Universidade de Brasília

Instituto de Física

Theoretical Foundations of the Sunyaev-Zel'dovich Effect

Fundamentos teóricos do efeito Sunyaev-Zel'dovich

Raul Grande Quartieri

DISSERTAÇÃO DE MESTRADO

PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA - PPGFÍS

Brasília

2025

FICHA CATALOGRÁFICA

Grande Quartieri, Raul.

Theoretical Foundations of the Sunyaev-Zel'dovich Effect / Raul Grande Quartieri; orientadora Mariana Penna Lima Vitenti. – Brasília, 2025.

69 p.

Dissertação de Mestrado (Programa de Pós-Graduação em Física - PPGFÍS) – Universidade de Brasília, 2025.

1. Sunyaev-Zeldovich Effect. 2. Inverse Compton Scattering. 3. Boltzmann Equation. 4. Cosmic Microwave Background. 5. Quantum Field Theory.I. Penna Lima Vitenti, Mariana, orient. II. Título.

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Orientadora: Prof. Dra. Mariana Penna Lima Vitenti

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Agradecimentos

Agradeço, em primeiro lugar, à minha família, por sempre acreditar em mim e me apoiar incondicionalmente. Especialmente aos meus pais, Sílvia e Leonardo, pelo amor, paciência e pelos valores que me transmitiram, fundamentais para minha formação pessoal e profissional. À minha irmã Lívia, pelo companheirismo e pelas palavras de incentivo nos momentos mais desafiadores. E aos pequenos irmãos, Alice e Gael, por me inspirarem a ser uma pessoa melhor a cada dia.

Aos meus primos, Gabriel e Laís, por sempre serem companheiros de aventuras e estarem ao meu lado também nos momentos difíceis. Ao Daniel, por todo o carinho, apoio e pela presença constante como parte essencial da nossa família. E ao Ícaro, por sua amizade e companhia ao longo desses anos.

Aos amigos João Vítor, Nathan, Carina, Gabriel, Pedro e João Victor, que foram presença constante, mesmo nos momentos de silêncio e distância. Obrigado por compartilharem alegrias, dúvidas, discussões e por tornarem essa jornada mais leve e significativa. A convivência com vocês foi essencial para manter o equilíbrio e a motivação.

In memoriam, ao professor Dr. Álvaro Gomes dos Santos, agradeço imensamente pela oportunidade de ter trabalhado com você e por ter reacendido minha paixão pela Física.

À minha orientadora, Dra. Mariana Penna Lima Vitenti, expresso minha profunda gratidão pela confiança, orientação, dedicação e entusiasmo. Sua competência acadêmica, rigor intelectual e generosidade foram inspirações constantes ao longo de todo este trabalho. Sou muito grato por ter tido a oportunidade de aprender com você.

Agradeço a todos que, de alguma forma, contribuíram para que este trabalho fosse possível. Cada gesto, conversa ou ajuda teve um papel importante nesta trajetória.

Por fim, agradeço ao apoio financeiro da CAPES.

Abstract

The cosmic microwave background (CMB) is a powerful observational tool for probing the formation and evolution of structures in our universe, encoding information through both primary anisotropies, imprinted by primordial fluctuations, and secondary anisotropies, which arise from interactions with large-scale structures of the universe. Among the secondary anisotropies, a particularly powerful probe of ionized gas, such as that found in the intra-cluster medium (ICM) of galaxy clusters, is the Sunyaev-Zel'dovich effect (SZE), i.e., the spectral distortion induced in the CMB spectrum by the interaction of CMB photons with free electrons in the ICM via inverse Compton scattering.

Although the phenomenology of both the thermal and kinetic components of the SZE is well understood, a rigorous derivation of this effect is rarely presented in the literature. Therefore, in this work, we provide such a derivation by formulating the Boltzmann equation in a curved spacetime framework and applying tools from quantum field theory to describe the underlying microscopic interactions responsible for the SZ effect.

Keywords: Sunyaev-Zeldovich Effect. Inverse Compton Scattering. Boltzmann Equation. Cosmic Microwave Background. Quantum Field Theory.

Resumo

A radiação cósmica de fundo em micro-ondas (CMB) é uma poderosa ferramenta observacional para investigar a formação e a evolução de estruturas em nosso universo, codificando informações tanto por meio de anisotropias primárias, impressas por flutuações primordiais, quanto por meio de anisotropias secundárias, que surgem de interações com estruturas de grande escala do universo. Entre as anisotropias secundárias, uma sonda particularmente poderosa para gás ionizado, como a encontrada no meio intra-aglomerado (ICM) de aglomerados de galáxias, é o efeito Sunyaev-Zel'dovich (SZE), ou seja, a distorção espectral induzida no espectro da CMB pela interação de fótons da CMB com elétrons livres no ICM via espalhamento Compton inverso.

Embora a fenomenologia dos componentes térmico e cinético da SZE seja bem compreendida, uma derivação rigorosa desse efeito raramente é apresentada na literatura. Portanto, neste trabalho, realizamos tal derivação formulando a equação de Boltzmann em uma estrutura de espaço-tempo curvo e aplicando ferramentas da teoria quântica de campos para descrever as interações microscópicas subjacentes responsáveis pelo efeito SZ.

Palavras-chave: Efeito Sunyaev-Zeldovich. Espalhamento Compton Inverso. Equação de Boltzmann. Radiação Cosmica de Fundo em Micro-ondas. Teoria Quântica de Campos.

List of figures

List of abbreviations and acronyms

CMB	Cosmic Microwave Background radiation
FLRW	Friedmann-Lemaître-Robertson-Walker
GR	General Relativity
ICM	intra-cluster medium
kSZE	Kinetic Sunyaev-Zeldovich Effect
LHS	Left hand side
NR	Non-Relativistic Limit
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
RHS	Right hand side
SZE	Sunyaev-Zeldovich Effect
tSZE	Thermal Sunyaev-Zeldovich Effect

Contents

1 Introduction

One of the most powerful observational probes in modern cosmology is the cosmic microwave background radiation (CMB). This radiation represents the first light emitted in the universe and was generated approximately 380,000 years after the Big Bang, during the epoch of recombination, when the universe cooled enough for protons and electrons to combine into neutral hydrogen atoms. As a result, photons decoupled from matter and began to travel freely through space, carrying information about the physical conditions and primordial density fluctuations of the early universe. These imprints are observed today as tiny temperature and polarization anisotropies in the CMB (known as *primary anisotropies*) which provide a snapshot of the universe at that early epoch, before the formation of large-scale structures.

In addition to these primordial imprints, the CMB also contains the so-called *secondary anisotropies*, which arises from interactions between the CMB photons and matter as they travel through the evolving universe. These include gravitational effects, such as the Integrated Sachs-Wolfe effect and gravitational lensing, as well as scattering processes such as the Sunyaev-Zel'dovich effect (SZE), where CMB photons gain energy by interacting with hot electrons in the intracluster medium (ICM) of galaxy clusters. By analysing both primary and secondary anisotropies, we can extract a wealth of information about the universe's composition, geometry, and evolution from the physics of the early universe to the growth of cosmic structures.

Of particular interest in this work is the SZE, which was first proposed by Sunyaev and Zel'dovich in (Zeldovich; Sunyaev, 1969), and which we aim to investigate more deeply, not merely from a phenomenological point of view, which is already well established through both theoretical modelling and observational confirmation (Planck Collaboration, 2016; Bleem *et al.*, 2015), but rather from a more fundamental, microscopic perspective. While this study does not aim to introduce new physical results or propose novel mechanisms, it seeks to revisit the SZE within a rigorous quantum field theoretical and statistical framework, clarifying the assumptions and derivations that underlie its standard description. This effort contributes by bridging the gap between phenomenological treatments and more fundamental approaches, offering a systematic derivation of the Boltzmann collision term and exploring how quantum-level interactions manifest in macroscopic observables. In this sense, although the effect itself is well known, the present analysis adds value by deepening our conceptual understanding and highlighting the theoretical consistency behind the observable signatures of the SZE in the CMB.

To achieve this goal, we adopt a two-fold approach. First, in Chapter 2, we formulate the Boltzmann equation within the framework of general relativity, accounting for the curvature of spacetime in a spatially flat, expanding universe. This formulation allows for a consistent description of the gravitational effects on the photon distribution function. Second, still in Chapter 2, we employ tools from quantum field theory (QFT) to construct a general form of the collision operator for n-particle interactions. This development leads us to Chapter 3, where we apply quantum electrodynamics (QED) to model the microscopic process of Compton scattering between CMB photons and electrons. By integrating these elements, we derive the collision term that governs the SZE, encapsulating the net effect of scattering processes on the evolution of the photon distribution. Finally, in Chapter 4, we bring together all these ingredients to derive how the photon distribution is modified by inverse Compton scattering and to demonstrate the resulting distortion in the CMB spectrum.

2 The Fundamentals of the Boltzmann equation

In 1872, Boltzmann published the paper (Boltzmann, 2003), in which he investigated how to describe the behaviour of gas molecules in a thermodynamic system. This work led to the formulation of what is now known as the Boltzmann equation. To achieve this, he introduced the distribution function $f(\zeta,t)$ representing the number of molecules with energy ζ at a time t, and derived a partial differential equation for f, considering how the distribution changes during a short time interval as the result of collisions between the molecules that make up the gas. In addition to that, he considered that there were no external forces and that the conditions were uniform throughout the gas. With these considerations, he successfully derived the equation for monatomic gas molecules:

$$\frac{\partial f(\zeta,t)}{\partial t} = \int_0^{\infty}\int_0^{\zeta+\zeta'}\left[\frac{f(\xi,t)}{\xi^{\frac{1}{2}}}\frac{f(\xi',t)}{(\xi')^{\frac{1}{2}}} - \frac{f(\zeta,t)}{\zeta^{\frac{1}{2}}}\frac{f(\zeta',t)}{\zeta'^{\frac{1}{2}}}\right](\zeta\zeta')^{\frac{1}{2}}\psi(\zeta,\zeta',\xi)\zeta' d\xi, \tag{2.1}$$

where ζ and ζ' are the energy of the molecules before the interaction (collision), and ξ and $\xi' = \zeta + \zeta' - \xi$ are the energy of the molecules after the interaction. The term $\psi(\zeta,\zeta',\xi)$ describes the interaction.

Later, Boltzmann generalized this result for polyatomic gases, but we do not need to discuss these results here; the important fact is that his work shows us how to properly write a differential equation for the evolution of the distribution function in thermodynamic systems. Although the equation used today differs in appearance, the idea behind it is the same as that proposed by Boltzmann. Thus, we now focus on deriving the contemporary version using quantum field theory (QFT).

2.1 The Boltzmann equation

In contrast to Boltzmann's original formulation, which described particle distributions as functions of time and energy, modern relativistic kinetic theory describes them through a phase-space distribution function $f = f(z^M)$, defined on the cotangent bundle of spacetime, whose coordinates are given by $z^M = (x^\mu, p_\mu)$ (Acuña-Cárdenas; Gabarrete; Sarbach, 2022). As usual, we choose Greek indices to vary from zero to 3, and $x^\mu = (t,\mathbf{x})$, $p_\mu = (-E,\mathbf{p})$, which leads to the mass-shell condition $p^\mu p_\mu = -m^2 = -E^2 + p^2$ (note that $|\mathbf{p}| = p$, and we choose a metric with signature $(-,+,+,+)$).

The use of the cotangent bundle (rather than the tangent bundle) is natural for a Hamiltonian formulation, since the momenta are canonically conjugate to positions.

However, the metric provides a natural isomorphism between the tangent-cotangent bundles, so both formulations are, in principle, equivalent.

Despite these technicalities, the core idea remains the same as in Boltzmann's original work: to understand how the distribution of particles evolves under the influence of interactions. However, in a more elegant way, this evolution is formulated using operator equations acting on the distribution function $f(z^M)$. Following (Enomoto *et al.*, 2023), we will write the Boltzmann equation in the form:

$$\mathbf{L}[f] = \mathbf{C}[f]. \tag{2.2}$$

Here, $\mathbf{L}$ is the Liouville operator, which encapsulates the geometric evolution in phase-space (i.e. the gravitational effect on the distribution of particles, in the case of the general relativity framework), while $\mathbf{C}$ is the collision operator, which quantifies the evolution of the distribution function given the microscopic interactions. Crucially, constructing $\mathbf{C}$ requires the QFT formalism.

2.1.1 The Liouville operator in the context of General Relativity

In classical mechanics, the Liouville operator governs the evolution of distribution functions in phase space, driven by a flow generated by a Hamiltonian $\mathcal{H}$. In curved spacetime, as described by general relativity (GR), this is analogous to taking the Lie derivative of the distribution function along the Hamiltonian vector field $X_{\mathcal{H}}$, sometimes referred to as the Liouville vector field (Acuña-Cárdenas; Gabarrete; Sarbach, 2022). We can write the vector field $X_{\mathcal{H}}$ as:

$$X^M_{\mathcal{H}} = \left(\frac{\partial \mathcal{H}}{\partial p_\mu}, -\frac{\partial \mathcal{H}}{\partial x^\mu}\right). \tag{2.3}$$

On the other hand, we can describe the path that particles take in a curved spacetime by the parameter λ, which makes it possible to write Hamilton's equations of motion in the form:

$$\frac{dx^\mu}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_\mu}, \quad \frac{dp_\mu}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^\mu}. \tag{2.4}$$

Therefore, the vector field (Equation (2.3)) can be written as:

$$X^M_{\mathcal{H}} = \left(\frac{dx^\mu}{d\lambda}, \frac{dp_\mu}{d\lambda}\right). \tag{2.5}$$

Now, expressing the action of the Liouville operator on the distribution function $f = f(z^M)$ as the Lie derivative of the distribution along $X^M_{\mathcal{H}}$, we have:

$$\mathbf{L}[f] = \mathcal{L}_{X^M_{\mathcal{H}}} f = X^M_{\mathcal{H}} \frac{\partial f}{\partial z^M} = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{dp_\mu}{d\lambda}\frac{\partial f}{\partial p_\mu}. \tag{2.6}$$

A proof of the Liouville theorem using the Lie derivative as the Liouville operator can be found in chapter X of the (Choquet-Bruhat, 2009) book and in (Acuña-Cárdenas; Gabarrete; Sarbach, 2022).

In the context of GR, we can further simplify equation (2.6) using the fact that free particles follow a geodesic path, in which the parameter λ is the affine parameter of the geodesic. Furthermore, in the Hamiltonian formalism of GR, this geodesic path is associated with the Hamiltonian (Acuña-Cárdenas; Gabarrete; Sarbach, 2022):

$$\mathcal{H}(x,p) = \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu, \tag{2.7}$$

where $g^{\mu\nu}$ is the metric tensor, and $p^\mu p_\mu = -m^2$ is the mass-shell condition that enforces relativistic consistency. Consequently, the first equation of motion in equation (2.4) results in:

$$\begin{aligned}
\frac{dx^\mu}{d\lambda} &= \frac{\partial \mathcal{H}}{\partial p_\mu} = \frac{\partial}{\partial p_\mu}\left(\frac{1}{2}g^{\alpha\beta}(x)p_\alpha p_\beta\right) \\
&= \frac{1}{2}g^{\alpha\beta}(\delta^\mu_\alpha p_\beta + \delta^\mu_\beta p_\alpha) \\
&= g^{\mu\alpha}p_\alpha
\end{aligned}$$

$$\therefore \frac{dx^\mu}{d\lambda} = p^\mu. \tag{2.8}$$

On the other hand, the second equation of motion gives:

$$\begin{aligned}
\frac{dp_\mu}{d\lambda} &= -\frac{\partial \mathcal{H}}{\partial x^\mu} = -\frac{\partial}{\partial x^\mu}\left(\frac{1}{2}g^{\alpha\beta}(x)p_\alpha p_\beta\right) \\
&= -\frac{1}{2}\left(\frac{\partial g^{\alpha\beta}(x)}{\partial x^\mu}\right)p_\alpha p_\beta.
\end{aligned} \tag{2.9}$$

However, it is more usual, and convenient to write the Liouville operator in terms of the affine connection:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right). \tag{2.10}$$

To do this, we have to relate the derivative of the metric to the connection. This relation is given by:

$$\frac{\partial g^{\alpha\beta}(x)}{\partial x^\mu} = -2g^{\beta\nu}\Gamma^\alpha_{\mu\nu}. \tag{2.11}$$

Proof. To prove this relation, we begin by differentiating the identity

$$g^{\alpha\beta}g_{\beta\nu} = \delta^\alpha_\nu.$$

Therefore:

$$\frac{\partial}{\partial x^\mu}\left(g^{\alpha\beta}g_{\beta\nu}\right) = g_{\beta\nu}\frac{\partial g^{\alpha\beta}}{\partial x^\mu} + g^{\alpha\beta}\frac{\partial g_{\beta\nu}}{\partial x^\mu} = 0.$$

Now we multiply both sides of this equation by $g^{\rho\nu}$, which gives after some manipulations:

$$\frac{\partial g^{\alpha\beta}}{\partial x^\mu} = -g^{\alpha\rho}g^{\beta\nu}\frac{\partial g_{\rho\nu}}{\partial x^\mu}.$$

Now, we contract the contravariant index of $\Gamma^{\sigma}_{\mu\nu}$ and rewrite equation (2.10) as:

$$\Gamma_{\rho\mu\nu} = \frac{1}{2}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right),$$

which allows us to write:

$$\frac{\partial g_{\rho\nu}}{\partial x^{\mu}} = \Gamma_{\rho\mu\nu} + \Gamma_{\nu\mu\rho},$$

and consequently, using the symmetry relation of the indices of the connection, we have:

$$\begin{aligned}\frac{\partial g^{\alpha\beta}}{\partial x^{\mu}} &= -g^{\alpha\rho}g^{\beta\nu}\left(\Gamma_{\rho\mu\nu} + \Gamma_{\nu\mu\rho}\right)\\ &= -2\Gamma^{\alpha\beta}_{\mu}.\end{aligned}$$

On the other hand $\Gamma^{\alpha\beta}_{\mu} = g^{\beta\nu}\Gamma^{\alpha}_{\mu\nu}$, which results in:

$$\frac{\partial g^{\alpha\beta}(x)}{\partial x^{\mu}} = -2g^{\beta\nu}\Gamma^{\alpha}_{\mu\nu}.$$

□

This relation allows us to write the second equation of motion (equation (2.9)) as:

$$\begin{aligned}\frac{dp_{\mu}}{d\lambda} &= -\frac{1}{2}\left(-2g^{\beta\nu}\Gamma^{\alpha}_{\mu\nu}\right)p_{\alpha}p_{\beta}\\ &= \Gamma^{\alpha}_{\mu\nu}p_{\alpha}p^{\nu}.\end{aligned} \tag{2.12}$$

Finally, we can write the Liouville operator (equation (2.6)) using equations (2.8) and (2.12) in the form:

$$\mathbf{L}[f] = p^{\mu}\frac{\partial f}{\partial x^{\mu}} + \Gamma^{\alpha}_{\mu\nu}p_{\alpha}p^{\nu}\frac{\partial f}{\partial p_{\mu}}. \tag{2.13}$$

which makes it possible to write the Boltzmann equation (2.2) as:

$$p^{\mu}\frac{\partial f}{\partial x^{\mu}} + \Gamma^{\alpha}_{\mu\nu}p_{\alpha}p^{\nu}\frac{\partial f}{\partial p_{\mu}} = \mathbf{C}[f]. \tag{2.14}$$

Later in this chapter, we will express the Liouville operator explicitly using the metric of Friedmann-Lemaître-Robertson-Walker (FLRW), which is spatially homogeneous and isotropic. For now, however, we will focus on deriving an explicit expression for the collision operator.

2.1.2 The collision operator

We define the collision operator to be the rate of change of the distribution function over the geodesic by the microscopic processes (M.P.) (Enomoto *et al.*, 2023), that is:

$$\mathbf{C}[f] \equiv \left.\frac{df}{d\lambda}\right|_{\text{M.P}} = E\left.\frac{df}{dt}\right|_{\text{M.P}} \approx E\frac{\delta f}{\delta t}. \tag{2.15}$$

The second equality can be independently verified using the time component of equation (2.8), with the subsequent approximation arising from the vast difference between the rapid timescale of microscopic collisions and the much slower timescale of macroscopic evolution.

To find an explicit formula for (2.15), we consider the following statements:

- The M.P. can be described by QFT.
- The M.P. can be evaluated in the Minkowski space, since the Liouville operator carries the information of the gravitational effect.

The first of these considerations implies that the distribution function can be understood as the expectation value of all possible occupation numbers with their probabilities, and the second implies that these probabilities, i.e., the transition amplitude for the effect we want to study, can be calculated in flat space using standard QFT.

2.1.2.1 QFT Foundations

To begin, we define a multi-particle state as:

$$|\{n\}\rangle = \bigotimes_{s,\mathbf{p}} |n_s(\mathbf{p})\rangle \,, \tag{2.16}$$

where s represents the species of the particle, and $\bigotimes$ represents the tensor product in all independent single-particle modes. We construct this multi-particle state to be an eigenstate of the occupation number operator $\hat{n}_s(\mathbf{p})$:

$$\hat{n}_s(\mathbf{p})\,|\{n\}\rangle = n_s(\mathbf{p})\,|\{n\}\rangle \,, \tag{2.17}$$

i.e., the occupation number operator selects the particles of type s and momentum $\mathbf{p}$ and returns how many of them are in the volume of our system $V \equiv \int d^3x = (2\pi)^3\delta^{(3)}(\mathbf{p}=0)$. Therefore, this is very similar to what we usually do for a Harmonic oscillator in QFT, and consequently, we can define the number operator as:

$$\hat{n}_s(\mathbf{p}) \equiv \frac{1}{V} a_{\mathbf{p}}^{(s)\dagger} a_{\mathbf{p}}^{(s)}. \tag{2.18}$$

As usual, $a_{\mathbf{p}}^{s}$ and $a_{\mathbf{p}}^{(s)\dagger}$ are the annihilation and creation operators, respectively. Consequently, these operators follow the commutation/anti-commutation relations:

$$\begin{aligned} \left[a_{\mathbf{p_1}}^{(s_1)}, a_{\mathbf{p_2}}^{(s_2)\dagger}\right] &= \delta^{s_1 s_2}(2\pi)^3\delta^{(3)}(\mathbf{p_1}-\mathbf{p_2}), \quad \text{for Bosons} \\ \left\{a_{\mathbf{p_1}}^{(s_1)}, a_{\mathbf{p_2}}^{(s_2)\dagger}\right\} &= \delta^{s_1 s_2}(2\pi)^3\delta^{(3)}(\mathbf{p_1}-\mathbf{p_2}), \quad \text{for Fermions.} \end{aligned} \tag{2.19}$$

From equations (2.18) and (2.19), we can easily show that:

$$\begin{aligned} &a_{\mathbf{p}}^{(s)\dagger} a_{\mathbf{p}}^{(s)}\,|\{n\}\rangle = V n_s(\mathbf{p})\,|\{n\}\rangle \\ &a_{\mathbf{p}}^{(s)} a_{\mathbf{p}}^{(s)\dagger}\,|\{n\}\rangle = V\,(1 \pm n_s(\mathbf{p}))\,|\{n\}\rangle\,; \quad + \text{for bosons and} - \text{for fermions.} \end{aligned} \tag{2.20}$$

This allows us to find the normalization constants $c_\pm$ for the decreased state $|\{n\}, \mathbf{p}_s^{(-1)}\rangle$ and the increased state $|\{n\}, \mathbf{p}_s^{(+1)}\rangle$, which are such that:

$$\begin{aligned} a_{\mathbf{p}}^{(s)} |\{n\}\rangle &= c_- |\{n\}, \mathbf{p}_s^{(-1)}\rangle\,, \\ a_{\mathbf{p}}^{(s)\dagger} |\{n\}\rangle &= c_+ |\{n\}, \mathbf{p}_s^{(+1)}\rangle\,. \end{aligned} \tag{2.21}$$

Normalizing the first equation of (2.21) we have:

$$\begin{aligned} \langle\{n\}| \, a_{\mathbf{p}}^{(s)\dagger} a_{\mathbf{p}}^{(s)} |\{n\}\rangle &= |c_-|^2 \\ \langle\{n\}| \, a_{\mathbf{p}}^{(s)\dagger} a_{\mathbf{p}}^{(s)} |\{n\}\rangle &= V n_s, \end{aligned}$$

$$\therefore\ c_- = \sqrt{V n_s}. \tag{2.22}$$

On the other hand, for the second equation of (2.21) we have:

$$\begin{aligned} \langle\{n\}| \, a_{\mathbf{p}}^{(s)} a_{\mathbf{p}}^{(s)\dagger} |\{n\}\rangle &= |c_+|^2 \\ \langle\{n\}| \, |a_{\mathbf{p}}^{(s)} a_{\mathbf{p}}^{(s)\dagger} |\{n\}\rangle &= V\,(1 \pm n_s(\mathbf{p}))\,, \end{aligned}$$

$$\therefore\ c_+ = \sqrt{V\,(1 \pm n_s(\mathbf{p}))}. \tag{2.23}$$

Given equations (2.22) and (2.23), we can write (2.21) in the form:

$$\begin{aligned} |\{n\}, \mathbf{p}_s^{(-1)}\rangle &= \frac{1}{\sqrt{V n_s}} a_{\mathbf{p}}^{(s)} |\{n\}\rangle\,, \\ |\{n\}, \mathbf{p}_s^{(+1)}\rangle &= \frac{1}{\sqrt{V\,(1 \pm n_s(\mathbf{p}))}} a_{\mathbf{p}}^{(s)\dagger} |\{n\}\rangle\,; \quad + \text{ for bosons and } - \text{ for fermions.} \end{aligned} \tag{2.24}$$

As a consequence of this normalization, we can write the multi-particle state as follows:

$$|\{n\}\rangle = \prod_{s,\mathbf{p}} \left(\frac{1}{\sqrt{n_s! V^{n_s}}} \left(a_{\mathbf{p}}^{(s)\dagger} \right)^{n_s} \right) |\{0\}\rangle\,, \tag{2.25}$$

where $|\{0\}\rangle$ is the ground state for $|\{n\}\rangle$.

