac{-d_6}{d_4} & \frac{-d_6}{d_3} & \frac{d_6}{d_2} & \frac{d_6 + d_6 x_{1,2}}{d_1} \\ 0 & 1 & 0 & 0 & \frac{d_5}{d_2} & \frac{d_5 x_{1,2}}{d_1} \\ 0 & 0 & 1 & \frac{d_4}{d_3} & 0 & \frac{-d_4}{d_1} \\ 0 & 0 & 0 & 1 & 0 & \frac{-d_3}{d_1} \\ 0 & 0 & 0 & 0 & 1 & \frac{d_2 x_{1,2}}{d_1} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Assume that the extremal automorphism $u \in \mathcal{U}$ is given by

$$xe_{1,2} \to \mathbb{1} + (x(\varepsilon_{1,2} + \lambda\varepsilon_{2,6}) + \frac{\lambda x^2}{2}\varepsilon_{1,6}),$$
$$xe_{5,6} \to \mathbb{1} + (x(\varepsilon_{5,6} + \mu\varepsilon_{1,5}) + \frac{\mu x^2}{2}\varepsilon_{1,6}),$$

for λ, μ running through $\mathbb{Q}$. And all other generators remain fixed. We note that modulo γ_4 the extremal automorphisms act like the identity (for more details see Lemma 13 of [17]). So this type of automorphism only modifies the $(1,6)$, $(2,6)$, $(1,5)$ entries of $(x_{i,j})^{dwu}$ so that the $(5,6)$, $(4,6)$, $(3,4)$, $(2,5)$, $(1,3)$, $(1,2)$, $(2,4)$, $(3,5)$ entries remain fixed. In other words, $(x_{i,j})^{dwu}$ coincide with $(x_{i,j})^{dw}$ in entries $(5,6)$, $(4,6)$, $(3,4)$, $(2,5)$, $(1,3)$, $(1,2)$,

$(2,4)$, $(3,5)$. So

$$(x_{i,j})^{dwu} = \begin{pmatrix} 1 & \frac{d_6}{d_5} & \frac{-d_6}{d_4} & \frac{-d_6}{d_3} & * & * \\ 0 & 1 & 0 & 0 & \frac{d_5}{d_2} & * \\ 0 & 0 & 1 & \frac{d_4}{d_3} & 0 & \frac{-d_4}{d_1} \\ 0 & 0 & 0 & 1 & 0 & \frac{-d_3}{d_1} \\ 0 & 0 & 0 & 0 & 1 & \frac{d_2 x_{1,2}}{d_1} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the star-marked entries depend on u.

Assume that $k \in \mathcal{J}$ is the conjugation by unitriangular matrix $A = (a_{i,j})$, then

$$(x_{i,j})^{dwuk} = (y_{i,j}) \implies A(x_{i,j})^{dwu}A^{-1} = (y_{i,j}) \implies A(x_{i,j})^{dwu} = (y_{i,j})A.$$

Denote $(x'_{ij}) = (x_{i,j})^{dwu}$. We compare the elements of matrices $A(x'_{ij})$ and $(y_{i,j})A$ at positions:

$$\begin{aligned} (1,2) &: x'_{1,2} = y_{1,2}, \\ (1,3) &: x'_{1,3} = y_{1,2}a_{2,3} + y_{1,3}, \\ (2,4) &: a_{2,3}x'_{3,4} = 0, \\ (2,5) &: x'_{2,5} = y_{2,5}, \\ (3,4) &: x'_{3,4} = y_{3,4}, \\ (3,5) &: 0 = y_{3,4}a_{4,5}, \\ (4,6) &: x'_{4,6} + a_{4,5}x'_{5,6} = y_{4,6}, \\ (5,6) &: x'_{5,6} = y_{5,6}. \end{aligned}$$

We can conclude that

$$\begin{aligned} &x'_{1,2} = y_{1,2}, \quad x'_{1,3} = y_{1,3}, \quad a_{2,3} = 0, \quad x'_{2,5} = y_{2,5}, \\ &x'_{3,4} = y_{3,4}, \quad a_{4,5} = 0, \quad x'_{4,6} = y_{4,6}, \quad x'_{5,6} = y_{5,6}. \end{aligned}$$

This means that $(x_{i,j})^{dwu}$ coincides with $(y_{i,j})$ in entries $(5,6)$, $(4,6)$, $(3,4)$, $(2,5)$, $(1,3)$, $(1,2)$,$(2,4)$, $(3,5)$. And consequently $(x_{i,j})^{dw}$ coincides with $(y_{i,j})$ at the above entries.

Now assume $w \in \mathcal{W}$ is trivial then we compare $(x_{i,j})^{dw}$ with $(y_{i,j})$ in entries

$$
\begin{aligned}
(1,2) &: \frac{d_2}{d_1}x_{1,2} = y_{1,2}, \\
(1,3) &: \frac{d_3}{d_1} = 1, \\
(2,5) &: \frac{d_5}{d_2} = 1, \\
(3,4) &: \frac{d_4}{d_3} = 1, \\
(4,6) &: \frac{d_6}{d_4} = 1, \\
(5,6) &: \frac{d_6}{d_5} = 1.
\end{aligned}
$$

From the relations above we obtain that $d_1 = d_3 = d_4 = d_6 = d_5 = d_2$ so the relations implies that $x_{1,2} = y_{1,2}$.

If w is not trivial, then we compare $(x_{i,j})^{dw}$ with $(y_{i,j})$ in entries

$$
\begin{aligned}
(1,2) &: \frac{d_6}{d_5} = y_{1,2}, \\
(1,3) &: -\frac{d_6}{d_4} = 1, \\
(2,5) &: \frac{d_5}{d_2} = 1, \\
(3,4) &: \frac{d_4}{d_3} = 1, \\
(4,6) &: -\frac{d_3}{d_1} = 1, \\
(5,6) &: \frac{d_2}{d_1}x_{1,2} = 1.
\end{aligned}
$$

From the relations above we obtain that

$$\frac{d_6}{d_5}\frac{d_2}{d_1}x_{1,2} = y_{1,2}, \quad \frac{d_3}{d_1}\frac{d_6}{d_4} = 1, \quad d_5 = d_2, \quad d_4 = d_3.$$

So $x_{1,2} = y_{1,2}$. This means that if the elements

$$(x_{i,j}) = \begin{pmatrix} 1 & x_{1,2} & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (y_{i,j}) = \begin{pmatrix} 1 & y_{1,2} & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in S,$$

are in the same automorphism orbit, then $x_{1,2} = y_{1,2}$, and so $(x_{i,j}) = (y_{i,j})$. This shows that S has infinitely many orbit representatives, once the set $\mathbb{Q}$ is infinite and any two elements in S with differing entries in $(1,2)$ belong to distinct automorphism orbits. This proves the Theorem for $n = 6$.

Now we extend to $n \geq 6$. One can view $UT_n(\mathbb{Q})$ as $\begin{pmatrix} UT_6(\mathbb{Q}) & \star \\ 0 & UT_{n-6}(\mathbb{Q}) \end{pmatrix}$ so that $\begin{pmatrix} X & \star \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} Y & \star \\ 0 & I \end{pmatrix}$ are in the same automorphism orbit in $UT_n(\mathbb{Q})$ if and only if X and Y are in the same automorphism orbit, for $X, Y \in UT_6(\mathbb{Q})$. So we have even fewer automorphisms to consider, as we can disregard flip and extremal automorphisms. Nevertheless, the same argument applies to prove that the number of automorphism orbits is infinite and Theorem E is proved. $\square$

We note that the result would still hold if we considered the action of $\mathrm{Aut}(UT_6(\mathbb{Q}))$ on the quotient $UT_6(\mathbb{Q})/\gamma_4$.

This concludes Chapter 3.

Final considerations

To conclude this work, we discuss possible questions to be addressed in future investigations.

Li and Zhu published a paper [18], in 2025, that classifies all finite p-groups with exactly three automorphism orbits, an odd prime p. With this case now settled, new classification questions have naturally arisen.

Question 3.1. Let G be a finite rank soluble group. Is it possible to classify all groups with $\omega(G) = 3$?

Question 3.2. Let G be a finite rank metabelian group. Is it possible to classify all groups with $\omega(G) = 3$?

In the case of finite groups, within the classification of solvable non-p-groups with 4 automorphism orbits, the groups with 3 automorphism orbits appear as part of the description. Given this recent result [18] and the fact that the classification of finite groups with 3 orbits is now complete, we are led to ask the following question:

Question 3.3. Let G be a finite soluble non-p-group. Is it possible to classify all groups with $\omega(G) = 5$?

Regarding Theorems D, E, and F, several additional questions naturally arise and deserve further investigation. For instance, in the proof of Theorem F we present a subset of $UT_6(\mathbb{Q})$ which contain infinitely many orbit representatives. It is natural to ask:

Question 3.4. Find a set of orbit representatives of $UT_6(\mathbb{Q})$.

The unitriangular groups $UT_n(\mathbb{Q})$ form an important example of nilpotent groups, in which the terms of the lower central series can be easily described. Note that $UT_n(\mathbb{Q})$ also is torsion-free of finite rank. The study of $UT_n(\mathbb{Q})$ and its automorphims motivated us to ask,

Question 3.5. Is the nilpotency class for torsion-free groups of finite rank with finitely many automorphism orbits of bounded?

Another approach is to study linear groups. Some well-known results in this direction are the following:

Theorem 3.5 ([19], 3.1.2)**.** (Tits Alternative) Let G be a linear group over a field F.

1. If $\mathrm{char}(F) = 0$, then G is virtually solvable or G contains a nonabelian free subgroup of rank 2;

2. If $\mathrm{char}(F) \neq 0$ and G is finitely generated, the same conclusion holds.

Theorem 3.6 ([19], 3.1.6)**.** (Mal'cev) Let V be a finite-dimensional vector space of dimension n over an algebraically closed field F, and let G be a solvable subgroup of $GL(V,F)$.