Proof. To show (2.25) we begin by applying the occupation operator $\hat{n}_s(\mathbf{p})$ for all the species s n_s times, i.e.:

$$\prod_{s\in\{n\},\mathbf{p}} (\hat{n}_s(\mathbf{p}))^{n_s} |\{n\}\rangle = \prod_{s\in\{n\},\mathbf{p}} n_s(\mathbf{p})!\,|\{n\}\rangle\,.$$

On the other hand:

$$\begin{aligned} \prod_{s\in\{n\},\mathbf{p}} (\hat{n}_s(\mathbf{p}))^{n_s} |\{n\}\rangle &= \frac{1}{V^{n_s}} (a_{\mathbf{p}}^{(s)\dagger})^{n_s} (a_{\mathbf{p}}^{(s)})^{n_s} |\{n\}\rangle \\ &= \frac{1}{V^{n_s}} (a_{\mathbf{p}}^{(s)\dagger})^{n_s} \sqrt{V^{n_s}} \sqrt{n_s(\mathbf{p})!}\,|\{0\}\rangle\,. \end{aligned}$$

Joining these two equations, we have:

$$\prod_{s\in\{n\},\mathbf{p}} n_s(\mathbf{p})!\,|\{n\}\rangle = \frac{1}{V^{n_s}}(a_{\mathbf{p}}^{(s)\dagger})^{n_s}\sqrt{V^{n_s}}\sqrt{n_s(\mathbf{p})!}\,|\{0\}\rangle\,.$$

Therefore:

$$|\{n\}\rangle = \prod_{s\in\{n\},\mathbf{p}} \frac{1}{\sqrt{n_s!}\sqrt{V^{n_s}}}(a_{\mathbf{p}}^{(s)\dagger})^{n_s}\,|\{0\}\rangle\,.$$

□

Now that we have defined everything we need, we can calculate the transition probability for the interactions in the gas.

2.1.2.2 Transition Probability

We consider an interaction process:

$$\mathrm{I} = \{A,B,\ldots\} \to \mathrm{F} = \{X,Y,\ldots\}, \tag{2.26}$$

where the species in the initial set I are destroyed, and the species in the final set F are created. The complete set of interacting species is $\mathrm{S} = \mathrm{I} \cup \mathrm{F}$, while $\mathrm{W} = \mathrm{S} \cup \{\mathrm{bkg}\}$ includes both interacting species and background species that remain uncoupled from the process. Therefore, we can construct our quantum states in such a way that it is possible to factorize as follows:

$$\begin{aligned} |i\rangle &= |\{n_{\mathrm{S}}\}\rangle \otimes |\mathrm{bkg}\rangle \\ |f\rangle &= |\{n'_{\mathrm{S}}\}\rangle \otimes |\mathrm{bkg}\rangle\,, \end{aligned} \tag{2.27}$$

in which $|\{n_{\mathrm{S}}\}\rangle$ and $|\{n'_{\mathrm{S}}\}\rangle$ describe the interacting subsystem and $|\mathrm{bkg}\rangle$ represents background species. The updated occupation numbers for species $s \in \mathrm{S}$ are:

$$n'_s(\mathbf{p}) = n_s(\mathbf{p}) - \delta_{s\in\mathrm{I}} + \delta_{s\in\mathrm{F}} \tag{2.28}$$

with

$$\delta_{s\in\mathrm{I}} = \begin{cases} 1 & s \in \mathrm{I} \\ 0 & s \notin \mathrm{I} \end{cases}, \quad \delta_{s\in\mathrm{F}} = \begin{cases} 1 & s \in \mathrm{F} \\ 0 & s \notin \mathrm{F} \end{cases}. \tag{2.29}$$

Fundamentally, the construction of n'_{S} reflects the nature of the interaction: particles in the initial set I are annihilated, while those in the final set F are created. In particular, elastic processes, such as $(A(\mathbf{p}_1) + B(\mathbf{p}_2) \to A(\mathbf{p}_3) + B(\mathbf{p}_4)$), involve species that appear in both sets. These cases are naturally handled by the formalism, as occupation numbers for such species are updated accordingly. We do not restrict $\mathrm{I} \cap \mathrm{F}$, allowing for a unified treatment of both elastic and inelastic processes within the same framework.

As is customary, the scattering matrix operator $\hat{S}$ can be decomposed into two terms: the identity operator $\hat{1}$, which represents the non-interacting (free) evolution of particles, and the transition operator $\hat{T}$, which accounts for interactions (Peskin; Schroeder, 1995; Weinberg, 1995). We thus write:

$$\hat{S} = \hat{1} + i\hat{T}. \tag{2.30}$$

In the Appendix A we show that this is a result of the interaction picture in QFT, and that $\hat{T}$ has to be treated perturbatively when we in fact compute the transition amplitude. However, for now, equation (2.30) is sufficient.

Given that the process in hand involves the annihilation of initial-state particles and the creation of final-state particles, the transition operator can be defined as:

$$\hat{T} := (2\pi)^4\delta^{(4)}\left(\sum_{\mathrm{i}\in\mathrm{I}} p_\mathrm{i} - \sum_{\mathrm{f}\in\mathrm{F}} p_\mathrm{f}\right) C : \prod_{\mathrm{f}\in\mathrm{F}} a^{(\mathrm{f})\dagger}_{\mathbf{p}_\mathrm{f}} \prod_{\mathrm{i}\in\mathrm{I}} a^{(\mathrm{i})}_{\mathbf{p}_\mathrm{i}} :, \tag{2.31}$$

where C is a c-number coefficient (typically involving coupling constants), and the Dirac delta function ensures conservation of total four-momentum. The colons denote normal ordering of the creation and annihilation operators.

Now, we are finally able to calculate the transition amplitude for such a process:

$$\begin{aligned}\mathcal{M}_{(\mathrm{I}\to\mathrm{F})} &= \langle f|\, i\hat{T}\, |i\rangle = \langle \{n'\}_{\text{process}}|\, i\hat{T}\, |\{n\}_{\text{process}}\rangle \\ &= (2\pi)^4\delta^{(4)}\left(\sum_{\mathrm{i}\in\mathrm{I}} p_\mathrm{i} - \sum_{\mathrm{f}\in\mathrm{F}} p_\mathrm{f}\right) C\sqrt{V^{|\mathrm{I}|}}\sqrt{V^{|\mathrm{F}|}} \prod_{\mathrm{i}\in\mathrm{I}} \sqrt{n_\mathrm{i}(\mathbf{p}_\mathrm{i})} \prod_{\mathrm{f}\in\mathrm{F}} \sqrt{1 \pm n_\mathrm{f}(\mathbf{p}_\mathrm{f})}.\end{aligned} \tag{2.32}$$

On the other hand, we can define a Lorentz-invariant transition amplitude $\mathcal{A}$ as:

$$\langle \{\mathbf{p}_f\}|_{\text{inv}}\, i\hat{T}\, |\{\mathbf{p}_i\}\rangle_{\text{inv}} = \mathcal{A}\,, \tag{2.33}$$

where:

$$|\{\mathbf{p}_i\}\rangle_{\text{inv}} \equiv \prod_{i\in I} \sqrt{2E_{p_i}}\, a^{(i)\dagger}_{\mathbf{p}_i} |0\rangle\ , \tag{2.34}$$

and the factor $\sqrt{2E_{p_i}}$ ensures Lorentz invariance of the state normalization. The amplitude $\mathcal{A}$ must preserve energy-momentum conservation, which is already guaranteed by the operator $\hat{T}$. We can factor out this conservation law from $\mathcal{A}$ by introducing the Lorentz-invariant amplitude $\mathcal{M}$. Note that $\mathcal{M}$ is not the same as $\mathcal{M}_{(I\to F)}$, which is a non-invariant amplitude specific to our system. Therefore, we have:

$$\begin{aligned}\mathcal{A} &= \langle 0| \left(\prod_{f\in F} \sqrt{2E_f}\, a^{(f)}_{\mathbf{p}_f}\right) i\hat{T} \left(\prod_{i\in I} \sqrt{2E_i}\, a^{(i)\dagger}_{\mathbf{p}_i}\right) |0\rangle \\ &= (2\pi)^4\delta^{(4)}\left(\sum_{i\in I} p_i - \sum_{f\in F} p_f\right) i\mathcal{M}\,.\end{aligned} \tag{2.35}$$

For each particle species, we have:

$$a^\dagger_{\mathbf{p}_A} |0\rangle = \sqrt{V}\, |\mathbf{p}_A\rangle\,, \quad \langle 0|\, a_{\mathbf{p}_X} = \langle \mathbf{p}_X|\, \sqrt{V},$$

and consequently this implies:

$$
\begin{aligned}
i\mathcal{M} &= \prod_{\mathrm{i}\in\mathrm{I}} \sqrt{2E_{p_i}} \prod_{\mathrm{f}\in\mathrm{F}} \sqrt{2E_{p_\mathrm{f}}} \langle\{\mathbf{p}_\mathrm{f}\}| \left(C : \prod_{\mathrm{f}\in\mathrm{F}} a^{(\mathrm{f})\dagger}_{\mathbf{p}_\mathrm{f}} \prod_{\mathrm{i}\in\mathrm{I}} a^{(\mathrm{i})}_{\mathbf{p}_\mathrm{i}} : \right) |\{\mathbf{p}_\mathrm{i}\}\rangle \\
&= \prod_{\mathrm{i}\in\mathrm{I}} \sqrt{2E_{p_i}} \prod_{\mathrm{f}\in\mathrm{F}} \sqrt{2E_{p_\mathrm{f}}} C \sqrt{V^{|\mathrm{I}|}} \sqrt{V^{|\mathrm{F}|}}\,.
\end{aligned}
\tag{2.36}
$$

Note that the delta functions for energy-momentum conservation appear on both sides, which cancel out. Thus, we can express equation (2.32) in the form:

$$
i\mathcal{M}_{(\mathrm{I}\to\mathrm{F})} = (2\pi)^4\delta^{(4)}\left(\sum_{\mathrm{i}\in\mathrm{I}} p_\mathrm{i} - \sum_{\mathrm{f}\in\mathrm{F}} p_\mathrm{f}\right) i\mathcal{M} \prod_{\mathrm{i}\in\mathrm{I}} \frac{\sqrt{n_\mathrm{i}(\mathbf{p}_\mathrm{i})}}{\sqrt{2E_\mathrm{i}}} \prod_{\mathrm{f}\in\mathrm{F}} \frac{\sqrt{1 \pm n_\mathrm{f}(\mathbf{p}_\mathrm{f})}}{\sqrt{2E_\mathrm{f}}}. \tag{2.37}
$$

With the result of the equation (2.37), we can calculate the transition probability given the particles present in the initial state, that is, the particles in the set W. For this, we have to integrate all momenta in the phase space and sum over quantum numbers (g_s) for every particle in the interaction set S, i.e.:

$$
\begin{aligned}
\mathcal{P}_{(\mathrm{I}\to\mathrm{F})\mathrm{W}} &= \prod_{\mathrm{s}\in\mathrm{S}} \int \frac{d^3p_\mathrm{s}}{(2\pi)^3} \sum_{g_\mathrm{s}} |\mathcal{M}_{(\mathrm{I}\to\mathrm{F})}|^2 \\
&= \Bigg\{ \prod_{\mathrm{s}\in\mathrm{S}} \int \frac{d^3p_\mathrm{s}}{(2\pi)^3 2E_\mathrm{s}} \sum_{g_\mathrm{s}} |\mathcal{M}|^2 \left|(2\pi)^4\delta^{(4)}\left(\sum_{\mathrm{i}\in\mathrm{I}} p_\mathrm{i} - \sum_{\mathrm{f}\in\mathrm{F}} p_\mathrm{f}\right)\right|^2 \times \\
&\quad \times \prod_{\mathrm{i}\in\mathrm{I}} n_\mathrm{i}(\mathbf{p}_\mathrm{i}) \prod_{\mathrm{f}\in\mathrm{F}} (1 \pm n_\mathrm{f}(\mathbf{p}_\mathrm{f})) \Bigg\},
\end{aligned}
\tag{2.38}
$$

where $\frac{d^3p_\mathrm{s}}{(2\pi)^3 2E_\mathrm{s}} = d^3\Pi_\mathrm{s}$ is the Lorentz invariant phase-space measure. The product $\prod_{\mathrm{i},\mathrm{f}} \frac{1}{(2E_\mathrm{i})(2E_\mathrm{f})}$ combines into $\prod_\mathrm{s} \frac{1}{2E_\mathrm{s}}$. Furthermore, the squared delta function in equation (2.38) is regularized by interpreting it in a finite spacetime volume $V \times \delta t$, in which δT is the time scale of our interaction:

$$
\left|(2\pi)^4\delta^{(4)}\left(\sum_\mathrm{i} p_\mathrm{i} - \sum_\mathrm{f} p_\mathrm{f}\right)\right|^2 \to (2\pi)^4\delta^{(4)}\left(\sum_\mathrm{i} p_\mathrm{i} - \sum_\mathrm{f} p_\mathrm{f}\right) V\delta t. \tag{2.39}
$$

This shows that we can write equation (2.38) in the form:

$$
\begin{aligned}
\mathcal{P}_{(\mathrm{I}\to\mathrm{F})\mathrm{W}} = \Bigg\{ & V\delta t \prod_{\mathrm{s}\in\mathrm{S}} \int d^3\Pi_\mathrm{s} \sum_{g_\mathrm{s}} |\mathcal{M}|^2 (2\pi)^4\delta^{(4)}\left(\sum_{\mathrm{i}\in\mathrm{I}} p_\mathrm{i} - \sum_{\mathrm{f}\in\mathrm{F}} p_\mathrm{f}\right) \times \\
& \times \prod_{\mathrm{i}\in\mathrm{I}} n_\mathrm{i}(\mathbf{p}_\mathrm{i}) \prod_{\mathrm{f}\in\mathrm{F}} (1 \pm n_\mathrm{f}(\mathbf{p}_\mathrm{f})) \Bigg\}.
\end{aligned}
\tag{2.40}
$$

On the other hand, the probability of realizing the initial state $|i\rangle = |\{n\}\rangle$ (as defined in equation (2.27)) given the particles present in the set W, is given by the product of independent probabilities for each particle species and momentum mode:

$$
P_\mathrm{W} \equiv \prod_{\mathrm{w},\mathbf{p}} \rho(n_\mathrm{w}(\mathbf{p})); \quad \sum_{n_\mathrm{w}=0}^{\infty} \rho(n_\mathrm{w}(\mathbf{p})) = 1. \tag{2.41}
$$

Here, $\rho(n_{\mathrm{w}}(\mathbf{p}))$ is the probability of the occupation number n_{w} of species w with momentum $\mathbf{p}$ in the initial state. Again, by the definition of the state $|i\rangle$, this probability separates into two probabilities $P_{\{\mathrm{W}\}} = P_{\mathrm{S}} P_{\{bkg\}}$, which helps us to write the average transition probability as:

$$
\begin{aligned}
\langle \mathcal{P}_{(\mathrm{i}\to\mathrm{f})_{\mathrm{W}}} \rangle &\equiv \sum_{n_{\mathrm{w}}} P_{\mathrm{W}} \mathcal{P}_{(\mathrm{I}\to\mathrm{F})_{\mathrm{W}}} \\
&= \sum_{n_{\{\mathrm{bkg}\}}} P_{\{bkg\}} \sum_{n_{\mathrm{s}}} P_{\mathrm{S}} \mathcal{P}_{(\mathrm{I}\to\mathrm{F})_{\mathrm{W}}} \\
&= \sum_{n_{\mathrm{s}}} P_{\mathrm{S}} \mathcal{P}_{(\mathrm{I}\to\mathrm{F})_{\mathrm{W}}} \\
&= \left\{ V\delta t \prod_{\mathrm{s}\in\mathrm{S}} \int d^3\Pi_{\mathrm{s}} \sum_{g_{\mathrm{s}}} |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)} \left(\sum_{\mathrm{i}\in\mathrm{I}} p_{\mathrm{i}} - \sum_{\mathrm{f}\in\mathrm{F}} p_{\mathrm{f}} \right) \times \right. \\
&\left. \times \prod_{\mathrm{i}\in\mathrm{I}} f_{\mathrm{i}} \prod_{\mathrm{f}\in\mathrm{F}} (1 \pm f_{\mathrm{f}}) \right\}.
\end{aligned}
\tag{2.42}
$$

Note that in the third line we use the fact that $\mathcal{P}_{(\mathrm{i}\to\mathrm{f})_{\mathrm{W}}}$ does not depend on the particles in the background and $\sum_{n_{\{\mathrm{bkg}\}}} P_{\{bkg\}} = 1$. Furthermore, we define the distribution function as the average occupation numbers, that is:

$$
f_{\mathrm{s}} \equiv \langle n_{\mathrm{s}} \rangle = \sum_{n_{\mathrm{s}}} \rho(n_{\mathrm{s}}(\mathbf{p})) n_{\mathrm{s}}(\mathbf{p}). \tag{2.43}
$$

Since our objective is to understand the evolution of one particular distribution of particle species, we can define from equation (2.42), the partial average probability $d_\psi[\mathcal{P}_{(\mathrm{I}\to\mathrm{F})_{\mathrm{W}}}]$ for the species $\psi \in \mathrm{I}$ by (Enomoto *et al.*, 2023):

$$
\begin{aligned}
d_\psi[\mathcal{P}_{(\mathrm{I}\to\mathrm{F})_{\mathrm{W}}}] &= \frac{\langle \mathcal{P}_{(\mathrm{I}\to\mathrm{F})_{\mathrm{W}}} \rangle}{V \frac{d^3 p_\psi}{(2\pi)^3} \sum_{g_\psi}} \\
&= \left\{ \frac{\delta t}{2E_\psi} \prod_{\substack{\mathrm{s}\in\mathrm{S} \\ \mathrm{i}\neq\psi}} \int d^3\Pi_{\mathrm{s}} \sum_{\substack{g_{\mathrm{s}} \\ g_{\mathrm{i}}\neq g_\psi}} |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)} \left(\sum_{\mathrm{i}\in\mathrm{I}} p_{\mathrm{i}} - \sum_{\mathrm{f}\in\mathrm{F}} p_{\mathrm{f}} \right) \times \right. \\
&\left. \times \prod_{\mathrm{i}\in\mathrm{I}} f_{\mathrm{i}} \prod_{\mathrm{f}\in\mathrm{F}} (1 \pm f_{\mathrm{f}}) \right\}.
\end{aligned}
\tag{2.44}
$$

For simplicity, we also define:

$$
\overline{|\mathcal{M}|^2} = \sum_{\substack{g_{\mathrm{s}} \\ g_{\mathrm{i}}\neq g_\psi}} |\mathcal{M}|^2. \tag{2.45}
$$

Consequently, we can write equation (2.44) in a more compact form:

$$
\begin{aligned}
d_\psi[\mathcal{P}_{(\mathrm{I}\to\mathrm{F})_{\mathrm{W}}}] = &\left\{ \frac{\delta t}{2E_\psi} \prod_{\substack{\mathrm{s}\in\mathrm{S} \\ \mathrm{i}\neq\psi}} \int d^3\Pi_{\mathrm{s}} \overline{|\mathcal{M}|^2} (2\pi)^4 \delta^{(4)} \left(\sum_{\mathrm{i}\in\mathrm{I}} p_{\mathrm{i}} - \sum_{\mathrm{f}\in\mathrm{F}} p_{\mathrm{f}} \right) \times \right. \\
&\left. \times \prod_{\mathrm{i}\in\mathrm{I}} f_{\mathrm{i}} \prod_{\mathrm{f}\in\mathrm{F}} (1 \pm f_{\mathrm{f}}) \right\}.
\end{aligned}
\tag{2.46}
$$

Finally, we have all the necessary components to write an explicit formula to calculate the collision term as defined in equation (2.15).

2.1.2.3 The Collision operator

From our partial average probability (equation (2.46)), we define the variation of the distribution function by the difference between the forward process I $\longrightarrow$ F and the reverse process F $\longrightarrow$ I, that is:

$$\delta f_\psi(\mathbf{p}_\psi) \approx \sum_{\text{all process}} \delta N_\psi \left[d_\psi [\mathcal{P}_{(\mathrm{I}\to\mathrm{F})_\mathrm{W}}] - d_\psi [\mathcal{P}_{(\mathrm{F}\to\mathrm{I})_\mathrm{W}}] \right], \tag{2.47}$$

where δN_ψ is equal to -1, since our process annihilates particles of species ψ. Using equation (2.46), we can write this variation of the distribution function as:

$$\begin{aligned} \delta f_\psi(\mathbf{p}_\psi) \approx & \left\{ -\frac{\delta T}{2E_\psi} \sum_{\text{all process}} \prod_{\substack{\mathrm{s}\in\mathrm{S} \\ \mathrm{i}\neq\psi}} \int d^3\Pi_\mathrm{s} \overline{|\mathcal{M}|^2} (2\pi)^4 \delta^{(4)} \left(\sum_{\mathrm{i}\in\mathrm{I}} p_\mathrm{i} - \sum_{\mathrm{f}\in\mathrm{F}} p_\mathrm{f} \right) \times \right. \\ & \left. \times \left(\prod_{\mathrm{i}\in\mathrm{I}} f_\mathrm{i} \prod_{\mathrm{f}\in\mathrm{F}} (1 \pm f_\mathrm{f}) - \prod_{\mathrm{f}\in\mathrm{F}} f_\mathrm{f} \prod_{\mathrm{i}\in\mathrm{I}} (1 \pm f_\mathrm{i}) \right) \right\}, \end{aligned} \tag{2.48}$$

and finally, we write the collision operator (Equation (2.15)) in its final form:

$$\begin{aligned} C[f_\psi(p_\psi)] \approx & \left\{ \frac{1}{2} \sum_{\text{all process}} \prod_{\substack{\mathrm{s}\in\mathrm{S} \\ s\neq\psi}} \int d^3\Pi_\mathrm{s} \overline{|\mathcal{M}|^2} (2\pi)^4 \delta^{(4)} \left(\sum_{\mathrm{i}\in\mathrm{I}} p_\mathrm{i} - \sum_{\mathrm{f}\in\mathrm{F}} p_\mathrm{f} \right) \times \right. \\ & \left. \times \left(\prod_{\mathrm{f}\in\mathrm{F}} f_\mathrm{f} \prod_{\mathrm{i}\in\mathrm{I}} (1 \pm f_\mathrm{i}) - \prod_{\mathrm{i}\in\mathrm{I}} f_\mathrm{i} \prod_{\mathrm{f}\in\mathrm{F}} (1 \pm f_\mathrm{f}) \right) \right\}. \end{aligned} \tag{2.49}$$

The collision operator $C[f_\psi]$ has a clear physical interpretation:

- $\prod_{f\in F} f_f \prod_{i\in I}(1 \pm f_i)$ represents the **gain rate** in f_ψ due to $I \longrightarrow F$ processes creating ψ-particles.
- $\prod_{i\in I} f_i \prod_{f\in F}(1 \pm f_f)$ represents the **loss rate** from inverse processes ($F \longrightarrow I$).

The $(1 \pm f)$ factors incorporate quantum statistics: $(1 + f)$ for bosons (stimulated emission) and $(1 - f)$ for fermions (Pauli blocking).