1. If G is irreducible, then G has a normal diagonalizable subgroup D such that $[G:D] \leq g(n)$ for some function g;

2. In general, G has a normal triangulable subgroup T such that $[G:T] \leq f(n)$ for some function f.

We have considered the following question.

Question 3.6. Determine the characteristic subgroups of $UT_n(\mathbb{Q})$ with finitely many orbits under $\mathrm{Aut}(UT_n(\mathbb{Q}))$ (and under $T_n(\mathbb{Q})$).

This concludes the Final considerations chapter.

References

[1] J. Alperin and R. Bell. *Groups and Representations*. Graduate Texts in Mathematics. 162. Springer-Verlag. New York. 1995.

[2] R. Bastos and A. Dantas. *FC-groups with finitely many automorphism orbits*. J. Algebra, **516**:401–413, 2018.

[3] R. Bastos, A. Dantas and M. Garonzi, *Finite groups with six or seven automorphism orbits*. J. Group Theory, **20(5)**:945–954, 2017.

[4] R. Bastos, A. Dantas and E. de Melo. *Soluble groups with few orbits under automorphisms*. Geom. Dedicata, **209**:119-123, 2020.

[5] R. Bastos, A. Dantas and E. de Melo. *Virtually nilpotent groups with finitely many orbits under automorphisms*. Archiv der Mathematik, **116**:261-270, 2021.

[6] J. Bidwell, M. Curran and D. McCaughan. *Automorphisms of direct products of finite groups*. Archiv der Mathematik, **86**: 481–489, 2006.

[7] A. Bors and S. P. Glasby. *Finite 2-groups with exactly three automorphism orbits*. Journal of Group Theory, 2025.

[8] W. Burnside. *Theory of groups of finite order*. Dover, 1955.

[9] GAP-Group. GAP – groups, algorithms, and programming, version 4.10.1, 2019. https://www.gap-system.org [last accessed June 2025].

[10] S. Glasby. *Subgroups of the upper-triangular matrix group with maximal derived length and a minimal number of generators*. arXiv:1410.5052, 2014.

[11] D. Gorenstein. *Finite Groups*. Second edition, Chelsea Publishing Co., New York, 1980.

[12] F. Gross. *2-automorphic 2-groups*. J. Algebra, **40(2)**:348-353, 1976.

[13] G. Higman. *Suzuki 2-groups*. Illinois J. Math., **7**:79-96, 1963.

[14] K. Hoffman and R. Kunze. *Linear Algebra*. 2nd Edition, Prentice Hall Inc., Upper Saddle River, 1971.

[15] L. Jafari, S. Kohl, and E. O'Brien. *Automorphism group orbits on finite simple groups*. Comm. Algebra, **49(8)**:3294-3300, 2021.

[16] T. J. Laffey and D. MacHale. *Automorphism orbits of finite groups*. J. Austral. Math. Soc. Ser. A, **40(2)**:253-260, 1986.

[17] V. Levchuk. *Connections between the unitriangular group and certain rings. II. Groups of automorphisms*. Siberian Journal of Mathematics , **24(4)**:543–557, 1983.

[18] C. Li and Y. Zhu. *The finite groups with three automorphism orbits*. Journal of Algebra, **678**:677-705, 2025.

[19] J. Lennox and D. Robinson. *The Theory of Infinite Soluble Groups*, Clarendon Press, Oxford (2004).

[20] A. Mahalanobis. *The automorphism group of the group of unitriangular matrices over a field*. International Journal of Algebra, **7(15)**: 723-733, 2013.

[21] H. Mäurer and M. Stroppel. *Groups that are almost homogeneous*. Geom. Dedicata, **68**:229-243, 1997.

[22] J. Maginnis. *Outer automorphism of upper triangular matrices*. Journal of Algebra **161**:267-270, 1993.

[23] D. Robinson. A course in the theory of groups, 2nd edition, Springer Verlag, New York, 1996.

[24] SageMath, the Sage Mathematics Software System (Version 9.3), The Sage Developers, 2023, https://www.sagemath.org.

[25] M. Schwachhöfer and M. Stroppel. *Finding representatives for the orbits under the automorphism group of a bounded abelian group*. J. Algebra, **211**: 225-239, 1999.

[26] E. Shult. *On finite automorphic algebras*. Illinois J. Math., **13**:625–653, 1969.

[27] M. Stroppel. *Locally compact groups with few orbits under automorphisms*. Top. Proc. **26(2)**:819-842, 2002.

[28] A. Weir. *Sylow p subgroup of the general linear group over finite fields of characteristic p*. Proceedings of the American Mathematical Society, **6(3)**:454-464, 1955.

[29] A. E. Zalesskii. *A remark on the triangular linear group*. Vesti Acad. Navuk BSSR, ser. fiz. mat. navuk, **2**:129-131, 1968.

[30] J. Zhang. *On finite groups all of whose elements of the same order are conjugate in their automorphism groups*. J. Algebra, **153(1)**:22–36, 1992.