2.1.3 The Full Boltzmann equation

With the results obtained so far, namely equations (2.14) and (2.49), we can now write the full Boltzmann equation (2.2) in its final form, before making further considerations about the spacetime geometry and our M.P system:

$$p^{\mu}\frac{\partial f_{\psi}}{\partial x^{\mu}}+\Gamma^{\alpha}_{\mu\nu}p_{\alpha}p^{\nu}\frac{\partial f_{\psi}}{\partial p_{\mu}}=\left\{\frac{1}{2}\sum_{\text{all process}}\prod_{\substack{s\in S\\ s\neq\psi}}\int d^3\Pi_s\overline{|\mathcal{M}|^2}(2\pi)^4\delta^{(4)}\left(\sum_{i\in I}p_i-\sum_{f\in F}p_f\right)\times\right.$$
$$\left.\times\left(\prod_{f\in F}f_f\prod_{i\in I}(1\pm f_i)-\prod_{i\in I}f_i\prod_{f\in F}(1\pm f_f)\right)\right\}. \tag{2.50}$$

2.2 The Boltzmann equation for the FLRW universe

2.2.1 The FLRW universe

As we discussed in the chapter 1, for modelling the SZE it is sufficient to work with the FLRW universe without perturbation in the metric. That is, we work with a fully homogeneous and isotropic universe described by the metric:

$$\begin{aligned} g_{00}(\mathbf{x},t)&=-1;\\ g_{0i}(\mathbf{x},t)&=g_{i0}(\mathbf{x},t)=0;\\ g_{ij}(\mathbf{x},t)&=a^2(t)\delta_{ij}, \end{aligned} \tag{2.51}$$

which has inverse:

$$\begin{aligned} g^{00}(\mathbf{x},t)&=-1;\\ g^{0i}(\mathbf{x},t)&=g^{i0}(\mathbf{x},t)=0;\\ g^{ij}(\mathbf{x},t)&=a^{-2}(t)\delta^{ij}. \end{aligned} \tag{2.52}$$

Note that we used the flat FLRW metric, since current observational data strongly support a spatially flat universe (Räsänen; Bolejko; Finoguenov, 2015; Ade *et al.*, 2014). An important component of this metric is the scale factor $a(t)$.

Using this metric, we can calculate the connections that appear in the Boltzmann equation (2.50), using the definition of the affine metric (equation (2.10)). The result of this calculation results in the following non-null connection elements:

$$\begin{aligned} \Gamma^0_{ij}&=\delta_{ij}\dot{a}a,\\ \Gamma^i_{0j}&=\Gamma^i_{j0}=\delta^i_jH, \end{aligned} \tag{2.53}$$

where $H=\frac{\dot{a}}{a}$ is the Hubble parameter.

Proof. We begin by noticing that the spatial derivatives of the metric vanish due to homogeneity:

$$\frac{\partial}{\partial x^k}g_{\mu\nu}=0.$$

Also, since $g_{00}=-1$ is constant, we have:

$$\frac{\partial}{\partial x^0}g_{00}=0.$$

For $\alpha = 0$**:**

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2}g^{0\rho}\left(\frac{\partial}{\partial x^0}g_{\rho 0} + \frac{\partial}{\partial x^0}g_{\rho 0} - \frac{\partial}{\partial x^\rho}g_{00}\right) \\
&= \frac{1}{2}g^{00}(0+0-0) + \frac{1}{2}g^{0k}(0+0-0) = 0.
\end{aligned}$$

$$\begin{aligned}
\Gamma^0_{0i} = \Gamma^0_{i0} &= \frac{1}{2}g^{0\rho}\left(\frac{\partial}{\partial x^i}g_{\rho 0} + \frac{\partial}{\partial x^0}g_{\rho i} - \frac{\partial}{\partial x^\rho}g_{i0}\right) \\
&= \frac{1}{2}g^{00}(0+0-0) + \frac{1}{2}g^{0k}(0+0-0) = 0.
\end{aligned}$$

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{0\rho}\left(\frac{\partial}{\partial x^j}g_{\rho i} + \frac{\partial}{\partial x^i}g_{\rho j} - \frac{\partial}{\partial x^\rho}g_{ij}\right) \\
&= \frac{1}{2}g^{00}(0+0-\frac{\partial}{\partial x^0}(a^2\delta_{ij})) + 0 \\
&= -\frac{1}{2}\frac{\partial}{\partial x^0}(a^2\delta_{ij}) \\
&= -\frac{1}{2}(2a\dot{a}\delta_{ij}) = -a\dot{a}\delta_{ij}.
\end{aligned}$$

For $\alpha = i$**:**

$$\Gamma^i_{00} = \frac{1}{2}g^{i\rho}\left(\frac{\partial}{\partial x^0}g_{\rho 0} + \frac{\partial}{\partial x^0}g_{\rho 0} - \frac{\partial}{\partial x^\rho}g_{00}\right) = 0.$$

$$\begin{aligned}
\Gamma^i_{0j} = \Gamma^i_{j0} &= \frac{1}{2}g^{ik}\left(\frac{\partial}{\partial x^j}g_{k0} + \frac{\partial}{\partial x^0}g_{kj} - \frac{\partial}{\partial x^k}g_{j0}\right) \\
&= \frac{1}{2}g^{ik}\frac{\partial}{\partial x^0}(a^2\delta_{kj}) = \frac{1}{2}a^{-2}\delta^{ik}(2a\dot{a}\delta_{kj}) \\
&= \frac{\dot{a}}{a}\delta^i_j = H\delta^i_j.
\end{aligned}$$

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial}{\partial x^k}g_{mj} + \frac{\partial}{\partial x^j}g_{mk} - \frac{\partial}{\partial x^m}g_{jk}\right) = 0.$$

□

2.2.2 The Boltzmann equation for the FLRW universe

Firstly, we should note that since we are working in an FLRW universe, we have to consider a homogeneous and isotropic distribution function; this means that we do not have a dependence on the spatial position in our distribution. Consequently, we can reduce our phase-space coordinates to $z^M = (x^0, p_\mu)$, and write the left-hand side (LHS), of equation (2.50), i.e., the Liouville operator, as:

$$L[f_\psi(z^M)] = p^0 \frac{\partial f_\psi}{\partial x^0} + \Gamma^\alpha_{\mu\nu} p_\alpha p^\nu \frac{\partial f_\psi}{\partial p_\mu}. \tag{2.54}$$

On the other hand, by equation (2.53), we can write this as:

$$\begin{aligned} L[f_\psi(z^M)] &= p^0 \frac{\partial f_\psi}{\partial x^0} + \Gamma^0_{ij} p_0 p^j \frac{\partial f_\psi}{\partial p_i} + \Gamma^i_{j0} p_i p^o \frac{\partial f_\psi}{\partial p_j} + \Gamma^i_{0j} p_i p^j \frac{\partial f_\psi}{\partial p_0} \\ &= p^0 \frac{\partial f_\psi}{\partial x^0} + \delta_{ij} \dot{a} a p_0 p^j \frac{\partial f_\psi}{\partial p_i} + H \delta^i_j p_i p^o \frac{\partial f_\psi}{\partial p_j} + H \delta^i_j p_i p^j \frac{\partial f_\psi}{\partial p_0}. \end{aligned}$$

Using equation (2.51), we have the identities:

$$p_0 = g_{00} p^0 = -p^0; \quad p_k = g_{kl} p^l = a^2 \delta_{kl} p^l; \quad p_j = g^{il} g_{lj} p_i = \delta^i_j p_i,$$

and can write:

$$\begin{aligned} L[f_\psi(z^M)] &= p^0 \frac{\partial f_\psi}{\partial x^0} + \frac{\dot{a} a}{a^2} p_0 p_i \frac{\partial f_\psi}{\partial p_i} - H p_j p_o \frac{\partial f_\psi}{\partial p_j} + H p_i p^i \frac{\partial f_\psi}{\partial p_0} \\ &= p^0 \frac{\partial f_\psi}{\partial x^0} + \cancel{H p_0 p_i \frac{\partial f_\psi}{\partial p_i}} - \cancel{H p_j p_o \frac{\partial f_\psi}{\partial p_j}} + H p_i p^i \frac{\partial f_\psi}{\partial p_0}. \end{aligned}$$

Finally, from the four-vector components that we indicated at the beginning of this chapter, we can write:

$$\therefore L[f_\psi(z^M)] = E \frac{\partial f_\psi}{\partial t} - H p^2 \frac{\partial f_\psi}{\partial E}. \tag{2.55}$$

Furthermore, the isotropy of our system implies that the distribution function f_ψ depends on the 3-momentum only through its magnitude, p. Therefore, to simplify the work with our differential equation (2.50), which has only integrals on the momentum and not in the energy, it is convenient to rewrite the energy derivative in the Liouville operator (equation (2.55)) in terms of a derivative with respect to p. This is accomplished using the mass-shell condition, which relates the energy of a particle to its momentum ($E^2 - m^2 = p^2$), and the chain rule:

$$\frac{\partial}{\partial E} = \frac{\partial}{\partial p} \frac{\partial p}{\partial E} = \frac{E}{p} \frac{\partial}{\partial p},$$

consequently, equation (2.55) is written as:

$$\begin{aligned} L[f_\psi(z^M)] &= E \frac{\partial f_\psi}{\partial t} - H \cancel{p^{2}} \frac{E}{\cancel{p}} \frac{\partial f_\psi}{\partial p} \\ &= E \left[\frac{\partial}{\partial t} - H p \frac{\partial}{\partial p} \right] f_\psi. \end{aligned} \tag{2.56}$$

Finally, the Boltzmann equation (2.50) has its final form:

$$\begin{aligned} \left[\frac{\partial}{\partial t} - H p \frac{\partial}{\partial p} \right] f_\psi = \Bigg\{ \frac{1}{2E_\psi} \sum_{\text{all process}} \prod_{\substack{s \in S \\ s \neq \psi}} \int d^3\Pi_s \overline{|\mathcal{M}|^2} (2\pi)^4 \delta^{(4)} \left(\sum_{i \in I} p_i - \sum_{f \in F} p_f \right) \times \\ \times \left(\prod_{f \in F} f_f \prod_{i \in I} (1 \pm f_i) - \prod_{i \in I} f_i \prod_{f \in F} (1 \pm f_f) \right) \Bigg\}, \end{aligned} \tag{2.57}$$

where we have to remember that ψ represents the species of particles that we are interested in studying.

In this chapter, we have derived the Boltzmann equation in a GR and QFT framework, ready for applications in cosmology using a flat FLRW metric. In the next chapter, we will focus on the specific interaction of interest: Compton scattering between photons and electrons, which plays a crucial role in the SZE. This will allow us to solve the LHS, that is, the collision term of the Boltzmann equation (2.57).

3 The Compton Scattering

As discussed in Chapter 1, the SZE arises from inverse Compton scattering of CMB photons with hot electrons in the ICM of galaxy clusters. The theoretical foundations of this interaction was established by Arthur Compton in 1923 (Compton, 1923), when he proposed a quantum hypothesis, based on Einstein's photoelectric work (Einstein, 1905), to explain the wavelength shift in scattered radiation. This effect is now known as Compton scattering.

The Compton scattering process is described by the process:

$$\gamma(p^{\mu}) + e^{-}(q^{\mu}) \rightarrow \gamma(p'^{\mu}) + e^{-}(q'^{\mu}), \tag{3.1}$$

with four-momentum conservation:

$$p^{\mu} + q^{\mu} = p'^{\mu} + q'^{\mu}. \tag{3.2}$$

The four-momentum for the photons are $p^{\mu} = E_{\gamma}(1, \hat{p})$ and $p'^{\mu} = E'_{\gamma}(1, \hat{p}')$, and the four-momentum for the electrons are $q^{\mu} = (E_e, \mathbf{p}_e) = \gamma_e m_e(1, \mathbf{v}_e)$ and $q'^{\mu} = (E'_e, \mathbf{p}'_e) = \gamma'_e m_e(1, \mathbf{v}'_e)$.[1] Here, $\hat{p}$ is a unitary vector that defines the direction of the photon momentum and $\mathbf{p}_e$ is the three-momentum of the electrons. Diagrammatically, this interaction in a generic frame is represented in Figure 3.1.

Compton Scattering

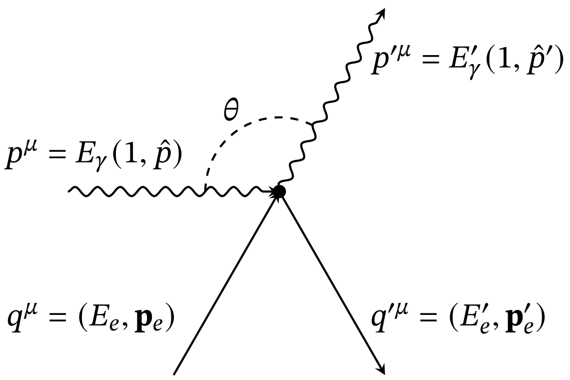


Figure 3.1 – Representation of the Compton scattering process in a generic frame. The scattering angle θ measure the final photon change of direction ($\hat{p} \cdot \hat{p}' = \cos\theta$). In the Laboratory frame, the initial electron is at rest ($\mathbf{p}_e = 0$).

[1] Here the subscript γ denotes the photon energy, whereas γ_e and γ'_e denote the electron Lorentz factors, to avoid notational confusion.

It is also useful to define our cinematic variables for the scattering process (3.1). Therefore, we define the Lorentz-invariant Mandelstam variables using four-momenta:

$$
\begin{aligned}
s &= (q+p)^2 = q^\mu q_\mu + p^\mu p_\mu + 2q^\mu p_\mu \\
t &= (p-p')^2 = p^\mu p_\mu + p'^\mu p'_\mu - 2p^\mu p'_\mu \\
u &= (q-p')^2 = q^\mu q_\mu + p'^\mu p'_\mu - 2q^\mu p'_\mu
\end{aligned}
\tag{3.3}
$$

where, from now on, we will denote the contractions between two four-vectors a^μ and b^μ as $a^\mu b_\mu = a \cdot b$, and $a^2 = a \cdot a$. Using the mass-shell condition, we have:

$$
\begin{aligned}
q^2 &= -m_e^2 = q'^2 \\
p^2 &= 0 = p'^2 .
\end{aligned}
\tag{3.4}
$$

This allows us to write equation (3.3) in the form:

$$
\begin{aligned}
s &= (q+p)^2 = -m_e^2 + 2(q \cdot p) , \\
t &= (p-p')^2 = -2(p \cdot p') , \\
u &= (q-p')^2 = -m_e^2 - 2(q \cdot p') .
\end{aligned}
\tag{3.5}
$$

In addition, for this interaction, since we are interested in the observation of the anisotropies in the CMB spectrum, the particle that we are interested in studying using the Boltzmann equation (2.57) is the incoming photon, that is, $f_\psi = f_\gamma = f_\gamma(p,t)$. Consequently, from this interaction, we can write the Boltzmann equation 2.57 as:

$$
\begin{aligned}
\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right] f_\gamma(p) = \Bigg\{ & \frac{1}{2E_\gamma} \int \frac{d^3 p'}{(2\pi)^3 2E'_\gamma} \int \frac{d^3 q}{(2\pi)^3 2E_e} \int \frac{d^3 q'}{(2\pi)^3 2E'_e} \times \\
& \times \overline{|\mathcal{M}|^2} (2\pi)^4 \delta^{(4)} \left(p + q - p' - q'\right) \times \\
& \times \Big[f_\gamma(p') f_e(q') \left(1 + f_\gamma(p)\right) \left(1 - f_e(q)\right) \\
& - f_\gamma(p) f_e(q) \left(1 + f_\gamma(p')\right) \left(1 - f_e(q')\right) \Big] \Bigg\} .
\end{aligned}
\tag{3.6}
$$

Note that we suppressed the dependency on time of the distribution functions.

For the remainder of this chapter we will work to improve our understanding of the Compton effect and calculate the transition amplitude $\overline{|\mathcal{M}|^2}$, which will facilitate our work to solve the RHS of equation (3.6) in Chapter 4. We begin this work by deriving the Compton formula in the next section.

3.1 Compton Formula

With four-momentum conservation (equation (3.2)), we can now derive the Compton formula, originally obtained in (Compton, 1923), but here derived in a covariant form within a generic frame as in Figure 3.1.

Using equation (3.4), we can rewrite the four-momentum conservation equation (3.2). Squaring both sides, we have:

$$(p - p')^2 = (q' - q)^2$$
$$p \cdot p' = m_e^2 + q \cdot q'.$$

Isolating q'_μ in equation (3.2) and substituting the four-momenta, we can write:

$$p \cdot p' = m_e^2 + q \cdot q' \rightarrow p \cdot p' = m_e^2 + q \cdot (p + q - p')$$
$$E_\gamma E'_\gamma (1 - \cos\theta) = \gamma_e m_e E_\gamma (1 - \mathbf{v_e} \cdot \mathbf{p}) - \gamma_e m_e E'_\gamma (1 - \mathbf{v_e} \cdot \mathbf{p}').$$

Rearranging the last equation, we obtain the Compton formula for a generic frame:

$$\frac{E'_\gamma}{E_\gamma} = \frac{(1 - \mathbf{v_e} \cdot \mathbf{p})}{\left((1 - \mathbf{v_e} \cdot \mathbf{p}') + \frac{E_\gamma}{\gamma_e m_e}(1 - \cos\theta)\right)}. \tag{3.7}$$

From now on, we will call the reference frame in which the initial electrons are at rest (i.e. $\mathbf{v}_e = 0$, which implies $\gamma_e = 1$), the *laboratory frame* (or simply the *lab frame*), for which equation (3.7) can be written as:

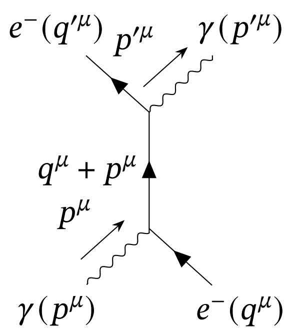
$$E'_\gamma = \frac{E_\gamma}{\left(1 + \frac{E_\gamma}{m_e}(1 - \cos\theta)\right)}, \tag{3.8}$$

which is the usual Compton formula.

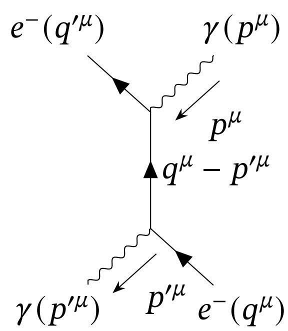
3.2 Transition amplitude and Klein-Nishina formula

To compute the transition amplitude $\overline{|\mathcal{M}|^2}$ and the differential cross-section $d\sigma$ for the Compton effect, that is, the Klein-Nishina formula (Klein; Nishina, 1929), we employ the Quantum Electrodynamics (QED) Feynman rules derived in Appendix A in the dominant term of the perturbative expansion of the scattering matrix operator $\hat{S}$, which is the tree-level. As derived in the Appendix A, for this level we have two topologically different diagrams, known as the $s-$ and $u-$ channels:

$i\mathcal{M} =$



$+$



(3.9)

The first diagram corresponds to the s-channel process, in which the incoming photon is first absorbed by the electron, forming a virtual electron with momentum $q^\mu + p^\mu$. This virtual electron then emits the outgoing photon, resulting in a final state with a scattered electron and photon. The second diagram represents the u-channel process. Here, the electron first emits the outgoing photon and then absorbs the incoming photon, again forming a virtual electron, now with momentum $q^\mu - p'^\mu$.[2]

From the Appendix A, we derive that such diagrams (3.9) have the full matrix element (A.53):

$$
\begin{aligned}
i\mathcal{M} = -\, ie^2 \bar{u}(\mathbf{q}', \sigma') \Bigg[& \not{\epsilon}^*(\mathbf{p}', \lambda') \frac{-i(\not{p} + \not{q}) + m}{(q+p)^2 + m^2} \not{\epsilon}(\mathbf{p}, \lambda) \\
& + \not{\epsilon}(\mathbf{p}, \lambda) \frac{-i(\not{q} - \not{p}') + m}{(q-p')^2 + m^2} \not{\epsilon}^*(\mathbf{p}', \lambda') \Bigg] u(\mathbf{q}, \sigma),
\end{aligned} \tag{3.10}
$$

where $\not{\epsilon}^*(\mathbf{p}', \lambda') = \gamma^\mu \varepsilon^*_\mu$ are the polarization vectors contracted with the respective gamma matrices (similar for $\not{p}$). $u_s(\mathbf{p})$ is the corresponding spinors, with the adjoint defined by $\bar{u}(\mathbf{p}, \sigma) = u_s(\mathbf{p}, \sigma)\beta$, with $\beta = i\gamma^0$ as defined in Appendix A. We can rewrite this using the Mandelstam variables (3.5) as:

$$
\begin{aligned}
i\mathcal{M} = -\, ie^2 \bar{u}(\mathbf{q}', \sigma') \Bigg[& \not{\epsilon}^*(\mathbf{p}', \lambda') \frac{-i\not{p} + (-i\not{q} + m)}{2q \cdot p} \not{\epsilon}(\mathbf{p}, \lambda) \\
& + \not{\epsilon}(\mathbf{p}, \lambda) \frac{i\not{p}' + (-i\not{q} + m)}{-2q \cdot p'} \not{\epsilon}^*(\mathbf{p}', \lambda') \Bigg] u(\mathbf{q}, \sigma).
\end{aligned} \tag{3.11}
$$

We can further simplify the numerators of this equation by using the identity $\{\gamma^\mu \gamma^\nu\} = 2\eta^{\mu\nu}$ (equation (A.3) of the Appendix A), and the Dirac equation for the spinor $(i\gamma^\alpha q_\alpha + m)u(\mathbf{q}, \sigma) = 0$ (equation (A.16) of the same appendix), which altogether will give $(-i\not{q} + m)\not{\epsilon}(\mathbf{p}, \lambda)u(\mathbf{q}, \sigma) = 0$, for both $\not{\epsilon}(\mathbf{p}, \lambda)$ or $\not{\epsilon}^*(\mathbf{p}', \lambda')$ if we choose the Lorentz gauge at the lab frame [3].

Proof. $(-i\not{q} + m)\not{\epsilon}(\mathbf{p}, \lambda)u(\mathbf{q}, \sigma) = 0$:

$$
\begin{aligned}
(-i\not{q} + m)\not{\epsilon}(\mathbf{p}, \lambda)u(\mathbf{q}, \sigma) &= (-iq_\mu \gamma^\mu + m)\gamma^\nu \varepsilon_\nu(\mathbf{p},\lambda)u(\mathbf{q}, \sigma) \\
&= (-iq_\mu \gamma^\mu \gamma^\nu + \gamma^\nu m)\varepsilon_\nu(\mathbf{p},\lambda)u(\mathbf{q}, \sigma) \\
&= -iq_\mu \{\gamma^\mu, \gamma^\nu\}\varepsilon_\nu(\mathbf{p},\lambda)u(\mathbf{q}, \sigma) + \not{\epsilon}(\mathbf{p}, \lambda)(i\not{q} + m)u(\mathbf{q}, \sigma).
\end{aligned}
$$

The second term vanishes by the Dirac equation for the spinor (A.16), and for the first

---

[2] As shown in the Appendix A, at this level we do not have a t-channel contribution, which exchange momentum via a virtual photon.

[3] Choosing the Lorentz gauge in the lab frame, where $q^\mu = (m, \mathbf{0})$, simplifies terms like $q^\nu \varepsilon_{\nu,\lambda} = 0$. However, the resulting amplitude is covariant, and the Klein-Nishina formula remains valid in any inertial frame after the proper transformation.

term, we use the identity (A.3). Therefore:

$$(-i\not{q}+m)\not{\epsilon}(\mathbf{p},\lambda)u(\mathbf{q},\sigma) = -2iq^{\nu}\varepsilon_{\nu}(\mathbf{p},\lambda)u(\mathbf{q},\sigma).$$

By our gauge choice (Lorentz gauge at the lab frame), the term $q^{\nu}\varepsilon_{\nu}(\mathbf{p},\lambda) = 0$, which means that:

$$(-i\not{q}+m)\not{\epsilon}(\mathbf{p},\lambda)u(\mathbf{q},\sigma) = 0. \tag{3.12}$$

□

Therefore, equation (3.11) can be written as:

$$i\mathcal{M} = -ie^2\bar{u}(\mathbf{q}',\sigma')\left[\not{\epsilon}^*(\mathbf{p}',\lambda')\frac{-i\not{p}}{2q\cdot p}\not{\epsilon}(\mathbf{p},\lambda) + \not{\epsilon}(\mathbf{p},\lambda)\frac{-i\not{p}'}{2q\cdot p'}\not{\epsilon}^*(\mathbf{p}',\lambda')\right]u(\mathbf{q},\sigma). \tag{3.13}$$

To simplify our calculations, we will define the operators:

$$A = \not{\epsilon}^*(\mathbf{p}',\lambda')\frac{\not{p}}{2q\cdot p}\not{\epsilon}(\mathbf{p},\lambda), \quad B = \not{\epsilon}(\mathbf{p},\lambda)\frac{\not{p}'}{2q\cdot p'}\not{\epsilon}^*(\mathbf{p}',\lambda'), \tag{3.14}$$

and consequently we can rewrite equation (3.13) in the form:

$$i\mathcal{M} = -e^2\bar{u}(\mathbf{q}',\sigma')\,[A+B]\,u(\mathbf{q},\sigma). \tag{3.15}$$

Now, taking the square of the equation (3.15), we have:

$$|\mathcal{M}|^2 = e^4\,[\bar{u}(\mathbf{q}',\sigma')(A+B)u(\mathbf{q},\sigma)]\left[\bar{u}(\mathbf{q},\sigma)\beta(A^{\dagger}+B^{\dagger})\beta u(\mathbf{q}',\sigma')\right], \tag{3.16}$$

where, using the property $\beta\gamma^{\mu}\beta = -\gamma^{\mu}$, we obtain:

$$\bar{A} = \beta A^{\dagger}\beta^{-1} = -\not{\epsilon}^*(\mathbf{p},\lambda)\frac{\not{p}}{2q\cdot p}\not{\epsilon}(\mathbf{p}',\lambda'), \quad \bar{B} = \beta B^{\dagger}\beta^{-1} = -\not{\epsilon}(\mathbf{p}',\lambda')\frac{\not{p}'}{2q\cdot p'}\not{\epsilon}^*(\mathbf{p},\lambda), \tag{3.17}$$

with A and B given by (3.14) and its Dirac adjoint by (3.17).

As we have seen in Chapter (2), we need the average square amplitude $\overline{|\mathcal{M}|^2}$ to compute the collision term in the Boltzmann equation for our process ((3.6))v. Despite needing an average over the polarization and spin of only three of the four particles involved in the processes, to continue our calculation, we will average over the four particles, which is more convenient for our calculation. Therefore, to compute this unpolarized squared amplitude $\langle|\mathcal{M}|^2\rangle$, we average over the two initial photon polarizations and the two initial electron spin states, and sum over the final ones. This results in a prefactor of $\frac{1}{4}$. Thus, using (3.16) we can write:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{1}{4}\sum_{\lambda,\lambda'}\sum_{\sigma,\sigma'}|\mathcal{M}|^2 \\ &= \frac{1}{4}\sum_{\lambda,\lambda'}\sum_{\sigma,\sigma'}e^4\left[\bar{u}(\mathbf{q}',\sigma')(A+B)u(\mathbf{q},\sigma)\bar{u}(\mathbf{q},\sigma)(\bar{A}+\bar{B})u(\mathbf{q}',\sigma')\right].\end{aligned} \tag{3.18}$$

To simplify the spin sums, we make use of the completeness relation for Dirac spinors (Weinberg, 1995):

$$\sum_{\sigma} u_{\alpha}(\mathbf{q},\sigma)\bar{u}_{\beta}(\mathbf{q},\sigma) = (-i\not{q} + m)_{\alpha\beta}, \tag{3.19}$$

and similarly for the outgoing spinor:

$$\sum_{\sigma'} u_{\mu}(\mathbf{q}',\sigma')\bar{u}_{\nu}(\mathbf{q}',\sigma') = (-i\not{q}' + m)_{\mu\nu}. \tag{3.20}$$

Now, we can rewrite equation (3.18) as (Weinberg, 1995):

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle &= \frac{e^4}{4}\sum_{\lambda,\lambda'}\sum_{\sigma\sigma'}(A+B)_{\alpha\beta}\left(u_{\beta}(\mathbf{q},\sigma)\bar{u}_{\gamma}(\mathbf{q},\sigma)\right)(\bar{A}+\bar{B})_{\gamma\delta}\left(u_{\delta}(\mathbf{q}',\sigma')\bar{u}_{\alpha}(\mathbf{q}',\sigma')\right) \\ &= \frac{e^4}{4}\sum_{\lambda,\lambda'}\mathrm{Tr}\left[(A+B)(-i\not{q}+m)(\bar{A}+\bar{B})(-i\not{q}'+m)\right]. \end{aligned} \tag{3.21}$$

Doing all products inside the trace and using the property that all traces with an odd number of gamma matrices are zero (i.e. Tr[odd number of γ's matrices] = 0), we have:

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle = \frac{-e^4}{4}\quad \mathrm{Tr}\Bigg[& A\not{q}\bar{A}\not{q}' + A\not{q}\bar{B}\not{q}' + B\not{q}\bar{A}\not{q}' + B\not{q}\bar{B}\not{q}' + \\ & -m^2\left(A\bar{A} + A\bar{B} + B\bar{A} + B\bar{B}\right)\Bigg]. \end{aligned} \tag{3.22}$$

Using (3.14) and (3.16), we have:

$$\begin{aligned} \langle|\mathcal{M}|^2\rangle = \frac{e^4}{16}\sum_{\lambda,\lambda'}\Bigg[& \frac{T_1}{(q\cdot p)^2} + \frac{T_2+T_3}{(q\cdot p)(q\cdot p')} + \frac{T_4}{(q\cdot p')^2} + \\ & - m^2\left(\frac{t_1}{(q\cdot p)^2} + \frac{t_2+t_3}{(q^{\sigma}p_{\sigma})(q\cdot p')} + \frac{t_4}{(q\cdot p')^2}\right)\Bigg], \end{aligned} \tag{3.23}$$

where:

$$\begin{aligned} T_1 &= \mathrm{Tr}\left[\not{\epsilon}^*(\mathbf{p}',\lambda')\not{p}\not{\epsilon}(\mathbf{p},\lambda)\not{q}\not{\epsilon}^*(\mathbf{p},\lambda)\not{p}\not{\epsilon}(\mathbf{p}',\lambda')\not{q}'\right], \\ T_2 &= \mathrm{Tr}\left[\not{\epsilon}^*(\mathbf{p}',\lambda')\not{p}\not{\epsilon}(\mathbf{p},\lambda)\not{q}\not{\epsilon}(\mathbf{p}',\lambda')\not{p}'\not{\epsilon}^*(\mathbf{p},\lambda)\not{q}'\right], \\ T_3 &= \mathrm{Tr}\left[\not{\epsilon}(\mathbf{p},\lambda)\not{p}'\not{\epsilon}^*(\mathbf{p}',\lambda')\not{q}\not{\epsilon}^*(\mathbf{p},\lambda)\not{p}\not{\epsilon}(\mathbf{p}',\lambda')\not{q}'\right], \\ T_4 &= \mathrm{Tr}\left[\not{\epsilon}(\mathbf{p},\lambda)\not{p}'\not{\epsilon}^*(\mathbf{p}',\lambda')\not{q}\not{\epsilon}(\mathbf{p}',\lambda')\not{p}'\not{\epsilon}^*(\mathbf{p},\lambda)\not{q}'\right], \\ t_1 &= \mathrm{Tr}\left[\not{\epsilon}^*(\mathbf{p}',\lambda')\not{p}\not{\epsilon}(\mathbf{p},\lambda)\not{\epsilon}^*(\mathbf{p},\lambda)\not{p}\not{\epsilon}(\mathbf{p}',\lambda')\right], \\ t_2 &= \mathrm{Tr}\left[\not{\epsilon}^*(\mathbf{p}',\lambda')\not{p}\not{\epsilon}(\mathbf{p},\lambda)\not{\epsilon}(\mathbf{p}',\lambda')\not{p}'\not{\epsilon}^*(\mathbf{p},\lambda)\right], \\ t_3 &= \mathrm{Tr}\left[\not{\epsilon}(\mathbf{p},\lambda)\not{p}'\not{\epsilon}^*(\mathbf{p}',\lambda')\not{\epsilon}^*(\mathbf{p},\lambda)\not{p}\not{\epsilon}(\mathbf{p}',\lambda')\right], \\ t_4 &= \mathrm{Tr}\left[\not{\epsilon}(\mathbf{p},\lambda)\not{p}'\not{\epsilon}^*(\mathbf{p}',\lambda')\not{\epsilon}(\mathbf{p}',\lambda')\not{p}'\not{\epsilon}^*(\mathbf{p},\lambda)\right]. \end{aligned} \tag{3.24}$$

Note that when we use (3.16), a negative sign appears in front of all T_i and t_i's, which inverts the sign of the equation (3.23).

To solve these traces, we assume real polarization vectors for the external photons, such that $\not\varepsilon = \not\varepsilon^*$ and $\not\varepsilon' = \not\varepsilon'^*$, corresponding to linearly polarized states, where now to simplify we denote $\not\varepsilon(\mathbf{p}',\lambda') = \not\varepsilon'$. Although we work in Feynman gauge, where unphysical polarizations may formally appear, only the physical transverse components contribute for on-shell external photons (i.e., the polarization vectors satisfy $\varepsilon^\mu p_\mu = 0$ and contain no longitudinal or scalar parts). In addition, since we adopt the gauge in the lab frame, we also use that $q^\nu \varepsilon_\nu = 0$ and $q'^\nu \varepsilon'_\nu = 0$, and, of course, we use energy-momentum conservation as given in equation (3.2). To do these calculations in a more efficient way, we used the *Wolfram Mathematica* (Research, 2023) software with the package *FeynCalc* (Mertig; Shtabovenko; Orellana, 2023), which results in the following traces:

$$
\begin{aligned}
T_1 =& -16(q\cdot p)(p\cdot\varepsilon')^2 + 8(q\cdot p')(q\cdot p)\,,\\
T_2 =& T_3 = 8(q\cdot p')(p\cdot\varepsilon')^2 - 8(q\cdot p)(p'\cdot\varepsilon)^2 + 16(q\cdot p)(q\cdot p')(\varepsilon\cdot\varepsilon')^2\\
&+ 8m_e^2(p\cdot p')(\varepsilon\cdot\varepsilon')^2 - 8m_e^2(p\cdot\varepsilon')(p'\cdot\varepsilon)(\varepsilon\cdot\varepsilon')\\
&- 4m_e^2(p\cdot p') - 8(q\cdot p)(q\cdot p')\,,\\
T_4 =& 16(q\cdot p')(p'\cdot\varepsilon)^2 + 8(q\cdot p)(q\cdot p')\,,\\
t_1 =& t_4 = 0\,,\\
t_2 =& t_3 = 8(p\cdot p')(\varepsilon\cdot\varepsilon')^2 - 8(p\cdot\varepsilon')(p'\cdot\varepsilon)(\varepsilon\cdot\varepsilon') - 4(p\cdot p')
\end{aligned}
\tag{3.25}
$$

Putting these results in (3.23) we obtain after some simplifications:

$$
\langle|\mathcal{M}|^2\rangle = \frac{e^4}{16}\sum_{\lambda,\lambda'}\left[8\left(\frac{(q\cdot p)^2 + (q\cdot p') - 2(q\cdot p)(q\cdot p')}{(q\cdot p)(q\cdot p')}\right) + 32(\varepsilon\cdot\varepsilon')^2\right],
$$

which can be written in the compact form:

$$
\langle|\mathcal{M}|^2\rangle = \frac{e^4}{16}\sum_{\lambda,\lambda'}\left[8\frac{(p\cdot p')^2}{(q\cdot p)(q\cdot p')} + 32(\varepsilon\cdot\varepsilon')^2\right]. \tag{3.26}
$$

In the lab frame, we have:

$$
\begin{aligned}
p\cdot p' &= -E_\gamma E'_\gamma(1-\cos\theta)\,,\\
q\cdot p &= -E_\gamma m_e\,,\\
q\cdot p' &= -E'_\gamma m_e\,.
\end{aligned}
$$

Using the Compton Formula (3.8) we can rewrite the first relation:

$$
\begin{aligned}
p\cdot p' =& E_\gamma E'_\gamma m_e\left(\frac{1}{E_\gamma} - \frac{1}{E'_\gamma}\right),\\
q\cdot p =& -E_\gamma m_e\,,\\
q\cdot p' =& -E'_\gamma m_e\,.
\end{aligned}
\tag{3.27}
$$

This allows us to rewrite equation (3.26) as:

$$\langle|\mathcal{M}|^2\rangle = \frac{e^4}{2}\sum_{\lambda,\lambda'}\left[\frac{E_\gamma}{E'_\gamma} + \frac{E'_\gamma}{E_\gamma} - 2 + 4(\varepsilon\cdot\varepsilon')^2\right] \tag{3.28}$$

Finally, we can do the polarization sum. For this, we use that $\sum_{\lambda,\lambda'}(\varepsilon\cdot\varepsilon')^2 = 1+\cos^2\theta$, and for numerical factors, we just multiply by 2 for each polarization that we are summing over:

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle &= \frac{e^4}{2}\left[4\left(\frac{E_\gamma}{E'_\gamma} + \frac{E'_\gamma}{E_\gamma} - 2\right) + 4(1+\cos^2\theta)\right]\\ &= 2e^4\left[\frac{E_\gamma}{E'_\gamma} + \frac{E'_\gamma}{E_\gamma} - 2 + 1 + \cos^2\theta\right].\end{aligned}$$

Therefore, we finally have our final form for the squared amplitude averaged over initial and summed over final polarizations and spins:

$$\therefore\ \langle|\mathcal{M}|^2\rangle = 2e^4\left[\frac{E_\gamma}{E'_\gamma} + \frac{E'_\gamma}{E_\gamma} - \sin^2\theta\right]. \tag{3.29}$$

3.2.1 Klein-Nishina Formula

Having obtained the transition amplitude for the Compton scattering (3.29), we now turn to the computation of the differential cross-section, which encodes the probability per unit flux and per unit final-state phase space that an initial set of particles undergoes a specified scattering process. For a process like ours (3.1), this is written as (Peskin; Schroeder, 1995):

$$d\sigma = \frac{1}{2m_e 2E_\gamma}u^{-1}\langle|\mathcal{M}|^2\rangle\int\frac{d^3p'}{(2\pi)^3 2E'_\gamma}\int\frac{d^3q'}{(2\pi)^3 2q'^0}(2\pi)^4\delta^{(4)}(q'+p'-q-p), \tag{3.30}$$

where $u = \frac{|q\cdot p|}{q^0p^0}$ is the relative velocity between the initial particles, which in the lab frame is set as $u = 1$.

To proceed, we simplify equation (3.30) by reducing the integrals over the final-state invariant phase space:

$$I = \int\frac{d^3p'}{(2\pi)^3 2E'_\gamma}\int\frac{d^3q'}{(2\pi)^3 2q'^0}(2\pi)^4\delta^{(4)}(q'+p'-q-p').$$

Performing the d^3q' integral using $\delta^{(3)}$, we set $\mathbf{q}' = \mathbf{q}+\mathbf{p}-\mathbf{p}'$, and obtain:

$$I = \frac{1}{2q'^0}\int\frac{d^3p'}{(2\pi)^2 2E'_\gamma}\delta^{(1)}(q'^0+p'^0-q^0-p^0),$$

with

$$\delta^{(1)}(q^0+p^0-q'^0-p'^0) = \delta^{(1)}(\sqrt{m^2+E_\gamma^2+E_\gamma'^2-2E_\gamma E'_\gamma\cos\theta}+E'_\gamma-m_e-E_\gamma).$$

This sets $\sqrt{m^2 + E_\gamma^2 + E_\gamma'^2 - 2E_\gamma E_\gamma' \cos\theta} = -E_\gamma' + m_e + E_\gamma$, which we can solve and re-obtain equation (3.8), which sets $E_\gamma' = E_\gamma(\theta)$. Therefore, we can write the delta function as:

$$\delta^{(1)}(q^0 + p^0 - q'^0 - p'^0) = \frac{\delta(E_\gamma' - E_\gamma(\theta))}{\frac{\partial\sqrt{m^2+E_\gamma^2+E_\gamma'^2-2E_\gamma E_\gamma' \cos\theta + E_\gamma'}}{\partial E_\gamma'}}$$
$$= \frac{q'^0 E_\gamma'}{m E_\gamma}\delta(E_\gamma' - E_\gamma(\theta))\,,$$

which implies that:

$$I = \frac{1}{2q'^0}\int \frac{d^3p'}{(2\pi)^2 2E_\gamma'}\frac{q'^0 E_\gamma'}{m E_\gamma}\delta(E_\gamma' - E_\gamma(\theta))\,.$$

We can expand the differential d^3p' in terms of the solid angle $d\Omega$ as:

$$d^3p' = E_\gamma'^2 dE_\gamma' d\Omega.$$

As a result, we have that our final-state phase space can be written as:

$$I = \frac{1}{16\pi^2 m_e E_\gamma}\int E_\gamma'^2 \delta(E_\gamma' - E_\gamma(\theta) dE_\gamma' d\Omega$$
$$= \frac{E_\gamma'^2}{16\pi^2 m_e E_\gamma} d\Omega\,.$$

Thus, equation (3.30) can be simplified in the lab frame as:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\frac{E_\gamma'^2}{32\pi^2 m_e^2 E_\gamma^2}\langle|\mathcal{M}|^2\rangle. \tag{3.31}$$

Finally, using our result from the last section, we can write the Klein-Nishina formula:

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2 m_e^2}\frac{E_\gamma'^2}{E_\gamma^2}\left[\frac{E_\gamma}{E_\gamma'} + \frac{E_\gamma'}{E_\gamma} - \sin^2\theta\right]. \tag{3.32}$$

3.3 Low-Energy Limit and Approximate Elasticity of Compton Scattering

It is quite convenient for us now to work in a low-energy limit, where the incident photon satisfies $E_\gamma \ll m_e$, since in our context we are trying to model the SZE and the CMB photons have very low energy. One important result of this limit is that the Compton formula (3.8) will result in:

$$E_\gamma' \approx E_\gamma\,, \tag{3.33}$$

as a direct consequence of $E_\gamma/m_e \to 0$. This behaviour implies that our process can be effectively treated as an elastic scattering.

As a consequence, the Klein-Nishina formula (3.32) can also be simplified:

$$\frac{d\sigma_T}{d\Omega} = \frac{e^4}{32\pi^2 m_e^2}(1+\cos^2\theta), \tag{3.34}$$

which is exactly the classical result derived by J.J. Thomson (Thomson, 1920) for the scattering of electromagnetic radiation by a free charged particle at rest. To determine the total scattering cross-section in this elastic regime, we can integrate (3.34) over the solid angle, which will result in the well-known result:

$$\sigma_T = \frac{e^4}{6m_e^2\pi}\,. \tag{3.35}$$

One last thing we can do for convenience is to write an expression relating the Thomson cross-section with the scattering amplitude averaged over the spin of the initial electron and summed over the final spin and polarization, that is, relating the $\overline{|\mathcal{M}|^2}$ defined in equation (2.45) and necessary to compute (3.6). For this we compare the equations (3.29), (3.31) and (3.32) with our result for the low-energy scattering (3.34), which allows us to write for this limit:

$$\frac{d\sigma_T}{d\Omega} = \frac{1}{64\pi^2 m_e^2}\langle|\mathcal{M}|^2\rangle\,, \quad \langle|\mathcal{M}|^2\rangle = 2e^4(1+\cos^2\theta)\,.$$

We can multiply the expression for $\langle|\mathcal{M}|^2\rangle$ by $\frac{6m_e^2\pi}{6m_e^2\pi}$, which will give:

$$\langle|\mathcal{M}|^2\rangle = 12\pi m_e^2\sigma_T(1+\cos^2\theta)\,. \tag{3.36}$$

To relate this to $\overline{|\mathcal{M}|^2}$, we undo the average that we have made in (3.18) over the initial electron spin, and maintain the average over the initial photon polarization. This guarantees that, effectively, we only sum over three particles as required by equation (2.45), which will allow us to write:

$$\overline{|\mathcal{M}|^2} = 2\langle|\mathcal{M}|^2\rangle = 24\pi m_e^2\sigma_T(1+\cos^2\theta)\,. \tag{3.37}$$

Since our goal in the next chapter is to model the SZE, which arises from the interaction of CMB photons with electrons in an approximately isotropic radiation field, we can simplify the analysis by averaging over the scattering angle θ, neglecting the explicit angular dependence:

$$\frac{1}{2}\int_0^{\pi}(1+\cos^2\theta)\sin\theta d\theta = \frac{4}{3}.$$

Therefore, we have the angle-averaged squared amplitude:

$$\overline{|\mathcal{M}|^2} = 32\pi m_e^2\sigma_T. \tag{3.38}$$

With all the necessary components now established, we are ready to proceed in our exploration of the SZE, culminating in the derivation of the equation that governs the evolution of CMB photons following interactions of the type discussed throughout this chapter.

4 The Sunyaev-Zel'dovich effect

As discussed in previous chapters, the SZE arises from the interaction between CMB photons and hot electrons in the ICM. In this context, we can safely consider the low-energy limit, since the energy of CMB photons is much smaller than that of the ICM electrons, leading to approximately elastic scattering. Moreover, we neglect the Pauli blocking and stimulated emission factors, which would otherwise introduce terms of the form $(1 - f_e)$ and $(1 + f_\gamma)$ in the collision integral. The justification is twofold: first, the electron occupation numbers f_e are minimal in the ICM, as a consequence of the epoch of electron-positron annihilation in the early universe, rendering Pauli blocking negligible (Dodelson; Schmidt, 2020); second, as we will see in the following derivation, the stimulated emission contributions cancel out upon integration over the full scattering process.

In addition, we work in the non-relativistic limit (NR) for the electron, which is justified by the fact that despite being called hot (with temperatures $T_e \sim$ keV) (Birkinshaw, 1999) their average velocities (thermal velocity) are still small compared to the speed of light, implying $T_e \ll m_e$. As a consequence of this and the conservation of energy for elastic scattering, the initial and final energy of the electrons are simply $E_e \approx E'_e \approx m_e$.

With all these considerations, we write the Boltzmann equation (3.6) as:

$$\begin{aligned}\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right] f_\gamma(p) = &\left\{\frac{1}{2p}\int \frac{d^3p'}{(2\pi)^3 2p'} \int \frac{d^3q}{(2\pi)^3 2m_e} \int \frac{d^3q'}{(2\pi)^3 2m_e} \times\right. \\ &\times \overline{|\mathcal{M}|^2}(2\pi)^4\delta^{(4)}\left(p + q - p' - q'\right) \times \\ &\left.\times \left[f_\gamma(p')f_e(q') - f_\gamma(p)f_e(q)\right]\right\}.\end{aligned} \tag{4.1}$$

Note that we are using the fact that $E_\gamma = p$ and the same for the final photon.

Under these assumptions, in the rest of this chapter, we compute the collision term on the RHS of equation (4.1), and from it, derive a simplified form of the Boltzmann equation that describes how the CMB spectrum is distorted by its interaction with the ICM.

4.1 The Collision term for Compton scattering in a low energy non-relativistic limit

We begin to solve the collision term of equation (4.1) using $\delta^{(3)}$ to integrate over d^3q', which will result in:

$$C[f_\gamma(p)] = \left[\frac{\pi}{2m_e p} \int \frac{d^3q}{(2\pi)^3 2m_e} \int \frac{d^3p'}{(2\pi)^3 2p'} \delta^{(1)}\left(p + \frac{q^2}{2m_e} - p' - \frac{(\mathbf{q}+\mathbf{p}-\mathbf{p}')^2}{2m_e}\right) \right. \\ \left. \times \overline{|\mathcal{M}|^2} \left[f_e(q+p-p') f_\gamma(p') - f_e(q) f_\gamma(p) \right] \right]. \tag{4.2}$$

Again, $\delta^{(3)}$ ensures that $\mathbf{q}' = \mathbf{q}+\mathbf{p}-\mathbf{p}'$. In addition, $\delta^{(1)}\left(p + \frac{q^2}{2m_e} - p' - \frac{(\mathbf{q}+\mathbf{p}-\mathbf{p}')^2}{2m_e}\right)$ is the delta of energy conservation.

We can simplify the energy conservation delta using the fact that in NR the momentum of the photons is much smaller than the momentum of the electrons ($p, p' \ll q$). This allows us to approximate:

$$\frac{q^2}{2m_e} - \frac{(\mathbf{q}+\mathbf{p}-\mathbf{p}')^2}{2m_e} \approx \frac{\mathbf{q}\cdot(\mathbf{p}'-\mathbf{p})}{m_e}. \tag{4.3}$$

Therefore, we proceed to expand the delta function to the second order, which gives:

$$\begin{aligned}
\delta^{(1)}\left(p + \frac{q^2}{2m_e} - p' - \frac{(\mathbf{q}+\mathbf{p}-\mathbf{p}')^2}{2m_e}\right) \approx& \delta^{(1)}\left(p - p' + \frac{\mathbf{q}\cdot(\mathbf{p}'-\mathbf{p})}{m_e}\right) \\
&\approx \delta^{(1)}(p-p') + \frac{\mathbf{q}\cdot(\mathbf{p}'-\mathbf{p})}{m_e} \frac{\partial}{\partial(p'-p)} \delta^{(1)}(p-p') \\
&+ \frac{1}{2}\left(\frac{\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}{m_e}\right)^2 \frac{\partial^2}{\partial(p-p')^2} \delta^{(1)}(p-p') \\
&\approx \delta^{(1)}(p-p') + \frac{\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}{m_e} \frac{\partial}{\partial p'} \delta^{(1)}(p-p') \\
&+ \frac{1}{2}\left(\frac{\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')}{m_e}\right)^2 \frac{\partial^2}{\partial p'^2} \delta^{(1)}(p-p'),
\end{aligned} \tag{4.4}$$

where we have used the fact that $\frac{\partial}{\partial(p-p')} = -\frac{\partial}{\partial p'}$. In addition, the NR also allows us to approximate $f_e(\mathbf{q}+\mathbf{p}-\mathbf{p}') \approx f_e(q)$.

With all these results and considerations, in addition to the angle-averaged square amplitude (3.38), we can rewrite equation (4.2) as:

$$
\begin{aligned}
C[f_\gamma(p)] =& \Bigg[\frac{32 m_e^2 \pi^2 \sigma_T}{2 m_e p} \int \frac{d^3q}{(2\pi)^3 2m_e} f_e(q) \int \frac{d^3p'}{(2\pi)^3 2p'} \\
& \times \left(\delta^{(1)}(p-p') + \frac{\mathbf{q}\cdot(\mathbf{p}-\mathbf{p'})}{m_e} \frac{\partial}{\partial p'} \delta^{(1)}(p-p') + \frac{1}{2}\left(\frac{\mathbf{q}\cdot(\mathbf{p}-\mathbf{p'})}{m_e}\right)^2 \frac{\partial^2}{\partial p'^2}\delta^{(1)}(p-p') \right) \\
& \times \left[f_\gamma(p') - f_\gamma(p) \right] \Bigg].
\end{aligned}
\tag{4.5}
$$

Here, we should note that if we had considered stimulated emission, the extra terms that appear would simply cancel each other out. We can now drop the zeroth-order integral over d^3p', because $\int d^3p' \delta(p-p') \left[f_\gamma(p') - f_\gamma(p) \right] = 0$. Therefore, we can write the collision term as:

$$
\begin{aligned}
C[f_\gamma(p)] =& \Bigg[\frac{4 m_e \pi^2 \sigma_T}{p} \int \frac{d^3q}{(2\pi)^3 m_e} f_e(q) \\
& \times \Bigg(\int \frac{d^3p'}{(2\pi)^3 p'} \left(\frac{\mathbf{q}\cdot(\mathbf{p}-\mathbf{p'})}{m_e} \frac{\partial}{\partial p'} \delta^{(1)}(p-p') \right) \left[f_\gamma(p') - f_\gamma(p) \right] \\
& + \int \frac{d^3p'}{(2\pi)^3 p'} \left(\frac{1}{2}\left(\frac{\mathbf{q}\cdot(\mathbf{p}-\mathbf{p'})}{m_e}\right)^2 \frac{\partial^2}{\partial p'^2}\delta^{(1)}(p-p') \right) \left[f_\gamma(p') - f_\gamma(p) \right] \Bigg) \Bigg].
\end{aligned}
\tag{4.6}
$$

For our next step, it is easier to solve the d^3q of (4.6):

$$
\begin{aligned}
I_1(\mathbf{q}) &= \int \frac{d^3q}{(2\pi)^3} f_e(q) \frac{\mathbf{q}}{m_e}, \\
I_2(q) &= \int \frac{d^3q}{(2\pi)^3} f_e(q) \frac{q^2}{m_e}.
\end{aligned}
\tag{4.7}
$$

To solve these integrals, we assume the electron gas is in thermal equilibrium and adopt a Maxwell-Boltzmann distribution shifted by the bulk velocity [1], where $\mathbf{q} = m_e(\mathbf{v_e} + \mathbf{u_b})$, $\mathbf{v_e}$ is the thermal random velocity of the electrons and $\mathbf{u_b}$ is the bulk velocity of the electron gas to the CMB rest frame. Usually, the bulk velocity is much smaller than its thermal velocity (Birkinshaw, 1999); however, we will consider the contribution of this component here because it provides a source of correction to the standard SZE. The shifted Maxwell-Boltzmann distribution is given by:

$$
f_e(q) = \frac{n_e}{2}(2\pi)^3 \left(\frac{1}{2\pi m_e T_e} \right)^{\frac{3}{2}} \exp\left\{ \left(-\frac{(\mathbf{q} - m_e \mathbf{u_b})^2}{2 m_e T_e} \right) \right\},
\tag{4.8}
$$

[1] As mentioned before, in the ICM $T_e \ll m_e$, so the gas is extremely nondegenerate; quantum (FermiDirac) corrections are negligible.

where the factor $\frac{1}{2}$ appears to account for the two spin states of the electron. For convenience, we will introduce two new variables:

$$\begin{aligned} a =& \frac{1}{2m_e T_e}\,, \\ \mathbf{y} =& \mathbf{q} - m_e \mathbf{u_b} \to d^3q = d^3y\,, \end{aligned} \tag{4.9}$$

which allows us to rewrite 4.8 as:

$$f_e(q) = f_e(y) = \frac{n_e}{2\pi^{\frac{3}{2}}} a^{\frac{3}{2}} (2\pi)^3 e^{-ay^2}\,. \tag{4.10}$$

Now, let's solve the first d^3q integral:

$$\begin{aligned} I_1(\mathbf{q}) =& \int \frac{d^3q}{(2\pi)^3} f_e(q) \frac{\mathbf{q}}{m_e} = \int \frac{d^3y}{(2\pi)^3} f_e(\mathbf{y}) \frac{(\mathbf{y} + m_e \mathbf{u_b})}{m_e} \\ =& \frac{n_e}{2m_e \pi^{\frac{3}{2}}} a^{\frac{3}{2}} \int d^3y (\mathbf{y} + m_e \mathbf{u_b}) e^{(-ay^2)}\,. \end{aligned}$$

The first integral that appears here, that is, $\int d^3y\, \mathbf{y} e^{-ay^2}$ is the integral of an odd function integrated over all the symmetrical space; therefore, this term vanishes, leaving us with:

$$\begin{aligned} I_1(\mathbf{q}) =& \frac{n_e \mathbf{u_B}}{2\pi^{\frac{3}{2}}} a^{\frac{3}{2}} \int e^{(-ay^2)} d^3y \\ =& \frac{n_e \mathbf{u_B}}{2\pi^{\frac{3}{2}}} a^{\frac{3}{2}} 4\pi \int_0^\infty y^2 e^{(-ay^2)} dy\,, \end{aligned}$$

where 4π comes from the integral over the solid angle. Using the Feynman trick, we define an integral:

$$I(a) = \int_0^\infty e^{-ay^2} dy = \frac{1}{2} \int_{-\infty}^\infty e^{-ay^2} dy = \frac{1}{2} \left(\frac{\pi}{a}\right)^{\frac{1}{2}}\,,$$

whose derivative is:

$$\frac{dI}{da} = -\int_0^\infty y^2 e^{-ay^2} dy\,.$$

Consequently, we have:

$$\int_0^\infty y^2 e^{-ay^2} dy = -\frac{dI}{da} = -\frac{d}{da}\left(\frac{\pi}{a}\right)^{\frac{1}{2}} = \frac{\pi^{\frac{1}{2}}}{4} a^{-\frac{3}{2}}\,.$$

Thus, we can finally write $I_1(\mathbf{q})$ as:

$$I_1(\mathbf{q}) = \frac{n_e \mathbf{u_B}}{2} \frac{a^{\frac{3}{2}}}{\pi^{\frac{3}{2}}} 4 \frac{a^{-\frac{3}{2}} \pi^{\frac{3}{2}}}{4}$$

$$\therefore\ I_1(\mathbf{q}) = \int \frac{d^3q}{(2\pi)^3} f_e(q) \frac{\mathbf{q}}{m_e} = \frac{n_e \mathbf{u_B}}{2}, \tag{4.11}$$

which represents the electron number flux density due to the bulk motion of the gas.

Moving on to the second d^3q integral and using a similar procedure, we have:

$$
\begin{aligned}
I_2(q) &= \int \frac{d^3q}{(2\pi)^3} f_e(q) \frac{q^2}{m_e} = \int \frac{d^3y}{(2\pi)^3} f_e(y) \frac{(\mathbf{y} + m_e \mathbf{u_b})^2}{m_e} \\
&= \frac{n_e a^{\frac{3}{2}}}{2 m_e \pi^{\frac{3}{2}}} \int d^3y (\mathbf{y} + m_e \mathbf{u_b})^2 e^{-ay^2} \\
&= \frac{n_e a^{\frac{3}{2}}}{2 m_e \pi^{\frac{3}{2}}} \left[\int d^3y \left(y^2 + 2m_e(\mathbf{u_b} \cdot \mathbf{y}) + m_e^2 u_b^2 \right) e^{-ay^2} \right] \\
&= \frac{n_e a^{\frac{3}{2}}}{2 m_e \pi^{\frac{3}{2}}} 4\pi \left[\int_0^\infty dy \left(y^4 e^{-ay^2} + y^2 e^{-ay^2} \right) \right] .
\end{aligned}
$$

Note that, as usual, $u_b = |\mathbf{u_b}|$. More importantly, the term involving $\mathbf{u_b} \cdot \mathbf{y}$ vanishes upon integration due to isotropy. Again, 4π appears from the integrations over the solid angles. We can also note that we already solved the last integral of the last line in our resolution of $I_1(\mathbf{q})$ and obtained $\int_0^\infty dy\, y^2 e^{-ay^2} = \frac{\pi^{\frac{1}{2}} a^{\frac{-3}{2}}}{4}$. On the other hand, for the first integral in the last line, we again use the Feynman trick and obtain $\int_0^\infty dy\, y^4 e^{-ay^2} = \frac{\pi^{\frac{1}{2}}}{4} \frac{3}{2} a^{-\frac{5}{2}}$. Thus, we finally have:

$$
I_2(q) = \int \frac{d^3q}{(2\pi)^3} f_e(q) \frac{q^2}{m_e} = \frac{n_e a^{\frac{3}{2}}}{2 m_e \pi^{\frac{3}{2}}} 4\pi \left(\frac{\pi^{\frac{1}{2}}}{4} \frac{3}{2} a^{-\frac{5}{2}} + m_e^2 u_b^2 \frac{\pi^{\frac{1}{2}} a^{\frac{-3}{2}}}{4} \right)
$$

$$
\therefore I_2(q) = \int \frac{d^3q}{(2\pi)^3} f_e(q) \frac{q^2}{m_e} = \frac{n_e}{2} \left(3T_e + m_e u_b^2 \right), \tag{4.12}
$$

which represents the total kinetic energy density of the electron gas.

Finally, using 4.11 and 4.12 we can write 4.6 as:

$$
\begin{aligned}
C[f_\gamma(p)] =& \frac{2 m_e \pi^2 \sigma_T n_e}{p} \left[\int \frac{d^3p'}{(2\pi)^3 p'} \left(\frac{\mathbf{u_b} \cdot (\mathbf{p} - \mathbf{p}')}{m_e} \frac{\partial}{\partial p'} \delta^{(1)}(p - p') \right) \left[f_\gamma(p') - f_\gamma(p) \right] \right. \\
&\left. + \left(3T_e + m_e u_b^2 \right) \int \frac{d^3p'}{(2\pi)^3 p'} \left(\frac{1}{2} \left(\frac{\hat{q} \cdot (\mathbf{p} - \mathbf{p}')}{m_e} \right)^2 \frac{\partial^2}{\partial p'^2} \delta^{(1)}(p - p') \right) \left[f_\gamma(p') - f_\gamma(p) \right] \right] .
\end{aligned}
$$

The last thing missing is to solve the d^3p' integrals, which are a little more complicated than the d^3q. To simplify our work, we will define two more integrals:

$$
\begin{aligned}
I_1(p) &= \int \frac{d^3p'}{(2\pi)^3 p'} \left(\frac{\mathbf{u_b} \cdot (\mathbf{p} - \mathbf{p}')}{m_e} \frac{\partial}{\partial p'} \delta^{(1)}(p - p') \right) \left[f_\gamma(p') - f_\gamma(p) \right] , \\
I_2(p) &= \int \frac{d^3p'}{(2\pi)^3 p'} \left(\frac{1}{2} \left(\frac{\hat{q} \cdot (\mathbf{p} - \mathbf{p}')}{m_e} \right)^2 \frac{\partial^2}{\partial p'^2} \delta^{(1)}(p - p') \right) \left[f_\gamma(p') - f_\gamma(p) \right] .
\end{aligned} \tag{4.13}
$$

Consequently, we write the collision term as:

$$
C[f(\mathbf{p})] = \frac{2 m_e \pi^2 \sigma_T n_e}{p} \left[I_1(p) + \left(3T_e + m_e u_b^2 \right) I_2(p) \right] . \tag{4.14}
$$

For the $I_1(p)$ integral, we start by expanding the differential in its radial and angular parts, that is, $d^3p' = p'^2 dp' d\Omega$, and using the product $\mathbf{u_b} \cdot (\mathbf{p} - \mathbf{p}')$ to write two separated integrals:

$$I_1(p) = \int \frac{d\Omega'}{(2\pi)^3} \left[\int_0^\infty dp'\, p' \left(\frac{\mathbf{u_b} \cdot \mathbf{p}}{m_e} \frac{\partial}{\partial p'} \delta^{(1)}(p - p') \right) [f(p') - f_\gamma(p)] \right.$$
$$\left. - \int_0^\infty dp'\, p' \left(\frac{\mathbf{u_b} \cdot \mathbf{p}'}{m_e} \frac{\partial}{\partial p'} \delta^{(1)}(p - p') \right) [f(p') - f_\gamma(p)] \right].$$

Similarly as in the integration process of $I_2(q)$, we have a term $\mathbf{u_b} \cdot \mathbf{p}'$ that vanishes upon integration by isotropy, leaving us with:

$$I_1(p) = \frac{4\pi}{(2\pi)^3} \frac{\mathbf{u_b} \cdot \mathbf{p}}{m_e} \int_0^\infty dp' p' \left[f_\gamma(p') - f_\gamma(p) \right] \left(\frac{\partial}{\partial p'} \delta^{(1)}(p - p') \right).$$

Now, we integrate by parts setting $u = p' \left[f_\gamma(p') - f_\gamma(p) \right]$ and $dv = \frac{\partial}{\partial p'} \delta^{(1)}(p - p') dp'$. In addition, the boundary term $uv|_0^\infty$ vanishes due to the rapid decay of $f_\gamma(p')$ for large momenta and its regularity at the origin, which leads to:

$$\begin{aligned} I_1(p) &= -\frac{4\pi}{(2\pi)^3} \frac{\mathbf{u_b} \cdot \mathbf{p}}{m_e} \int_0^\infty dp' \delta(p - p') \frac{\partial}{\partial p'} \left(p' \left[f_\gamma(p') - f_\gamma(p) \right] \right) \\ &= -\frac{4\pi}{(2\pi)^3} \frac{\mathbf{u_b} \cdot \mathbf{p}}{m_e} \frac{\partial}{\partial p'} \left(p' \left[f_\gamma(p') - f_\gamma(p) \right] \right)_{p'=p} \theta(p), \end{aligned}$$

where $\theta(p)$ is the Heaviside step function that is equal to 1 when $p \geq 0$. Thus, we finally have:

$$\therefore I_1(p) = -\frac{4\pi}{(2\pi)^3} \frac{\mathbf{u_b} \cdot \hat{p}}{m_e} p^2 \frac{\partial}{\partial p} f_\gamma(p). \tag{4.15}$$

Moving on to the $I_2(p)$, our strategy starts in a similar way as for $I_1(p)$; we separate d^3p' into its radial and angular parts, and for the radial integral we integrate by parts twice as follows. For the first integration, we choose $u_1 = p'(\hat{q} \cdot (\mathbf{p} - \mathbf{p}'))^2 \left[f_\gamma(p') - f_\gamma(p) \right]$ and $dv_1 = \frac{\partial^2}{\partial p'^2} \delta^{(1)}(p - p') dp'$, and for the second we choose $u_2 = \frac{\partial}{\partial p'} u_1$ and $dv_2 = \frac{\partial}{\partial p'} \delta^{(1)}(p - p') dp'$. Again, the $uv|_0^\infty$ terms vanish in both our integrations, leaving us with:

$$\begin{aligned} I_2(p) &= \int \frac{d^3p'}{(2\pi)^3 p'} \left(\frac{1}{2} \left(\frac{\hat{q} \cdot (\mathbf{p} - \mathbf{p}')}{m_e} \right)^2 \frac{\partial^2}{\partial p'^2} \delta^{(1)}(p - p') \right) \left[f_\gamma(p') - f_\gamma(p) \right] \\ &= \frac{1}{2m_e^2} \int \frac{d\Omega'}{(2\pi)^3} \left(\frac{\partial^2}{\partial p'^2} \left(p' (\hat{q} \cdot (\mathbf{p} - \mathbf{p}'))^2 \left[f_\gamma(p') - f_\gamma(p) \right] \right) \right)_{p'=p}. \end{aligned}$$

Note that in the second line we neglect the Heaviside step function as we have done for the $I_1(p)$ integral. Now, solving the derivatives with respect to p', we obtain:

$$I_2(p) = \frac{1}{2m_e^2} \int \frac{d\Omega'}{(2\pi)^3} p^2 \left[(\hat{q} \cdot (\hat{p} - \hat{p}'))^2 \left(2 \frac{\partial}{\partial p} f_\gamma(p) + p \frac{\partial^2}{\partial p^2} f_\gamma(p) \right) \right.$$
$$\left. - 4 (\hat{q} \cdot (\hat{p} - \hat{p}'))(\hat{q} \cdot \hat{p}') \frac{\partial}{\partial p} f_\gamma(p) \right].$$

Using again that $\mathbf{q} = m_e(\mathbf{v_e} + \mathbf{u_b})$, we can write $\hat{q} = \frac{\mathbf{v_e}+\mathbf{u_b}}{|\mathbf{v_e}+\mathbf{u_b}|}$. Since, as mentioned, we usually have $u_b \ll v_e$, we will approximate $|\mathbf{v_e} + \mathbf{u_b}| \approx v_e$ and consequently $\hat{q} \approx \hat{v}_e + \frac{\mathbf{u_b}}{v_e}$. This allows us to write the dot products as:

$$
\begin{aligned}
(\hat{q}\cdot(\hat{p}-\hat{p}'))^2 &= \left(\frac{\mathbf{v_e}+\mathbf{u_b}}{|\mathbf{v_e}+\mathbf{u_b}|}\cdot(\hat{p}-\hat{p}')\right)^2 \approx \left(\left(\hat{v}_e + \frac{\mathbf{u_b}}{v_e}\right)\cdot(\hat{p}-\hat{p}')\right)^2 \\
&= (\hat{v}_e\cdot(\hat{p}-\hat{p}'))^2 + 2(\hat{v}_e\cdot(\hat{p}-\hat{p}'))\left(\frac{\mathbf{u_b}}{v_e}\cdot(\hat{p}-\hat{p}')\right) + O\left(\frac{u_b^2}{v_e^2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(\hat{q}\cdot(\hat{p}-\hat{p}'))(\hat{q}\cdot\hat{p}') &\approx \left(\hat{v}_e + \frac{\mathbf{u}_b}{v_e}\right)\cdot(\hat{p}-\hat{p}')\left(\hat{v}_e + \frac{\mathbf{u}_b}{v_e}\right)\cdot\hat{p}' \\
&= (\hat{v}_e\cdot(\hat{p}-\hat{p}'))(\hat{v}_e\cdot\hat{p}') + \frac{(\mathbf{u}_b\cdot(\hat{p}-\hat{p}'))(\hat{v}_e\cdot\hat{p}')}{v_e} \\
&+ \frac{(\mathbf{u}_b\cdot\hat{p}')(\hat{v}_e\cdot(\hat{p}-\hat{p}'))}{v_e} + O\left(\frac{u_b^2}{v_e^2}\right).
\end{aligned}
$$

Because the electrons are in thermal equilibrium, any directional dependence in our integrand must be removed by averaging over their orientations. Since the bulk velocity $\mathbf{u_b}$ is fixed, this is equivalent to averaging over the thermal-velocity unit vector $\hat{v}_e$ (i.e. $\hat{q} \to \hat{v}_e$). We therefore employ the standard isotropic-average identities [2]:

$$
\left\langle \hat{v}_e^i \right\rangle = 0, \qquad \left\langle \hat{v}_e^i \hat{v}_e^j \right\rangle = \frac{1}{3}\,\delta^{ij}, \qquad \left\langle (\hat{v}_e\cdot A)\,(\hat{v}_e\cdot B) \right\rangle = \frac{1}{3}\,(A\cdot B)
$$

where the indices i and j are the spatial coordinates indices. In addition, we neglect all terms of order $\left(\frac{u_b}{v_e}\right)^2$. This approximation is justified because, as we have already mentioned ($u_b \ll v_e$), rendering these quadratic terms physically insignificant. Since any terms linear in $\frac{u_b}{v_e}$ vanish upon isotropic averaging as per the identities above, these $O(\frac{u_b^2}{v_e^2})$ contributions are the first non-zero, but negligible, correction. Crucially, this differs from the main kinematic term that also scales with u_b^2 in equation (4.14); what is being dropped here is a small, higher-order correction to the anisotropy of the scattering process, whereas the $m_e u_b^2$ term is retained in (4.14) represents a fundamental contribution to the scalar kinetic energy of the system, a primary physical effect that survives all averaging. Therefore, we can write:

$$
\begin{aligned}
\left\langle (\hat{q}\cdot(\hat{p}-\hat{p}'))^2 \right\rangle &\approx \frac{2}{3}(1-\hat{p}\cdot\hat{p}') \\
\langle (\hat{q}\cdot(\hat{p}-\hat{p}'))(\hat{q}\cdot\hat{p}') \rangle &\approx -\frac{1}{3}(1-\hat{p}\cdot\hat{p}').
\end{aligned}
\tag{4.16}
$$

Consequently, averaging over the electron directions allows us to write $I_2(p)$ as simply:

$$
I_2(p) = \frac{1}{2m_e^2}\int \frac{d\Omega'}{(2\pi)^3} p^2 \left[\frac{8}{3}(1-\hat{p}\cdot\hat{p}')\frac{\partial}{\partial p}f_\gamma(p) + \frac{2}{3}p(1-\hat{p}\cdot\hat{p}')\frac{\partial^2}{\partial p^2}f_\gamma(p)\right].
$$

[2] To proof this identities we just integrate over the solid angle of the possible directions of the electron's thermal velocity vector.

Finally, we can solve the $I_2(p)$ integral over $d\Omega'$. For this, we will consider that the incoming photons are in the direction $\hat{p} = \hat{z}$, so that $\hat{p} \cdot \hat{p} = \cos(\theta')$. This finally gives us:

$$I_2(p) = \frac{4\pi}{3m_e^2(2\pi)^3}\left(4p^2\frac{\partial}{\partial p}f_\gamma(p) + p^3\frac{\partial^2}{\partial p^2}f_\gamma(p)\right). \tag{4.17}$$

Lastly, with the results of the equations (4.15) and (4.17), we can write (4.14) as:

$$\begin{aligned}\therefore\ C[f_\gamma(p)] =& -\sigma_T n_e(\mathbf{u_b}\cdot\hat{p})p\frac{\partial}{\partial p}f_\gamma(p,t) + \frac{\sigma_T n_e u_b^2}{3}\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^4\frac{\partial}{\partial p}f_\gamma(p,t)\right)\\ &+\frac{\sigma_T n_e T_e}{m_e}\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^4\frac{\partial}{\partial p}f_\gamma(p,t)\right).\end{aligned} \tag{4.18}$$

With this simplified form of the collision term, we now pass to write the Boltzmann equation and derive the SZE.

4.2 The Boltzmann equation for photon interacting via Compton scattering in the flat FLRW universe and the SZE

Now that we have obtained the collision term for the Compton scattering (4.18), we can write the Boltzmann equation (4.1) for the CMB photons in a flat FLRW universe as:

$$\begin{aligned}\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]f_\gamma(p,t) =& -\sigma_T n_e(\mathbf{u_b}\cdot\hat{p})p\frac{\partial}{\partial p}f_\gamma(p,t) + \frac{\sigma_T n_e u_b^2}{3}\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^4\frac{\partial}{\partial p}f_\gamma(p,t)\right)\\ &+\frac{\sigma_T n_e T_e}{m_e}\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^4\frac{\partial}{\partial p}f_\gamma(p,t)\right).\end{aligned} \tag{4.19}$$

We can further simplify this equation by introducing two new variables. The first variable we introduce is $x = \frac{p}{T(t)}$, where $T(t) = \frac{T_0}{a}$ is the photon temperature as a function of time. This variable will simplify the LHS of (4.19) in such a way that the only term left on this side will be the time derivative term:

Proof. To see this, we first rewrite the time derivative

$$\begin{aligned}\frac{\partial}{\partial t}f_\gamma(p,t) =& \frac{\partial}{\partial t}f_\gamma(x,t) + \frac{\partial x}{\partial t}\frac{\partial}{\partial x}f_\gamma(x,t)\\ =& \frac{\partial}{\partial t}f_\gamma(x,t) + \left(\frac{1}{T^2}\left(T\frac{\partial p}{\partial t} - p\frac{\partial T}{\partial t}\right)\right)\frac{\partial}{\partial x}f_\gamma(x,t)\\ =& \frac{\partial}{\partial t}f_\gamma(x,t) + \left(\frac{1}{T}\frac{\partial p}{\partial t} + xH\right)\frac{\partial}{\partial x}f_\gamma(x,t).\end{aligned}$$

Therefore, in the absence of external forces:

$$\frac{\partial}{\partial t}f_\gamma(p,t) = \frac{\partial}{\partial t}f_\gamma(x,t) + xH\frac{\partial}{\partial x}f_\gamma(x,t). \tag{4.20}$$

On the other hand, the momentum derivative can be written:

$$\frac{\partial}{\partial p} f_\gamma(p,t) = \frac{\partial x}{\partial p}\frac{\partial}{\partial x} f_\gamma(x,t) = \frac{1}{T}\frac{\partial}{\partial x} f_\gamma(x,t),$$

and finally, the LHS of equation (4.19) can be written as:

$$\begin{aligned}\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right] f_\gamma(p,t) &= \left[\frac{\partial}{\partial t} + xH\frac{\partial}{\partial x} - xH\frac{\partial}{\partial x}\right] f_\gamma(x,t) \\ &= \frac{\partial}{\partial t} f_\gamma(x,t)\,.\end{aligned} \tag{4.21}$$

□

Therefore, we can write (4.19) as:

$$\begin{aligned}\frac{\partial}{\partial t} f_\gamma(x,t) = &- \sigma_T n_e(\mathbf{u_b}\cdot\hat{x}) x \frac{\partial}{\partial x} f_\gamma(x,t) + \frac{\sigma_T n_e u_b^2}{3}\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x} f_\gamma(x,t)\right) \\ &+ \frac{\sigma_T n_e T_e}{m_e}\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x} f_\gamma(x,t)\right).\end{aligned} \tag{4.22}$$

The second variable we introduce to further simplify the Boltzmann equation (4.22) is a dimensionless evolution parameter, y, known as the Compton-y parameter, which combines the physical constants into a single variable. This parameter is defined by its differential:

$$dy = \frac{\sigma_T n_e T_e}{m_e} dt\,. \tag{4.23}$$

The integrated parameter y, obtained by integrating this differential along the photon's path, has a direct physical interpretation. It represents the product of the total scattering probability for a photon, given by the Compton optical depth (τ_e), and the mean fractional energy transfer per scattering, which is proportional to T_e/m_e.

Using the chain rule, the time derivative is transformed into a derivative with respect to y:

$$\frac{\partial}{\partial t} = \frac{dy}{dt}\frac{\partial}{\partial y} = \frac{\sigma_T n_e T_e}{m_e}\frac{\partial}{\partial y}.$$

Substituting this into equation (4.22) yields its final form, which is a generalized Kompaneets equation:

$$\frac{\partial}{\partial y} f(x,y) = -\frac{m_e(\mathbf{u_b}\cdot\hat{x})}{T_e} x\frac{\partial}{\partial x} f(x,y) + \left(\frac{m_e u_b^2}{3T_e} + 1\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x} f(x,y)\right). \tag{4.24}$$

4.2.1 The SZE effect

The last thing missing to understand how the CMB photon distribution changes after the interaction with the electron gas (of the ICM) is to solve the Boltzmann equation (4.24). For this, we use as our initial condition the Bose-Einstein distribution:

$$f(x,y=0) = f^{(0)}(x) = \left(e^x - 1\right)^{-1}\,. \tag{4.25}$$

Now, in a regime of small optical depth for the Compton scattering (equivalent to $y \ll 1$), which is reasonable for the ICM as shown by (Schiappucci *et al.*, 2023), we expand our solution to the Boltzmann equation $f(x,y)$ around $y = 0$, and truncate at its first order, that is, we suppose a solution of the form:

$$f(x,y) = f^{(0)}(x) + yf^{(1)}(x)\,. \tag{4.26}$$

Putting this into (4.24), we get:

$$\begin{aligned}\frac{\partial}{\partial y}(f^{(0)}(x) + yf^{(1)}(x)) = &-\frac{m_e(\mathbf{u_b}\cdot\hat{x})}{T_e}x\frac{\partial}{\partial x}(f^{(0)}(x) + yf^{(1)}(x))\\ &+\left(\frac{m_e u_b^2}{3T_e}+1\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x}(f^{(0)}(x) + yf^{(1)}(x))\right)\,.\end{aligned}$$

The first term does not depend on y. Therefore, it goes to zero, which gives:

$$\begin{aligned}f^{(1)}(x) = &-\frac{m_e(\mathbf{u_b}\cdot\hat{x})}{T_e}x\frac{\partial}{\partial x}(f^{(0)}(x) + yf^{(1)}(x))\\ &+\left(\frac{m_e u_b^2}{3T_e}+1\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x}(f^{(0)}(x) + yf^{(1)}(x))\right).\end{aligned}$$

Since we want to write $f(x,y)$ up to its first order, we write the term $yf^{(1)}(x)$ as:

$$yf^{(1)}(x) = -\frac{m_e(\mathbf{u_b}\cdot\hat{x})}{T_e}x\frac{\partial}{\partial x}f^{(0)}(x) + \left(\frac{m_e u_b^2}{3T_e}+1\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x}f^{(0)}(x)\right)\,. \tag{4.27}$$

Consequently, this result allows us to write our solution to the Boltzmann equation in terms only of $f^{(0)}(x)$ using (4.26) and (4.27) as:

$$\begin{aligned}f(x,y) &= \left[1 - y\frac{m_e(\mathbf{u_b}\cdot\hat{x})}{T_e}x\frac{\partial}{\partial x} + y\left(\frac{m_e u_b^2}{3T_e}+1\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x}\right)\right]f^{(0)}(x)\\ &= \left[1 - y\frac{m_e(\mathbf{u_b}\cdot\hat{x})}{T_e}x\frac{\partial}{\partial x} + y\left(\frac{m_e u_b^2}{3T_e}+1\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x}\right)\right](e^x-1)^{-1}\,.\end{aligned} \tag{4.28}$$

As already mentioned, the Compton-y parameter is related to the Compton optical depth by:

$$y = \tau_e\frac{T_e}{m_e}\,, \tag{4.29}$$

which allows us to write (4.28) in the form:

$$f(x,y) = \left[1 - \tau_e(\mathbf{u_b}\cdot\hat{x})x\frac{\partial}{\partial x} + \left(\tau\frac{u_b^2}{3}+y\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x}\right)\right](e^x-1)^{-1}\,. \tag{4.30}$$

On the other hand, the Doppler shift gives us the relation:

$$\frac{\Delta T}{T} = -\tau_e(\mathbf{u_b}\cdot\hat{x}) \tag{4.31}$$

and consequently we can write (4.30) as:

$$f(x,y) = \left[1 + x\frac{\Delta T}{T}\frac{\partial}{\partial x} + \left(\tau\frac{u_b^2}{3} + y\right)\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x}\right)\right](e^x - 1)^{-1} . \tag{4.32}$$

Finally, in equation (4.32) we have obtained how the CMB distribution changes after interactions with the electrons in the ICM via inverse Compton scattering in the low-energy NR limit for a small probability of interaction. It is easily seen that if the photon does not interact ($y = 0$), the distribution remains unchanged. The first term that alters the CMB distribution is:

$$x\frac{\Delta T}{T}\frac{\partial}{\partial x}(e^x - 1)^{-1} , \tag{4.33}$$

which corresponds to the kinetic Sunyaev-Zel'dovich effect (kSZE). This contribution arises due to the bulk motion of the galaxy cluster relative to the CMB rest frame. The term accounts for the Doppler shift correction caused by this motion, resulting in a first-order correction to the photon distribution. Next, the term:

$$y\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\frac{\partial}{\partial x}\right)(e^x - 1)^{-1} \tag{4.34}$$

corresponds to the thermal Sunyaev-Zel'dovich effect (tSZE), which is the leading term of the effect and arises from the Compton interaction. Lastly, the interesting aspect of our derivation is the emergence of the u_b^2 term, which is a relativistic correction due to the bulk motion of the cluster. This second-order correction agrees in form with those found in the literature (Challinor; Lasenby, 1998; Nozawa; Itoh; Kohyama, 2005), where similar u_b^2 terms arise from expanding the relativistic Boltzmann equation beyond leading order.

To visualize how the various SZE components modify the CMB spectrum, we plot the full solution (4.32) using the following physically motivated parameters that can be found in the literature:

- $\frac{\Delta T}{T} = 10^{-5}$ the typical amplitude of CMB temperature anisotropies (Dodelson; Schmidt, 2020);
- $y = 0.05$ a representative Compton-y parameter within our assumed regime (Birkinshaw, 1999);
- $\tau_e = 0.01$ a typical optical depth for ICM electrons (Carlstrom; Holder; Reese, 2002);
- $u_b = \frac{500\,\mathrm{km/s}}{c}$ a realistic bulk velocity for galaxy clusters, consistent with observational estimates in the range of 500 km/s – 1000 km/s (Birkinshaw, 1999);
- $\delta = \frac{\tau_e u_b^2}{c}$ a second-order bulk-motion correction that arises from retaining the $\mathbf{u}_b^2$ term in the electron momentum distribution.

The resulting spectra are shown in Figures 4.1 and 4.3. Figure 4.1 compares the undistorted CMB blackbody spectrum (4.25) to the spectrum modified by the total SZE distortion (4.32). The behaviour illustrated in this figure is the essence of the inverse Compton scattering: photons below $x \approx 3.83$ (or $\nu \approx 217$ GHz) lose intensity, while those above this threshold gain intensity. The point $x \approx 3.83$ is the null SZE, where the distorted and undistorted spectra coincide, and no net change in photon intensity is observed. This is

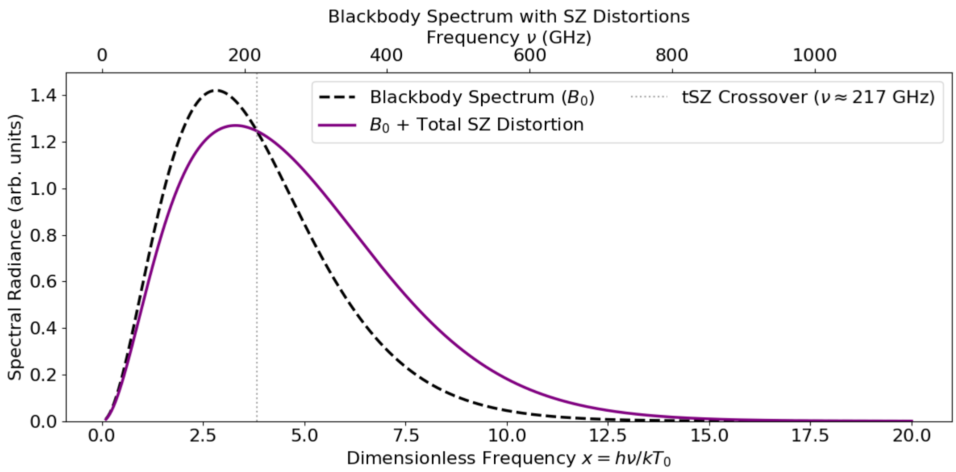


Figure 4.1 – Comparison between the undistorted CMB blackbody spectrum (4.25) (represented by B_0 - dashed black) and the spectrum including the full SZE distortion (solid purple) (4.32). The vertical dotted line marks at $x \approx 3.83$ (or $\nu \approx 217$ GHz), marks where the distorted spectrum intersects with the blackbody spectrum, that is, where we have a null SZE. The SZE reduces intensity below the null tSZE point and increases it above, producing a spectral distortion distinct from a simple blackbody shift.

the same behaviour observed in the Planck data result in Figure 4.2. Therefore, we observe excellent agreement between theory and observation of the SZE due to the up-scattered CMB photons by hot electrons in galaxy clusters.

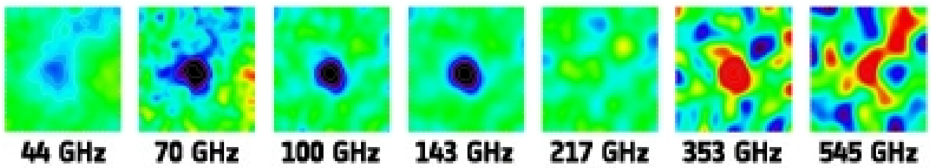


Figure 4.2 – Multi-frequency Planck observations of a galaxy cluster. The panels show the same region of sky at various frequencies ranging from 44 GHz to 545 GHz. The spectral evolution across bands reveals the SZE distortion. Image credit: (ESA / Planck Collaboration, 2019).

In contrast, Figure 4.3 separates the contributions from the tSZE, the kSZE, and the small second-order velocity correction associated with δ. From the decomposition in this figure, it is evident that the tSZE is the dominant contribution. The kSZE appears at sub-leading order, and the δ term arising from $\mathbf{u}_b^2$ acts as a second-order relativistic correction. Though small in amplitude, its spectral shape differs from both tSZE and kSZE, and its inclusion is crucial for a fully consistent relativistic treatment.

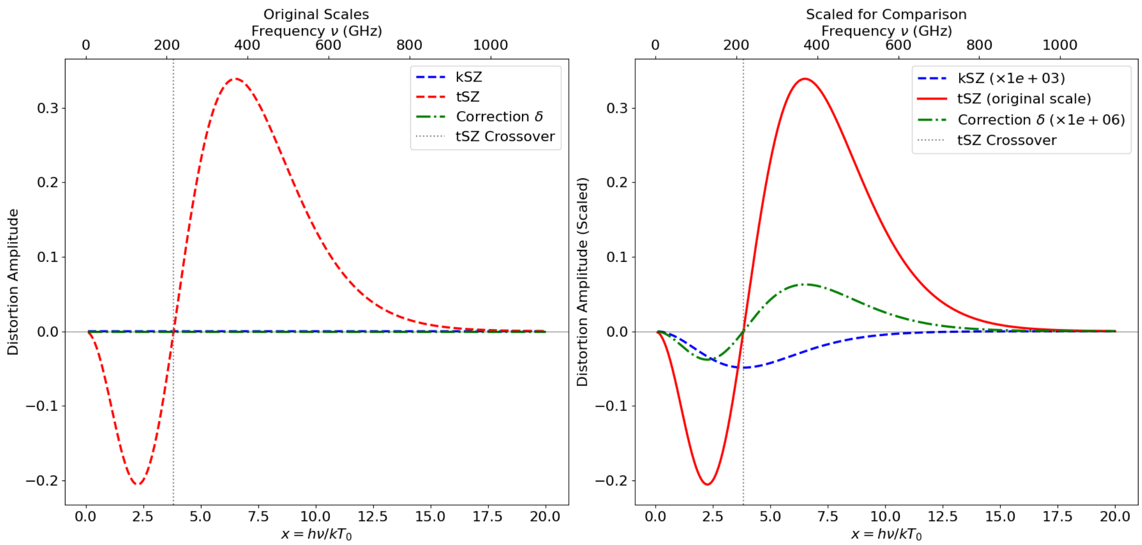


Figure 4.3 – Comparison of individual SZE components as functions of dimensionless frequency $x = \frac{h\nu}{k_B T_0}$, where T_0 is the present-day CMB temperature. The *left panel* shows the distortions at their natural amplitudes: the tSZE (4.34) (solid red) dominates, while the kSZE (4.33) (dashed blue) and the second-order velocity correction $\delta \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^4 \frac{\partial}{\partial x} \right) (e^x - 1)^{-1}$(dot-dashed green) are much smaller. The *right panel* rescales the subdominant terms for visibility: kSZE is scaled by 10^3, and δ by 10^6, while tSZE remains unscaled. The vertical dotted line again marks the null tSZE near $\nu \approx 217\,\text{GHz}$.

5 Final Remarks

Throughout this work, we have explored fundamental aspects of modern physics, culminating in the derivation of the spectral distortion of CMB photons due to inverse Compton scattering, as shown in equation (4.32). To achieve this, we employed tools from both GR, to account for gravitational effects on the photon distribution function, and QFT, to model the interaction kernel from first principles. Nevertheless, obtaining a compact analytical expression, such as the one presented for the SZE in Chapter 4, required the adoption of several physically motivated approximations. In particular, the low-energy and NR limits were essential, but they impose intrinsic limitations on the applicability of this standard SZE formalism.

In this context, a significant body of research has focused on extending the SZE framework beyond these approximations. It is crucial to distinguish between the two main sources of relativistic corrections. The first is the kSZE, which, as previously discussed, is a Doppler effect caused by the bulk velocity $\mathbf{u}_b$ of the electron gas relative to the CMB rest frame. The second, especially relevant for massive galaxy clusters, is the tSZE. Corrections to the tSZE become necessary when the electron gas is extremely hot ($k_B T_e \sim 10$ keV) (Challinor; Lasenby, 1998), making the individual electron motions relativistic, even in the absence of net bulk flow.

There are two primary strategies to incorporate these relativistic effects, both of which go beyond the near-elastic scattering approximation used in this work. The first is to apply a power-series expansion. This method builds on the Fokker-Planck approximation to simplify the collision integral (4.2) in the Boltzmann equation [1]. This approach is valid when the energy transfer per scattering is small, systematically introducing higher-order corrections to the standard kinetic framework. For example, the kSZE can be corrected by expanding in powers of the bulk velocity, while the tSZE can be refined via an expansion in the temperature parameter $\theta_e = k_B T_e / m_e c^2$ (Nozawa; Itoh; Kohyama, 2005; Challinor; Lasenby, 1998; Itoh; Kohyama; Nozawa, 1998; Sazonov; Sunyaev, 1998).

The second, more fundamental approach dispenses with the near-elastic approximation altogether. It starts from a fully relativistic description of Compton scattering, employing the complete Klein-Nishina cross section (3.32) and enforcing the conservation of relativistic four-momentum throughout the interaction. This method leads to an exact evaluation of the collision term in the Boltzmann equation (4.1), without relying on expansions or approx-

[1] This approximation replaces the full collision term with a differential operator that describes the combined effect of many small energy transfers during scattering(Lifshitz; Pitaevskii, 1981). When electrons have a bulk motion relative to the photons, these small changes add up, causing a Doppler shift in the photon spectrum. This is valid when momentum transfer per scattering is small and allows effects like the kSZE and tSZE to be treated using expansions in velocity and temperature.

imations. As demonstrated in recent work such as (Terzi *et al.*, 2024), this framework offers a fundamental re-derivation of the tSZE that is valid even for highly relativistic electron populations. A natural next step is to extend this fully relativistic treatment to also incorporate the kinematic contributions from cluster bulk motion, unifying both effects within a single coherent theory.

Finally, it is worth emphasizing that, even within the simplified framework used here, we were able to derive a non-trivial second-order correction involving the bulk velocity. By considering a shifted Maxwell-Boltzmann distribution and retaining the $\mathbf{u}_b^2$ term in the momentum integrals, we captured a contribution beyond the standard linear Doppler effect associated with the kSZE. While not traditionally classified as part of the kSZE, this second-order term reflects the interplay between the bulk motion and the thermal distribution of electrons, and is sometimes interpreted as a relativistic correction or a mixed kinetic-thermal effect.

References

ACUñA-CáRDENAS, R. O.; GABARRETE, C.; SARBACH, O. An introduction to the relativistic kinetic theory on curved spacetimes. **General Relativity and Gravitation**, Springer Science and Business Media LLC, v. 54, n. 3, Feb. 2022. ISSN 1572-9532. Available at: http://dx.doi.org/10.1007/s10714-022-02908-5. Cit. on pp. 12, 13, and 14.

ADE, P. A. R.; AGHANIM, N.; ARMITAGE-CAPLAN, C.; ARNAUD, M.; ASHDOWN, M.; ATRIO-BARANDELA, F.; AUMONT, J.; BACCIGALUPI, C.; BANDAY, A. J.; BARREIRO, R. B.; BARTLETT, J. G.; BATTANER, E.; BENABED, K.; BENOîT, A.; BENOIT-LéVY, A.; BERNARD, J.-P.; BERSANELLI, M.; BIELEWICZ, P.; BOBIN, J.; BOCK, J. J.; BONALDI, A.; BOND, J. R.; BORRILL, J.; BOUCHET, F. R.; BRIDGES, M.; BUCHER, M.; BURIGANA, C.; BUTLER, R. C.; CALABRESE, E.; CAPPELLINI, B.; CARDOSO, J.-F.; CATALANO, A.; CHALLINOR, A.; CHAMBALLU, A.; CHARY, R.-R.; CHEN, X.; CHIANG, H. C.; CHIANG, L.-Y.; CHRISTENSEN, P. R.; CHURCH, S.; CLEMENTS, D. L.; COLOMBI, S.; COLOMBO, L. P. L.; COUCHOT, F.; COULAIS, A.; CRILL, B. P.; CURTO, A.; CUTTAIA, F.; DANESE, L.; DAVIES, R. D.; DAVIS, R. J.; BERNARDIS, P. de; ROSA, A. de; ZOTTI, G. de; DELABROUILLE, J.; DELOUIS, J.-M.; DéSERT, F.-X.; DICKINSON, C.; DIEGO, J. M.; DOLAG, K.; DOLE, H.; DONZELLI, S.; DORé, O.; DOUSPIS, M.; DUNKLEY, J.; DUPAC, X.; EFSTATHIOU, G.; ELSNER, F.; ENSSLIN, T. A.; ERIKSEN, H. K.; FINELLI, F.; FORNI, O.; FRAILIS, M.; FRAISSE, A. A.; FRANCESCHI, E.; GAIER, T. C.; GALEOTTA, S.; GALLI, S.; GANGA, K.; GIARD, M.; GIARDINO, G.; GIRAUD-HéRAUD, Y.; GJERLøW, E.; GONZáLEZ-NUEVO, J.; GóRSKI, K. M.; GRATTON, S.; GREGORIO, A.; GRUPPUSO, A.; GUDMUNDSSON, J. E.; HAISSINSKI, J.; HAMANN, J.; HANSEN, F. K.; HANSON, D.; HARRISON, D.; HENROT-VERSILLé, S.; HERNáNDEZ-MONTEAGUDO, C.; HERRANZ, D.; HILDEBRANDT, S. R.; HIVON, E.; HOBSON, M.; HOLMES, W. A.; HORNSTRUP, A.; HOU, Z.; HOVEST, W.; HUFFENBERGER, K. M.; JAFFE, A. H.; JAFFE, T. R.; JEWELL, J.; JONES, W. C.; JUVELA, M.; KEIHäNEN, E.; KESKITALO, R.; KISNER, T. S.; KNEISSL, R.; KNOCHE, J.; KNOX, L.; KUNZ, M.; KURKI-SUONIO, H.; LAGACHE, G.; LäHTEENMäKI, A.; LAMARRE, J.-M.; LASENBY, A.; LATTANZI, M.; LAUREIJS, R. J.; LAWRENCE, C. R.; LEACH, S.; LEAHY, J. P.; LEONARDI, R.; LEóN-TAVARES, J.; LESGOURGUES, J.; LEWIS, A.; LIGUORI, M.; LILJE, P. B.; LINDEN-VøRNLE, M.; LóPEZ-CANIEGO, M.; LUBIN, P. M.; MACíAS-PéREZ, J. F.; MAFFEI, B.; MAINO, D.; MANDOLESI, N.; MARIS, M.; MARSHALL, D. J.; MARTIN, P. G.; MARTíNEZ-GONZáLEZ, E.; MASI, S.; MASSARDI, M.; MATARRESE, S.; MATTHAI, F.; MAZZOTTA, P.; MEINHOLD, P. R.; MELCHIORRI, A.; MELIN, J.-B.; MENDES, L.; MENEGONI, E.; MENNELLA, A.; MIGLIACCIO, M.; MILLEA, M.; MITRA, S.; MIVILLE-DESCHêNES, M.-A.; MONETI, A.; MONTIER, L.; MORGANTE,

G.; MORTLOCK, D.; MOSS, A.; MUNSHI, D.; MURPHY, J. A.; NASELSKY, P.; NATI, F.; NATOLI, P.; NETTERFIELD, C. B.; NøRGAARD-NIELSEN, H. U.; NOVIELLO, F.; NOVIKOV, D.; NOVIKOV, I.; ODWYER, I. J.; OSBORNE, S.; OXBORROW, C. A.; PACI, F.; PAGANO, L.; PAJOT, F.; PALADINI, R.; PAOLETTI, D.; PARTRIDGE, B.; PASIAN, F.; PATANCHON, G.; PEARSON, D.; PEARSON, T. J.; PEIRIS, H. V.; PERDEREAU, O.; PEROTTO, L.; PERROTTA, F.; PETTORINO, V.; PIACENTINI, F.; PIAT, M.; PIERPAOLI, E.; PIETROBON, D.; PLASZCZYNSKI, S.; PLATANIA, P.; POINTECOUTEAU, E.; POLENTA, G.; PONTHIEU, N.; POPA, L.; POUTANEN, T.; PRATT, G. W.; PRéZEAU, G.; PRUNET, S.; PUGET, J.-L.; RACHEN, J. P.; REACH, W. T.; REBOLO, R.; REINECKE, M.; REMAZEILLES, M.; RENAULT, C.; RICCIARDI, S.; RILLER, T.; RISTORCELLI, I.; ROCHA, G.; ROSSET, C.; ROUDIER, G.; ROWAN-ROBINSON, M.; RUBIñO-MARTíN, J. A.; RUSHOLME, B.; SANDRI, M.; SANTOS, D.; SAVELAINEN, M.; SAVINI, G.; SCOTT, D.; SEIFFERT, M. D.; SHELLARD, E. P. S.; SPENCER, L. D.; STARCK, J.-L.; STOLYAROV, V.; STOMPOR, R.; SUDIWALA, R.; SUNYAEV, R.; SUREAU, F.; SUTTON, D.; SUUR-USKI, A.-S.; SYGNET, J.-F.; TAUBER, J. A.; TAVAGNACCO, D.; TERENZI, L.; TOFFOLATTI, L.; TOMASI, M.; TRISTRAM, M.; TUCCI, M.; TUOVINEN, J.; TüRLER, M.; UMANA, G.; VALENZIANO, L.; VALIVIITA, J.; TENT, B. V.; VIELVA, P.; VILLA, F.; VITTORIO, N.; WADE, L. A.; WANDELT, B. D.; WEHUS, I. K.; WHITE, M.; WHITE, S. D. M.; WILKINSON, A.; YVON, D.; ZACCHEI, A.; ZONCA, A. Planck2013 results. xvi. cosmological parameters. **Astronomy & Astrophysics**, EDP Sciences, v. 571, p. A16, Oct. 2014. ISSN 1432-0746. Available at: http://dx.doi.org/10.1051/0004-6361/201321591. Cit. on p. 23.

BIRKINSHAW, M. The sunyaevzeldovich effect. **Physics Reports**, v. 310, n. 2, p. 97–195, 1999. ISSN 0370-1573. Available at: https://www.sciencedirect.com/science/article/pii/S0370157398000805. Cit. on pp. 37, 39, and 47.

BLEEM, L. E.; STALDER, B.; HAAN, T. de; ... Galaxy clusters discovered via the sunyaevzel'dovich effect in the 2500squaredegree sptsz survey. **Astrophysical Journal Supplement Series**, 2015. Cit. on p. 10.

BOLTZMANN, L. Further studies on the thermal equilibrium of gas molecules. *In*: ____. **The Kinetic Theory of Gases**. Imperial College Press and distributed by World Scientific Publishing Co., 2003. (History of Modern Physical Sciences, v. 1), p. 262–349. ISBN 978-1-86094-347-8. Reprint of the original paper published under the title "Weitere Studien über das Wörmegleichgewicht unter Gasmolekülen", in Sitzungsberichte Akad. Wiss., Vienna, part II, 66, 275370 (1872); reprinted in Boltzmann's Wissenschaftliche Abhandlungen, Vol. I, Leipzig, J. A. Barth, 1909, pp. 316402. Available at: https://doi.org/10.1142/9781848161337_0015. Cit. on p. 12.

Carlstrom, J. E.; Holder, G. P.; Reese, E. D. Cosmology with the Sunyaev-Zel'dovich Effect. , v. 40, p. 643–680, Jan. 2002. Cit. on p. 47.

CHALLINOR, A.; LASENBY, A. Relativistic corrections to the sunyaevzeldovich effect. **The Astrophysical Journal**, American Astronomical Society, v. 499, n. 1, p. 16, May 1998. ISSN 1538-4357. Available at: http://dx.doi.org/10.1086/305623. Cit. on pp. 47 and 50.

CHOQUET-BRUHAT, Y. **General Relativity and the Einstein Equations**. Oxford Univ Pr, 2009. (Oxford Mathematics Hardcover Unnumbered). ISBN 0199230722,9780199230723. Cit. on p. 13.

COMPTON, A. H. A quantum theory of the scattering of x-rays by light elements. **Phys. Rev.**, American Physical Society, v. 21, p. 483–502, May 1923. Available at: https://link.aps.org/doi/10.1103/PhysRev.21.483. Cit. on pp. 27 and 28.

D'INVERNO, R.; VICKERS, J. **Introducing Einstein's Relativity: A Deeper Understanding**. 2. ed. Oxford University Press, 2021. ISBN 0198862024; 9780198862024. No citations.

DODELSON, S.; SCHMIDT, F. **Modern Cosmology**. 2. ed. Academic Press, 2020. ISBN 0128159480,9780128159484. Cit. on pp. 37 and 47.

EINSTEIN, A. Über einen die erzeugung und verwandlung des lichtes betreffenden heuristischen gesichtspunkt. **Annalen der Physik**, v. 322, n. 6, p. 132–148, 1905. English: *On a Heuristic Viewpoint Concerning the Production and Transformation of Light*. Cit. on p. 27.

ENOMOTO, S.; SU, Y.-H.; ZHENG, M.-Z.; ZHANG, H.-H. **Boltzmann equation and its cosmological applications**. 2023. Available at: https://arxiv.org/abs/2301.11819. Cit. on pp. 13, 15, and 21.

ESA / Planck Collaboration. **Hunting Galaxy Clusters with Planck — The SunyaevZel'dovich Effect**. 2019. ESA Science Technology website. Accessed July 17 2025, updated 1 September 2019. Available at: https://sci.esa.int/web/planck/-/48227-hunting-galaxy-clusters-with-planck-the-sz-effect. Cit. on pp. 7 and 48.

FEYNMAN, R. P. **Quantum Electrodynamics**. 1. ed. Westview Press, 1998. ISBN 978-0201360758. Cit. on p. 59.

GRIFFITHS, D. J. **Introduction to Elementary Particles**. 2. ed. Wiley-VCH, 2008. ISBN 9783527406012. Cit. on p. 59.

HEINZL, T. **Quantum Field Theory I – Lecture Notes**. 2021. Lecture notes, Institut für Kernphysik, TU Darmstadt, SoSe 2021. Acessado em julho de 2025. Available at: https://theorie.ikp.physik.tu-darmstadt.de/qcd/21_QFTI.html. Cit. on p. 59.

ITOH, N.; KOHYAMA, Y.; NOZAWA, S. Relativistic corrections to the sunyaev-zeldovich effect for clusters of galaxies. **The Astrophysical Journal**, v. 502, n. 1, p. 7, jul 1998. Available at: https://dx.doi.org/10.1086/305876. Cit. on p. 50.

KLEIN, O.; NISHINA, Y. Über die streuung von strahlung durch freie elektronen nach der neuen relativistischen quantendynamik von dirac. **Zeitschrift für Physik**, Springer, v. 52, n. 1112, p. 853–868, 1929. Cit. on p. 29.

LIFSHITZ, E. M.; PITAEVSKII, L. P. **Physical Kinetics**. Oxford: Pergamon Press, 1981. v. 10. (Course of Theoretical Physics, v. 10). ISBN 9780750626354. Cit. on p. 50.

MERTIG, R.; SHTABOVENKO, V.; ORELLANA, F. **FeynCalc: Computer Algebra System for High Energy Physics**. 2023. Version 9.3 or later. Available at: https://feyncalc.github.io/. Cit. on p. 33.

NOZAWA, S.; ITOH, N.; KOHYAMA, Y. Relativistic corrections to the sunyaev-zeldovich effect for clusters of galaxies: effect of the motion of the observer. **Astronomy amp; Astrophysics**, EDP Sciences, v. 440, n. 1, p. 3944, Aug. 2005. ISSN 1432-0746. Available at: http://dx.doi.org/10.1051/0004-6361:20052923. Cit. on pp. 47 and 50.

PESKIN, M. E.; SCHROEDER, D. V. **An Introduction To Quantum Field Theory**. CRC Press, 1995. (Frontiers in Physics). ISBN 0201503972, 9780201503975. Cit. on pp. 19, 34, 59, and 60.

Planck Collaboration. Planck 2015 results. xxiv. cosmology from sunyaevzel'dovich cluster counts. **Astronomy & Astrophysics**, v. 594, p. A24, 2016. Cit. on p. 10.

RÄSÄNEN, S.; BOLEJKO, K.; FINOGUENOV, A. New test of the friedmann-lemaître-robertson-walker metric using the distance sum rule. **Phys. Rev. Lett.**, American Physical Society, v. 115, p. 101301, Sep 2015. Available at: https://link.aps.org/doi/10.1103/PhysRevLett.115.101301. Cit. on p. 23.

RESEARCH, W. **Wolfram Mathematica**. 2023. Version 13.3 or later. Available at: https://www.wolfram.com/mathematica/. Cit. on p. 33.

RYDER, L. H. **Quantum field theory**. 2nd ed. ed. Cambridge University Press, 1996. ISBN 0521478146,9780521478144,0521472423. Cit. on p. 59.

SAZONOV, S. Y.; SUNYAEV, R. A. Cosmic microwave background radiation in the direction of a moving cluster of galaxies with hot gas: Relativistic corrections. **The Astrophysical Journal**, v. 508, n. 1, p. 1, nov 1998. Available at: https://dx.doi.org/10.1086/306406. Cit. on p. 50.

SCHIAPPUCCI, E.; BIANCHINI, F.; AGUENA, M.; ARCHIPLEY, M.; BALKENHOL, L.; BLEEM, L. E.; CHAUBAL, P.; CRAWFORD, T. M.; GRANDIS, S.; OMORI, Y.; REICHARDT, C. L.; ROZO, E.; RYKOFF, E. S.; TO, C.; ABBOTT, T. M. C.; ADE, P. A. R.; ALVES, O.; ANDERSON, A. J.; ANDRADE-OLIVEIRA, F.; ANNIS, J.; AVVA, J. S.; BACON, D.; BENABED, K.; BENDER, A. N.; BENSON, B. A.; BERNSTEIN, G. M.; BERTIN, E.; BOCQUET, S.; BOUCHET, F. R.; BROOKS, D.; BURKE, D. L.; CARLSTROM, J. E.; ROSELL, A. C.; KIND, M. C.; CARRETERO, J.; CECIL, T. W.; CHANG, C. L.; CHICHURA, P. M.; CHOU, T.-L.; COSTANZI, M.; CUKIERMAN, A.; COSTA, L. N. da; DALEY, C.; HAAN, T. de; DESAI, S.; DIBERT, K. R.; DIEHL, H. T.; DOBBS, M. A.; DOEL, P.; DOUX, C.; DUTCHER, D.; EVERETT, S.; EVERETT, W.; FENG, C.; FERGUSON, K. R.; FERRERO, I.; FERTÉ, A.; FLAUGHER, B.; FOSTER, A.; FRIEMAN, J.; GALLI, S.; GAMBREL, A. E.; GARCÍA-BELLIDO, J.; GARDNER, R. W.;

GATTI, M.; GIANNANTONIO, T.; GOECKNER-WALD, N.; GRUEN, D.; GUALTIERI, R.; GUNS, S.; GUTIERREZ, G.; HALVERSON, N. W.; HINTON, S. R.; HIVON, E.; HOLDER, G. P.; HOLLOWOOD, D. L.; HOLZAPFEL, W. L.; HONSCHEID, K.; HOOD, J. C.; HUANG, N.; JAMES, D. J.; KNOX, L.; KORMAN, M.; KUEHN, K.; KUO, C.-L.; LAHAV, O.; LEE, A. T.; LIDMAN, C.; LIMA, M.; LOWITZ, A. E.; LU, C.; MARCH, M.; MENA-FERNÁNDEZ, J.; MENANTEAU, F.; MILLEA, M.; MIQUEL, R.; MOHR, J. J.; MONTGOMERY, J.; MUIR, J.; NATOLI, T.; NOBLE, G. I.; NOVOSAD, V.; OGANDO, R. L. C.; PADIN, S.; PAN, Z.; PAZ-CHINCHÓN, F.; PEREIRA, M. E. S.; PIERES, A.; MALAGÓN, A. A. P.; PRABHU, K.; PRAT, J.; QUAN, W.; RAHLIN, A.; RAVERI, M.; RODRIGUEZ-MONROY, M.; ROMER, A. K.; ROUBLE, M.; RUHL, J. E.; SANCHEZ, E.; SCARPINE, V.; SCHUBNELL, M.; SMECHER, G.; SMITH, M.; SOARES-SANTOS, M.; SOBRIN, J. A.; SUCHYTA, E.; SUZUKI, A.; TARLE, G.; THOMAS, D.; THOMPSON, K. L.; THORNE, B.; TUCKER, C.; UMILTA, C.; VIEIRA, J. D.; VINCENZI, M.; WANG, G.; WEAVERDYCK, N.; WELLER, J.; WHITEHORN, N.; WU, W. L. K.; YEFREMENKO, V.; YOUNG, M. R. Measurement of the mean central optical depth of galaxy clusters via the pairwise kinematic sunyaev-zel'dovich effect with spt-3g and des. **Phys. Rev. D**, American Physical Society, v. 107, p. 042004, Feb 2023. Available at: https://link.aps.org/doi/10.1103/PhysRevD.107.042004. Cit. on p. 46.

SCHWINGER, J. **Particles, Sources, and Fields**. Westview Press, 1998. v. 1. ISBN 9780738200538; 0738200530. Cit. on p. 59.

SCHWINGER, J. **Particles, Sources, and Fields**. Westview Press, 1998. v. 2. ISBN 9780738200545; 0738200549. Available at: libgen.li/file.php?md5=49d0f4743b113185d96caa4989eab847. Cit. on p. 59.

SEAN, C. **Spacetime and Geometry an introduction to General Relativity**. Cambridge, 2022. No citations.

SREDNICKI, M. A. **Quantum field theory**. Cambridge University Press, 2007. ISBN 9780521864497; 0521864496. Available at: https://web.physics.ucsb.edu/~mark/qft.html. Cit. on pp. 59, 61, and 62.

TERZI, B.; KRAFFT, G. A.; CLARK, W.; DEUR, A.; ROGERS, E.; SAKIOTIS, I.; VELASCO, B. Fully relativistic derivation of the thermal sunyaev-zeldovich effect. **Physics Letters B**, Elsevier BV, v. 859, p. 139119, Dec. 2024. ISSN 0370-2693. Available at: http://dx.doi.org/10.1016/j.physletb.2024.139119. Cit. on p. 51.

THOMSON, J. J. On some optical effects including refraction and rotation of the plane of polarization due to the scattering of light by electrons. **The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science**, Taylor & Francis, v. 40, n. 238, p. 713–735, December 1920. Accessed July 2025. Available at: https://archive.org/details/londonedinburg6401920lond/page/712/mode/2up. Cit. on p. 36.

WEINBERG, S. **The Quantum Theory of Fields (Volume 1)**. 1. ed. Cambridge University Press, 1995. ISBN 0521550017; 9780521550017. Cit. on pp. 19, 32, 59, 61, 62, and 63.

ZELDOVICH, Y. B.; SUNYAEV, R. A. The Interaction of Matter and Radiation in a Hot-Model Universe. , v. 4, n. 3, p. 301–316, Jul. 1969. Cit. on p. 10.

Appendices

Appendix A – QED Lagrangian and Feynman Rules

In this Appendix we will construct the Feynman Rules used in Chapter 3, for a Compton scattering (3.1):

$$\gamma(p^{\mu}) + e^{-}(q^{\mu}) \to \gamma(p'^{\mu}) + e^{-}(q'^{\mu}). \tag{A.1}$$

We will begin by showing how our metric convention can change the Lagrangian and after that obtain the invariant transition amplitude defined in equation (2.33) using perturbation theory and Wick's theorem.

A.1 Metric convention and QED Lagrangian

Conventionally, a common choice in QFT is to adopt the metric signature $\tilde{\eta}_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, as found in standard references such as (Peskin; Schroeder, 1995; Ryder, 1996; Griffiths, 2008; Feynman, 1998). However, throughout this dissertation, we adopt the opposite convention, $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, which is more commonly used in the context of general relativity and cosmology, as well as in some quantum field theory texts, such as (Weinberg, 1995; Schwinger, 1998a; Schwinger, 1998b; Srednicki, 2007), and lecture notes like (Heinzl, 2021).

This change in signature directly affects the Clifford algebra satisfied by the Dirac gamma matrices. In the $\tilde{\eta}_{\mu\nu}$ convention, the gamma matrices $\tilde{\gamma}^{\mu}$ obey the following:

$$\{\tilde{\gamma}^{\mu}, \tilde{\gamma}^{\nu}\} = 2\tilde{\eta}^{\mu\nu}. \tag{A.2}$$

In contrast, under the $\eta_{\mu\nu}$ metric, the gamma matrices γ^{μ} satisfy:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}. \tag{A.3}$$

Hence, the Clifford algebras associated with each signature are related by an overall sign:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = -\{\tilde{\gamma}^{\mu}, \tilde{\gamma}^{\nu}\}. \tag{A.4}$$

This relation implies that the gamma matrices in both conventions are connected through a redefinition that ensures consistency with the respective Clifford algebra:

$$\gamma^{\mu} = -i\tilde{\gamma}^{\mu}. \tag{A.5}$$

Importantly, this change in metric signature does not affect the structure of the Poincaré group, nor does it modify the generators of Lorentz transformations in the spinor representation:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] .$$

It is also important to emphasize that this redefinition of the gamma matrices does not alter the Pauli matrices σ^μ, which are used in constructing explicit representations of the gamma matrices (such as in the Dirac or Weyl bases). The Pauli matrices are intrinsic to the spin-$\frac{1}{2}$ representation of $\mathrm{SU}(2)$ and remain unchanged under changes in the spacetime metric signature, since they describe internal spin degrees of freedom rather than spacetime geometry.

However, since the change in metric signature modifies the Clifford algebra, and consequently the gamma matrices, it is straightforward to see that the QED Lagrangian, and the corresponding Feynman rules, will also be affected. These changes will be analyzed in the following sections.

A.2 QED Lagrangian

In the more conventional metric for particle physics $\tilde{\eta}_{\mu\nu}$, the QED Lagrangian is (Peskin; Schroeder, 1995):

$$\mathcal{L}_{\text{QED}} = \overline{\tilde{\Psi}}(i\tilde{\gamma}^\mu \partial_\mu - m)\Psi - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - e\overline{\tilde{\Psi}}\tilde{\gamma}^\mu \Psi \tilde{A}_\mu \,, \tag{A.6}$$

where Ψ is the Dirac field and A_μ the photon field. In addition, the first term is the Lagrangian of the free Dirac theory (i.e., free electrons), $\tilde{\mathcal{L}}_{\text{Dirac}} = \overline{\tilde{\Psi}}(i\tilde{\gamma}^\mu \partial_\mu - m)\Psi$; the second term is the Lagrangian of the free electromagnetic field, $\tilde{\mathcal{L}}_{\text{Maxwell}} = -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$, where $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$ is the usual electromagnetic field strength tensor; and finally, the last term describes the interaction between the fermion and gauge fields: $\tilde{\mathcal{L}}_{\text{int}} = -e\overline{\tilde{\Psi}}\tilde{\gamma}^\mu \Psi \tilde{A}_\mu$.

The first point to notice is that the covariant quantities $\tilde{A}_\mu$ and $\tilde{F}_{\mu\nu}$ are defined independently of the metric signature. Their definitions involve only partial derivatives and are not tied to the metric. Therefore, their components remain unchanged under a change of signature, and we may consistently identify:

$$\tilde{A}_\mu = A_\mu, \qquad \tilde{F}_{\mu\nu} = F_{\mu\nu}.$$

What does depend on the metric is the process of raising indices, as in $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$. However, this redefinition does not affect the structure of the kinetic term, since:

$$F_{\mu\nu}F^{\mu\nu} = \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{A.7}$$

because both sides are full contractions that absorb any change in the metric signature. Furthermore, the interaction term couples the fermionic current to the gauge field via A_μ, and not A^μ, and this contraction remains unaffected by metric conventions. Consequently, we may rewrite the Lagrangian (A.6) as:

$$\mathcal{L}_{\text{QED}} = \overline{\tilde{\Psi}}(i\tilde{\gamma}^\mu\partial_\mu - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\overline{\tilde{\Psi}}\tilde{\gamma}^\mu\Psi A_\mu\,. \tag{A.8}$$

Now, focusing on the fermionic sector of the QED Lagrangian (A.8), we substitute $\tilde{\gamma}^\mu = i\gamma^\mu$, using Eq. (A.5), and obtain:

$$\begin{aligned}\overline{\tilde{\Psi}}(i\tilde{\gamma}^\mu\partial_\mu - m)\Psi &= \overline{\Psi}\left(i(i\gamma^\mu)\partial_\mu - m\right)\Psi & \text{(A.9)}\\ &= -\overline{\Psi}\left(\gamma^\mu\partial_\mu + m\right)\Psi\,. & \text{(A.10)}\end{aligned}$$

where, all our Dirac adjoints are now $\overline{\tilde{\Psi}} = \Psi\tilde{\gamma}^0 = \Psi\beta = \overline{\Psi}$, with $\beta = i\gamma^0$. Applying the same redefinition to the interaction term yields:

$$-e\overline{\tilde{\Psi}}\tilde{\gamma}^\mu\Psi A_\mu = -ie\overline{\Psi}\gamma^\mu\Psi A_\mu\,. \tag{A.11}$$

Finally, the complete QED Lagrangian in the metric convention $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ becomes:

$$\boxed{\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \overline{\Psi}(\gamma^\mu\partial_\mu + m)\Psi - ie\overline{\Psi}\gamma^\mu A_\mu\Psi\,.} \tag{A.12}$$

This is the same expression found in QFT references that adopt the $\eta_{\mu\nu}$ metric, such as (Weinberg, 1995). Each term maintains its physical role, with only the Clifford algebra-dependent elements adjusted, that is:

$$\begin{aligned}&\mathcal{L}_{\text{Dirac}} = -\overline{\Psi}(\gamma^\mu\partial_\mu + m)\Psi\,, \quad \mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,\\ &\mathcal{L}_{int} = -ie\overline{\Psi}\gamma^\mu A_\mu\Psi\,.\end{aligned} \tag{A.13}$$

A.2.1 Equations of motion and Fields

A.2.1.1 Dirac equation

From the Dirac Lagrangian in (A.12), the Euler-Lagrange equation yields the Dirac equation for the spinor field (Weinberg, 1995):

$$(\gamma^\mu\partial_\mu + m)\Psi = 0. \tag{A.14}$$

As usual in QFT, the spinor field can be expanded in terms of creation and annihilation operators as (Weinberg, 1995; Srednicki, 2007):

$$\Psi(x) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\sum_{s=1}^{2}\left(u(\mathbf{q},\sigma)e^{iq\cdot x}a_{\mathbf{q},\sigma} + v(\mathbf{q},\sigma)e^{-iq\cdot x}b^\dagger_{\mathbf{q},\sigma}\right), \tag{A.15}$$

where $a_{\mathbf{q},\sigma}$ and $b^\dagger_{\mathbf{q},\sigma}$ are the annihilation and creation operators for electrons and positrons, respectively, and $u(\mathbf{q},\sigma)$, $v(\mathbf{q},\sigma)$ are the corresponding spinors with spin σ. Applying the Dirac equation (A.14) to the field (A.15) we obtain the equation for each spinor as:

$$\begin{aligned}&(i\gamma^\mu q_\mu + m)u(\mathbf{q},\sigma) = 0\\ &(-i\gamma^\mu q_\mu + m)v(\mathbf{q},\sigma) = 0\,.\end{aligned} \tag{A.16}$$

A.2.1.2 Maxwell Equation

We begin by gauge-fixing the Maxwell Lagrangian in (A.12) using the Lorentz-Feynman gauge condition $\partial_\mu A^\mu = 0$. This procedure does not alter the structure of the QED Lagrangian, but greatly simplifies calculations, particularly the photon propagator in perturbation theory. In this gauge, the Maxwell Lagrangian can be rewritten as:

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu). \tag{A.17}$$

The Euler-Lagrange equation derived from this Lagrangian yields the photon field equations:

$$\Box A^\mu = 0, \quad \text{with} \quad \Box = \partial_\rho \partial^\rho. \tag{A.18}$$

As usual in QFT, the photon field admits the mode expansion (Weinberg, 1995; Srednicki, 2007):

$$A_\mu(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=0}^{3} \left(e^{ip\cdot x} \varepsilon_\mu(\mathbf{p}, \lambda) a_{\mathbf{p},\lambda} + e^{-ip\cdot x} \varepsilon^*_\mu(\mathbf{p}, \lambda) a^\dagger_{\mathbf{p},\lambda} \right), \tag{A.19}$$

where $a_{\mathbf{k},\lambda}$ and $a^\dagger_{\mathbf{k},\lambda}$ are annihilation and creation operators for photons with polarization λ, and $\varepsilon_\mu(\mathbf{p}, \lambda)$ are the corresponding polarization vectors.

A.3 Feynman Rules for QED

Now that we have understood the relation between each Lagrangian in their respective metric convention, we will focus on deriving Feynman rules for QED in our $\eta_{\mu\nu}$ metric, that is, we will derive the rules from the Lagrangian (A.12), and obtain as a result the same rules as in (Weinberg, 1995).

Emergence of Feynman Diagrams via Wick's Theorem

One of the central objectives of QFT is to compute the scattering amplitude $\mathcal{A}$, which encodes the probability of transitions between the initial and final asymptotic states. As defined in Chapter 2 equation (2.33) we write this scattering amplitude as:

$$\langle \{\mathbf{p}_f\}|_{\text{inv}}\, i\hat{T}\, |\{\mathbf{p}_i\}\rangle_{\text{inv}} = \mathcal{A}\,, \tag{A.20}$$

where $|\{\mathbf{p}_i\}\rangle_{\text{inv}}$ is defined by equation (2.34). As we stated in equation (2.35), this transition amplitude must preserve energy-momentum conservation. Therefore, we can factorize $\mathcal{A}$ as:

$$\mathcal{A} = (2\pi)^4 \delta^{(4)} \left(\sum_{\text{i}} p_{\text{i}} - \sum_{\text{f}} p_{\text{f}} \right) i\mathcal{M}\,. \tag{A.21}$$

As expected, this definition depends on the scattering operator $\hat{S}$, which in Chapter 2 we defined by equation (2.30) as $\hat{S} = \hat{1} + \hat{T}$. However, this is a simplification. To be more precise, the interaction picture in QFT allows one to write the scattering operator as (Weinberg, 1995):

$$\hat{S} = \mathcal{T} \exp\left[-i \int d^4x \mathcal{L}_{\text{int}}\right], \tag{A.22}$$

where $\mathcal{T}$ is the time-ordering operator, and $\mathcal{L}_{\text{int}}$ is the Lagrangian that describes the interaction between the fields. When we expand this operator in a Dyson series, we can write:

$$\begin{aligned} \hat{S} &\equiv \hat{1} + i\hat{T}^{(1)} + i^2\hat{T}^{(2)} + i^3\hat{T}^{(3)} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \, \mathcal{T}\left[\mathcal{L}_{\text{int}}(x_1) \cdots \mathcal{L}_{\text{int}}(x_n)\right], \end{aligned} \tag{A.23}$$

where each Lagrangian gives a vertex of interaction. Consequently, the $\hat{T}$ that appears in (2.30) is actually $\hat{T} = \hat{T}^{(1)} + i\hat{T}^{(2)} + i^2\hat{T}^{(3)} + \cdots$. Each term of $\hat{T}$ is identified as:

$$i^n\hat{T}^{(n)} = \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \mathcal{T}\left[\mathcal{L}_{\text{int}}(x_1) \cdots \mathcal{L}_{\text{int}}(x_n)\right], \tag{A.24}$$

which means that each term of $\hat{T}$ involves time-ordered products of the interaction Lagrangian evaluated at different spacetime points.

For the interaction Lagrangian of QED in equation (A.12) (i.e., $\mathcal{L}_{\text{int}} = -ie\overline{\Psi}\gamma^\mu A_\mu \Psi$), the first-order term is:

$$i\hat{T}^{(1)} = e \int d^4x_1 \mathcal{T}\left[\overline{\Psi}(x_1)\gamma^\mu A_\mu(x_1)\Psi(x_1)\right]. \tag{A.25}$$

However, this vertex describes unphysical processes like $e^- \to e^-\gamma$ or $\gamma \to e^+e^-$ (two fermions plus one photon) that violate energy-momentum conservation: $e^- \to e^-\gamma$ breaks $q^2 = m^2$, and $\gamma \to e^+e^-$ requires $p^2 \geq 4m_e^2$ but real photons have $p^2 = 0$. Thus $\langle \mathbf{p}', \mathbf{q}' | \, i\hat{T}^{(1)} \, | \mathbf{p}, \mathbf{q} \rangle = 0$, leaving $\hat{T}^{(2)}$ as the leading contribution:

$$\begin{aligned} i^2\hat{T}^{(2)} &= \frac{-i^2}{2!} \int d^4x_1 d^4x_2 \mathcal{T}\left[\mathcal{L}_{\text{int}}(x_1)\mathcal{L}_{\text{int}}(x_2)\right] \\ &= \frac{-i^2}{2} \int d^4x_1 d^4x_2 \mathcal{T}\left[\left(-ie\overline{\Psi}(x_1)\gamma^\mu A_\mu(x_1)\Psi(x_1)\right)\left(-ie\overline{\Psi}(x_2)\gamma^\nu A_\nu(x_2)\Psi(x_2)\right)\right] \\ &= \frac{e^2}{2} \int d^4x_1 d^4x_2 \mathcal{T}\left[\left(\overline{\Psi}(x_1)\gamma^\mu A_\mu(x_1)\Psi(x_1)\right)\left(\overline{\Psi}(x_2)\gamma^\nu A_\nu(x_2)\Psi(x_2)\right)\right]. \end{aligned} \tag{A.26}$$

Therefore, for Compton scattering given by equation (A.1), the second-order transition amplitude term is given by:

$$\langle \mathbf{p}', \mathbf{q}' |_{\text{inv}} \, i^2\hat{T}^{(2)} \, | \mathbf{p}, \mathbf{q} \rangle_{\text{inv}} = i\mathcal{A}^{(2)}, \tag{A.27}$$

where

$$i\mathcal{A}^{(2)} = \frac{e^2}{2} \int d^4x_1 d^4x_2 \mathcal{K}', \tag{A.28}$$

with $\mathcal{K}'$ being the matrix element:

$$\mathcal{K}' = \langle \mathbf{p}', \mathbf{q}'|_{\text{inv}}\, \mathcal{T}\left[\overline{\Psi}(x_1)\gamma^{\mu}A_{\mu}(x_1)\Psi(x_1)\ \overline{\Psi}(x_2)\gamma^{\nu}A_{\nu}(x_2)\Psi(x_2)\right]|\mathbf{p}, \mathbf{q}\rangle_{\text{inv}}\,. \tag{A.29}$$

From equation (2.34) this gives:

$$\begin{aligned}\mathcal{K}' &= \sqrt{2E_{p'}}\sqrt{2E_{q'}}\sqrt{2E_p}\sqrt{2E_q}\times \\ &\times \langle 0|\, a_{\mathbf{p}',\lambda'}a_{\mathbf{q}',\sigma'}\left(\mathcal{T}\left[\overline{\Psi}(x_1)\gamma^{\mu}A_{\mu}(x_1)\Psi(x_1)\overline{\Psi}(x_2)\gamma^{\nu}A_{\nu}(x_2)\Psi(x_2)\right]\right)a^{\dagger}_{\mathbf{p},\lambda}a^{\dagger}_{\mathbf{q},\sigma}\,|0\rangle\,.\end{aligned} \tag{A.30}$$

Applying Wick's theorem [1], we can write:

$$\begin{aligned}\mathcal{K}' = \sqrt{2E_{p'}}\sqrt{2E_{q'}}\sqrt{2E_p}\sqrt{2E_q}\quad 2\times \Bigg[\\ = \underbrace{\langle 0|\, a_{\mathbf{q}',\sigma'}a_{\mathbf{p}',\lambda'}\overline{\Psi}(x_1)\gamma^{\mu}A_{\mu}(x_1)\Psi(x_1)\overline{\Psi}(x_2)\gamma^{\nu}A_{\nu}(x_2)\Psi(x_2)a^{\dagger}_{\mathbf{p},\lambda}a^{\dagger}_{\mathbf{q},\sigma}\,|0\rangle}_{\substack{\textbf{Diagram 1:}\text{ Photon absorbed at } x_2,\text{ emitted at } x_1. \\ \text{Electron propagates via internal line } \Psi(x_1)\,\overline{\Psi}(x_2)}} + \\ + \underbrace{\langle 0|\, a_{\mathbf{q}',\sigma'}a_{\mathbf{p}',\lambda'}\overline{\Psi}(x_1)\gamma^{\mu}A_{\mu}(x_1)\Psi(x_1)\overline{\Psi}(x_2)\gamma^{\nu}A_{\nu}(x_2)\Psi(x_2)a^{\dagger}_{\mathbf{p},\lambda}a^{\dagger}_{\mathbf{q},\sigma}\,|0\rangle}_{\substack{\textbf{Diagram 2:}\text{ Photon absorbed at } x_1,\text{ emitted at } x_2. \\ \text{Electron propagates via internal line } \Psi(x_1)\,\overline{\Psi}(x_2)}} \Bigg]\end{aligned} \tag{A.31}$$

The factor 2 arises because Wick's theorem generates two topologically distinct diagrams corresponding to the s-channel and u-channel processes. Each of these diagrams is symmetric under the exchange of interaction vertices ($x_1 \leftrightarrow x_2$). This symmetry does not yield new diagrams but rather reflects the redundancy in vertex labeling within the time-ordered product. The first diagram corresponds to the s-channel, and the second to the u-channel (see diagrams (3.9)). No t-channel diagram appears, as such a process would require an internal photon-photon propagator, which does not arise at second order in QED, where all interactions involve fermion-photon couplings.

We now compute the contractions using the field expansions. For the photon field,

[1] *Wick's theorem* allows us to expand time-ordered products of field operators into a sum over all possible contractions. In the vacuum expectation value, only the fully contracted terms contribute, as all uncontracted field operators annihilate the vacuum. This expansion forms the foundation for perturbative calculations in QFT and the diagrammatic representation via Feynman diagrams.

equation (A.19) gives [2]:

$$A_\mu a^\dagger_{\mathbf{p},\lambda}|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\sum_{\lambda'=0}^{3}\left(e^{ik\cdot x}\varepsilon_\mu(\mathbf{k},\lambda')a_{\mathbf{k},\lambda'} + e^{-ik\cdot x}\varepsilon^*_\mu(\mathbf{k},\lambda')a^\dagger_{\mathbf{k},\lambda'}\right)a^\dagger_{\mathbf{p},\lambda}|0\rangle$$

$$= \left[\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\sum_{\lambda'=0}^{3}e^{ik\cdot x}\varepsilon_\mu(\mathbf{k},\lambda')\underbrace{a_{\mathbf{k},\lambda'}a^\dagger_{\mathbf{p},\lambda}|0\rangle}_{\text{commutator}}\right.$$

$$\left. + \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E\mathbf{k}}}\sum_{\lambda'=0}^{3}e^{-ik\cdot x}\varepsilon^*_\mu(\mathbf{k},\lambda')\underbrace{a^\dagger_{\mathbf{k},\lambda'}a^\dagger_{\mathbf{p},\lambda}|0\rangle}_{\text{two-photon state}}\right].$$

The second term vanishes because it creates a two-photon state, which is orthogonal to the vacuum and does not contribute to the Compton scattering amplitude at tree level. For the first term, use the canonical commutation relation in the first equation of (2.19), which gives:

$$A_\mu(x)a^\dagger_{\mathbf{p},\lambda}|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\sum_{\lambda'=0}^{3}\varepsilon_\mu(\mathbf{k},\lambda')e^{ik\cdot x}(2\pi)^3\delta_{\lambda'\lambda}\delta^{(3)}(\mathbf{k}-\mathbf{p})|0\rangle\ .$$

Therefore:

$$A_\mu(x)a^\dagger_{\mathbf{p},\lambda}|0\rangle = \frac{1}{\sqrt{2E_\mathbf{p}}}\varepsilon_\mu(\mathbf{p},\lambda)e^{ip\cdot x}|0\rangle\ . \tag{A.32}$$

On the other hand, doing the same process for the fermionic field (A.15), we obtain:

$$\Psi(x_2)a^\dagger_{\mathbf{q},\sigma}|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\sum_{\sigma'=1}^{2}\left(e^{ik\cdot x_2}a_{\mathbf{k},\sigma'}u(\mathbf{k},\sigma') + b^\dagger_{\mathbf{k},\sigma'}v(\mathbf{k},\sigma')e^{-ik\cdot x_2}\right)a^\dagger_{\mathbf{q},\sigma}|0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\sum_{\sigma'=1}^{2}e^{ik\cdot x_2}u(\mathbf{k},\sigma')\underbrace{a_{\mathbf{k},\sigma'}a^\dagger_{\mathbf{q},\sigma}|0\rangle}_{\text{anticommutator}}$$

$$+ \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\sum_{\sigma'=1}^{2}e^{-ik\cdot x_2}v(\mathbf{k},\sigma')\underbrace{b^\dagger_{\mathbf{k},\sigma'}a^\dagger_{\mathbf{q},\sigma}|0\rangle}_{\text{two-particle state}}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\sum_{\sigma'=1}^{2}e^{ik\cdot x_2}u(\mathbf{k},\sigma')(2\pi)^3\delta_{\sigma'\sigma}\delta^{(3)}(\mathbf{k}-\mathbf{q})|0\rangle\ .$$

Therefore:

$$\Psi(x_2)a^\dagger_{\mathbf{q},s}|0\rangle = \frac{1}{\sqrt{2E_\mathbf{q}}}e^{iq\cdot x_2}u(\mathbf{q},\sigma)|0\rangle\ . \tag{A.33}$$

With these results, we can write equation (A.31) as:

$$\mathcal{K}' = \mathcal{K}'_s + \mathcal{K}'_u \tag{A.34}$$

[2] Here, λ' denotes a generic polarization, not necessarily the final polarization.

with

$$
\begin{aligned}
\mathcal{K}'_s &= 2\bar{u}(\mathbf{q}',\sigma')\gamma^\mu\varepsilon^*_\mu(\mathbf{p}',\lambda')\,S_F(x_1-x_2)\,\gamma^\nu\varepsilon_\nu(\mathbf{p},\lambda)u(\mathbf{q},\sigma)\,e^{-i(q'+p')\cdot x_1}e^{i(q+p)\cdot x_2}\\
\mathcal{K}'_u &= 2\bar{u}(\mathbf{q}',\sigma')\gamma^\mu\varepsilon_\mu(\mathbf{p},\lambda)\,S_F(x_1-x_2)\,\gamma^\nu\varepsilon^*_\nu(\mathbf{p}',\lambda')u(\mathbf{q},\sigma)\,e^{-i(p'-q)\cdot x_2}e^{-i(q'-p)\cdot x_1}\,,
\end{aligned}
\tag{A.35}
$$

where the label s indicates the $s-$ channel and the label u the $u-$ channel. We define the fermion propagator as:

$$
S_F(x_1-x_2) := \langle 0|\,\mathcal{T}\left\{\Psi(x_1)\overline{\Psi}(x_2)\right\}|0\rangle = \Psi(x_1)\overline{\Psi}(x_2)\,. \tag{A.36}
$$

Now, we are able to write (A.28) as:

$$
i\mathcal{A}^{(2)} = \frac{e^2}{2}\int d^4x_1d^4x_2\left(\mathcal{K}'_1+\mathcal{K}'_2\right)\,. \tag{A.37}
$$

To solve the d^4x_i integrals, we need to write the fermion propagator of each $\mathcal{K}'_i$ in the momentum space via a Fourier transformation:

$$
S(x_1-x_2) = \int\frac{d^4k}{(2\pi)^4}S(k)e^{-k\cdot(x_1-x_2)}\,. \tag{A.38}
$$

Now, we will calculate each $\mathcal{K}'_i$ integral.

Proof. For the $\mathcal{K}'_1$ integral we have:

$$
\begin{aligned}
\int d^4x_1d^4x_2\mathcal{K}'_1 =&2\int d^4x_1d^4x_2\Bigg[\bar{u}(\mathbf{q}',\sigma')\gamma^\mu\varepsilon^*_\mu(\mathbf{p}',\lambda')\times\\
&\times\int\frac{d^4k}{(2\pi)^4}S(k)e^{-ik(x_1-x_2)}\gamma^\nu\varepsilon_\nu(\mathbf{p},\lambda)u(\mathbf{q},\sigma)\,e^{-i(q'+p')\cdot x_1}e^{i(q+p)\cdot x_2}\Bigg]\,.
\end{aligned}
$$

We now bring together all the exponential terms, which allows us to write:

$$
\begin{aligned}
\int d^4x_1d^4x_2\mathcal{K}'_1 =\;& 2\int d^4x_1d^4x_2\Bigg[\bar{u}(\mathbf{q}',\sigma')\gamma^\mu\varepsilon^*_\mu(\mathbf{p}',\lambda')\times\\
&\times\int\frac{d^4k}{(2\pi)^4}S(k)\gamma^\nu\varepsilon_\nu(\mathbf{p},\lambda)u(\mathbf{q},\sigma)\,e^{-i(q'+p'+k)\cdot x_1}e^{i(q+p+k)\cdot x_2}\Bigg]\,.
\end{aligned}
$$

Integrating over both d^4x we have:

$$
\begin{aligned}
\int d^4x_1e^{-i(q'+p'+k)\cdot x_1} &= (2\pi)^4\delta^{(4)}(p'+q'+k),\\
\int d^4x_2e^{i(q+p+k)\cdot x_2} &= (2\pi)^4\delta^{(4)}(p+q+k).
\end{aligned}
$$

We use the first delta function to solve the d^4k integral, which results in:

$$\int d^4x_1 d^4x_2 \mathcal{K}_1' = 2(2\pi)^4\delta^{(4)}(p+q-p'-q')\Big[\bar{u}(\mathbf{q}',\sigma')\gamma^\mu\varepsilon^*_\mu(\mathbf{p}',\lambda')\times$$
$$\times S(-(p'+q'))\gamma^\nu\varepsilon_\nu(\mathbf{p},\lambda)u(\mathbf{q},\sigma)\Big].$$

Therefore, since $p+q=-(p'+q')$ we can write:

$$\begin{aligned}\int d^4x_1 d^4x_2 \mathcal{K}_1' =& (2\pi)^4\delta^{(4)}(p+q-p'-q')\times 2\Big[\\ & \bar{u}(\mathbf{q}',\sigma')\gamma^\mu\varepsilon^*_\mu(\mathbf{p}',\lambda')S(p+q))\gamma^\nu\varepsilon_\nu(\mathbf{p},\lambda)u(\mathbf{q},\sigma)\Big].\end{aligned} \tag{A.39}$$

For the $\mathcal{K}_2'$ integral we have:

$$\begin{aligned}\int d^4x_1 d^4x_2 \mathcal{K}_2' =& 2\int d^4x_1 d^4x_2\Big[\bar{u}(\mathbf{q}',\sigma')\gamma^\mu\varepsilon_\mu(\mathbf{p},\lambda)\times\\ & \times\int\frac{d^4k}{(2\pi)^4}S(k)e^{-ik(x_1-x_2)}\gamma^\nu\varepsilon^*_\nu(\mathbf{p}',\lambda')u(\mathbf{q},\sigma)\times\\ & \times e^{-i(q'-p)\cdot x_1}e^{-i(p'-q)\cdot x_2}\Big].\end{aligned}$$

Integrating over both d^4x we have:

$$\int d^4x_1 e^{-i(q'-p+k)\cdot x_1} = (2\pi)^4\delta^{(4)}(q'-p+k),$$
$$\int d^4x_2 e^{i(p'-q+k)\cdot x_2} = (2\pi)^4\delta^{(4)}(p'-q+k),$$

we use the second delta function to solve the d^4k integral, which results in:

$$\begin{aligned}\int d^4x_1 d^4x_2 \mathcal{K}_2' =& (2\pi)^4\delta^{(4)}(q'-p+q-p')\times 2\Big[\\ & \bar{u}(\mathbf{q}',\sigma')\gamma^\mu\varepsilon_\mu(\mathbf{p},\lambda)S(q-p')\gamma^\nu\varepsilon^*_\nu(\mathbf{p}',\lambda')u(\mathbf{q},\sigma)\Big].\end{aligned} \tag{A.40}$$

□

Finally, by equations (A.39) and (A.40) [3] we can write equation (A.37) as:

$$i\mathcal{A}^{(2)} = (2\pi)^4\delta^{(4)}\left(q+p-q'-p'\right)i\mathcal{M}^{(2)}, \tag{A.41}$$

[3] The delta function of both equations ensure the same overall energy-momentum conservation.

in which:

$$i\mathcal{M}^{(2)} = i\mathcal{M}_s^{(2)} + i\mathcal{M}_u^{(2)}, \tag{A.42}$$

where

$$\begin{aligned} i\mathcal{M}_s^{(2)} &= e^2\bar{u}(\mathbf{q}',\sigma')\gamma^\mu\varepsilon_\mu^*(\mathbf{p}',\lambda')S(p+q))\gamma^\nu\varepsilon_\nu(\mathbf{p},\lambda)u(\mathbf{q},\sigma), \\ i\mathcal{M}_u^{(2)} &= e^2\bar{u}(\mathbf{q}',\sigma')\gamma^\mu\varepsilon_\mu(\mathbf{p},\lambda)S(q-p')\gamma^\nu\varepsilon_\nu^*(\mathbf{p}',\lambda')u(\mathbf{q},\sigma). \end{aligned} \tag{A.43}$$

The only thing missing is the explicit form of the Fermionic propagator that we will derive next.

Feynman Propagators

We begin by considering a generic free field, $\Phi(x)$ that satisfies a linear differential equation of the form:

$$O_x\Phi(x) = 0, \tag{A.44}$$

with O_x being a linear differential operator acting on spacetime point x. Its adjoint field satisfies a similar relation, $\overline{\Phi}(y)O_y^\dagger = 0$.

The Feynman propagator (or two-point Green's function) is defined as the vacuum expectation value of the time-ordered product:

$$G_F(x-y) := \langle 0|\,\mathcal{T}\{\Phi(x)\overline{\Phi}(y)\}\,|0\rangle\ , \tag{A.45}$$

which represents the amplitude for propagation of the field from point y to point x.

To find the equation satisfied by the propagator, we act with the operator O_x on (A.45):

$$O_xG_F(x-y) = \langle 0|\,O_x\mathcal{T}\{\Phi(x)\overline{\Phi}(y)\}\,|0\rangle\,.$$

However, since O_x is a differential operator, it does not commute with the time-ordering operator due to singularities at coincident points. This leads to:

$$O_x\mathcal{T}\{\Phi(x)\overline{\Phi}(y)\} = \mathcal{T}\{O_x\Phi(x)\overline{\Phi}(y)\} + [O_x,\mathcal{T}]\{\Phi(x)\overline{\Phi}(y)\}.$$

Using the free field equation $O_x\Phi(x) = 0$, the first term vanishes. The remaining contribution comes from the commutator between the operator and the time-ordering, which generates a delta distribution at the coincidence point:

$$O_xG_F(x-y) = \langle 0|\,[O_x,\mathcal{T}]\{\Phi(x)\overline{\Phi}(y)\}\,|0\rangle = -i\delta^{(4)}(x-y). \tag{A.46}$$

This identity follows from the canonical (anti)commutation relations and holds universally for free fields.

Therefore, the Feynman propagator is the Green's function of the differential operator O_x:

$$\boxed{O_xG_F(x-y) = -i\delta^{(4)}(x-y)}. \tag{A.47}$$

Or, if we realize a Fourier transformation, we have in momentum space:

$$\boxed{O_pG_F(p) = -i\,.} \tag{A.48}$$

Fermionic Propagator

We have already determined the equation of motion for fermions (A.14), from which we have the operator $(\gamma^\mu \partial_\mu + m)$, that in momentum space ($\partial_\mu \to -ip_\mu$) can be written as $(-i\gamma^\mu p_\mu + m)$. We apply this to (A.48):

$$(-i\gamma^\mu p_\mu + m) S_F(p) = -i. \tag{A.49}$$

Thus:

$$S_F(p) = \frac{-i}{-i\gamma^\mu p_\mu + m} = \frac{-i(-i\gamma^\mu p_\mu + m)}{p^2 + m^2 - i\epsilon}. \tag{A.50}$$

The Feynman prescription $-i\epsilon$ ensures the correct treatment of the pole at $p^2 = -m^2$.

Photon Propagator

The same process is used to derive the photon propagator. However, since A^μ is a vector field, its propagator is a rank-2 tensor $G_F^{\mu\nu}(p)$, and we generalize (A.48) by writing:

$$p^2 G_F^{\mu\nu}(p) = -i\eta^{\mu\nu}. \tag{A.51}$$

Therefore:

$$G_F^{\mu\nu}(p) = \frac{-i\eta^{\mu\nu}}{p^2 - i\epsilon}, \tag{A.52}$$

where the prescription $-i\epsilon$ ensures the correct treatment of the pole at $p^2 = 0$.

A.4 Transition amplitude $i\mathcal{M}^{(2)}$ for the Compton Scattering

To simplify our notation, we will define for the Compton scattering $i\mathcal{M} = i\mathcal{M}^{(2)}$, since this is our leading order. Now, with equations (A.50) and (A.43), we can finally write the transition amplitude for the Compton scattering (equation (A.42)) as:

$$\begin{aligned} i\mathcal{M} =& e^2 \bar{u}(\mathbf{q}',\sigma') \not{\epsilon}^*(\mathbf{p}',\lambda') \frac{-i(-i(\not{p}+\not{q})+m)}{(p+q)^2+m^2-i\epsilon} \not{\epsilon}(\mathbf{p},\lambda) u(\mathbf{q},\sigma) \\ &+ e^2 \bar{u}_{s'}(\mathbf{q}') \not{\epsilon}(\mathbf{p},\lambda) \frac{-i(-i(\not{q}-\not{p}')+m)}{(q-p')^2+m^2-i\epsilon} \not{\epsilon}^*(\mathbf{p}',\lambda') u(\mathbf{q},\sigma) \end{aligned},$$

where we define $\gamma^\mu \varepsilon^*_{\mu,\lambda'} = \not{\epsilon}^*_{\lambda'}$ and $\gamma^\mu \varepsilon_{\mu,\lambda} = \not{\epsilon}_\lambda$. We can rewrite this as:

$$\begin{aligned} \therefore\ i\mathcal{M} =& -ie^2 \bar{u}(\mathbf{q}',\sigma') \Bigg[\not{\epsilon}^*(\mathbf{p}',\lambda') \frac{(-i(\not{p}+\not{q})+m)}{(p+q)^2+m^2} \not{\epsilon}(\mathbf{p},\lambda) \\ &+ \not{\epsilon}(\mathbf{p},\lambda) \frac{(-i(\not{q}-\not{p}')+m)}{(q-p')^2+m^2} \not{\epsilon}^*(\mathbf{p}',\lambda') \Bigg] u(\mathbf{q},\sigma)\,. \end{aligned} \tag{A.53}$$

Note, that we have dropped the $-i\epsilon$ in the denominators, since they will not vanish (we can see this by using equation (3.5)).